Chapter 8: Techniques of Integration

<u>Content Outline</u>

- Integration of $\mathrm{f}'(x)\left[\mathrm{f}(x)\right]^n$ (including $n=-1$), $\mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}$, $\sin^2 x$, $\cos^2 x$, $\tan^2 x$, $\sin mx\cos nx, \cos mx\cos nx, \sin mx\sin nx$, $\frac{1}{a^2+x^2}$, $\frac{1}{\sqrt{a^2-x^2}}$, $\frac{1}{a^2-x^2}$ and $\frac{1}{x^2-a^2}$
- Integration by a given substitution
- Integration by parts
- Evaluation of definite integrals
- Finding the approximate value of a definite integral using a graphic calculator

<u>References</u>

- Websites

1) http://integrals.wolfram.com/ [The Wolfram Integrator allows you to type in a function and obtain the indefinite integral almost immediately- good for checking answers in Tutorials.]	2) http://archives.math.utk.edu/visual.calculus/4/index.html [Contains numerous interactive tutorials on indefinite integrals, definite integrals and animations.]

- Books
  1) Catherine Berry, Val Hanrahan, Roger Porkess. "MEI Structured Mathematics Mathematics – Pure Mathematics 3", 2nd ed, Hodder & Stoughton. Call Number: 510 BER.
  2) Perkins & Perkins, "Advanced Mathematics – A Pure Course", Collins. Call Number: PER 510.000.
  3) Hugh Neill, Douglas Quadling, "OCR – Pure Mathematics 3", Cambridge University Press. Call Number 510 NEI.

Relevant Formulas Found in MF26

Integrals (Arbitrary constants are omitted; a denotes a positive constant.)

$\mathrm{f}(x)$	$\int \mathrm{f}(x)\,\mathrm{d}x$	
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	$(\lvert x\rvert < a)$
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left(\dfrac{x-a}{x+a}\right)$	$(x > a)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left(\dfrac{a+x}{a-x}\right)$	$(\lvert x\rvert < a)$
$\tan x$	$\ln(\sec x)$	$(\lvert x\rvert < \frac{1}{2}\pi)$
$\cot x$	$\ln(\sin x)$	$(0 < x < \pi)$
$\operatorname{cosec} x$	$-\ln(\operatorname{cosec} x + \cot x)$	$(0 < x < \pi)$
$\sec x$	$\ln(\sec x + \tan x)$	$(\lvert x\rvert < \frac{1}{2}\pi)$

Trigonometry

$$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$$
$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$
$$\sin 2A \equiv 2\sin A\cos A$$
$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$
$$\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A}$$
$$\sin P + \sin Q \equiv 2\sin\tfrac{1}{2}(P+Q)\cos\tfrac{1}{2}(P-Q)$$
$$\sin P - \sin Q \equiv 2\cos\tfrac{1}{2}(P+Q)\sin\tfrac{1}{2}(P-Q)$$
$$\cos P + \cos Q \equiv 2\cos\tfrac{1}{2}(P+Q)\cos\tfrac{1}{2}(P-Q)$$
$$\cos P - \cos Q \equiv -2\sin\tfrac{1}{2}(P+Q)\sin\tfrac{1}{2}(P-Q)$$

1. Antiderivatives and Indefinite Integrals

Definition

If the function $\mathrm{f}(x)$ is the derivative of a function $\mathrm{F}(x)$ with respect to x on an interval I,

i.e. $\boxed{\dfrac{\mathbf{d}}{\mathbf{d}x}[\mathbf{F}(x)] = \mathbf{f}(x)}$, for every value of x in I,

then the function $\mathrm{F}(x)$ is called an **antiderivative** of the function $\mathrm{f}(x)$.

Note:

Antiderivatives are **NOT** unique.

For example, if $\mathrm{f}(x) = 2x$, then $x^2,\ x^2+3,\ x^2-\pi$ are all antiderivatives of f.

1.1 The Indefinite Integral

If F is an antiderivative of f, we write (in integral notation):

$\boxed{\int \mathbf{f}(x)\ \mathbf{d}x = \mathbf{F}(x) + c}$, where c is an arbitrary constant.

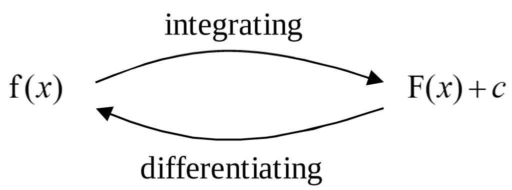
We call $\int \mathrm{f}(x)\ \mathrm{d}x$ the **indefinite integral** of f.

$\int$ - the integral sign

$\mathrm{d}x$ - (integrate) with respect to x

$\mathrm{f}(x)$ - the integrand

The process of finding antiderivatives is called **integration** and is the reverse process of differentiation, that is:



1.1.1 Properties of the Indefinite Integral

If the functions f and g are integrable on interval I, and k is a constant,

(i) $\int k\mathrm{f}(x)\ \mathrm{d}x = k\int \mathrm{f}(x)\ \mathrm{d}x$

(ii) $\int [\mathrm{f}(x) \pm \mathrm{g}(x)]\ \mathrm{d}x = \int \mathrm{f}(x)\ \mathrm{d}x \pm \int \mathrm{g}(x)\ \mathrm{d}x$

1.2 <u>The Definite Integral</u>

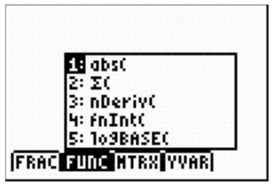
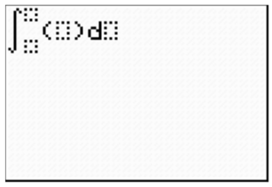
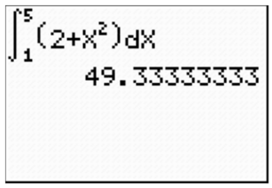
$\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ is referred to as the **definite integral** of f(x) from $x = a$ to $x = b$,
where a is the **lower limit**
and b is the **upper limit**.

If f is continuous on [a, b] and F is any antiderivative of f on [a, b], then $$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \left[\mathrm{F}(x)\right]_a^b = \mathrm{F}(b) - \mathrm{F}(a), \text{ where } \mathrm{F}(x) = \int \mathrm{f}(x)\,\mathrm{d}x..$$

e.g. $$\int_1^5 \left(2+x^2\right)\mathrm{d}x = \left[2x + \frac{x^3}{3}\right]_1^5 = \left(2(5) + \frac{5^3}{3}\right) - \left(2(1) + \frac{1^3}{3}\right) = \frac{148}{3}$$

We can use GC to evaluate $\int_1^5 (2+x^2)\,\mathrm{d}x$ as shown below:

Using the Home Screen:

1. Press alpha then window.	
2. Press 4 to select "**4: fnInt(**".	
3. Complete the expression and then press enter.	

1.2.1 Properties of the Definite Integral

If the functions f and g are integrable on [a, b], and k is a constant,

(i) $\int_a^a \mathrm{f}(x)\,\mathrm{d}x = 0$

(ii) $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = -\int_b^a \mathrm{f}(x)\,\mathrm{d}x$

Proof: $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \left[\mathrm{F}(x)\right]_a^b = \mathrm{F}(b) - \mathrm{F}(a) = -\left(\mathrm{F}(a) - \mathrm{F}(b)\right) = -\int_b^a \mathrm{f}(x)\,\mathrm{d}x$

(iii) $\int_a^b k\mathrm{f}(x)\,\mathrm{d}x = k\int_a^b \mathrm{f}(x)\,\mathrm{d}x$

(iv) $\int_a^b \left[\mathrm{f}(x) \pm \mathrm{g}(x)\right]\mathrm{d}x = \int_a^b \mathrm{f}(x)\,\mathrm{d}x \pm \int_a^b \mathrm{g}(x)\,\mathrm{d}x$

(v) $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \int_a^c \mathrm{f}(x)\,\mathrm{d}x + \int_c^b \mathrm{f}(x)\,\mathrm{d}x$, where $a < c < b$

2. Standard Integrals

2.1 Integrals Involving Algebraic Functions $(ax+b)^n,\ n \in \mathbb{R}$

Recall:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{n+1}) = (n+1)x^n, \text{ where } n \neq -1$$

$$\Rightarrow \frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}(x^{n+1}) = x^n$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{n+1}x^{n+1}\right) = x^n$$

Hence,
$$\boxed{\int x^n\,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + c}$$

And similarly,
$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$$

$$\boxed{\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c}$$

In general, $\dfrac{\mathrm{d}}{\mathrm{d}x}[(ax+b)^{n+1}]=(n+1)a(ax+b)^{n}$, where $n\neq -1$

$$\Rightarrow \boxed{\int (ax+b)^{n}\ \mathrm{d}x=\frac{(ax+b)^{n+1}}{a(n+1)}+c}$$

and $\dfrac{\mathrm{d}}{\mathrm{d}x}[\ln(ax+b)]=\dfrac{a}{ax+b}$

$$\Rightarrow \boxed{\int \frac{1}{ax+b}\ \mathrm{d}x=\frac{1}{a}\ln\left|ax+b\right|+c}$$

Summarizing the above results, we have

		In general
1.	$\int x^{n}\ \mathrm{d}x=\dfrac{x^{n+1}}{n+1}+c$, where $n\neq -1$	$\int (ax+b)^{n}\ \mathrm{d}x=\dfrac{(ax+b)^{n+1}}{a(n+1)}+c$, where $n\neq -1$
2.	$\int \dfrac{1}{x}\ \mathrm{d}x=\ln\left|x\right|+c$	$\int \dfrac{1}{ax+b}\ \mathrm{d}x=\dfrac{1}{a}\ln\left|ax+b\right|+c$

Example 1

Find the following integrals, without the use of a graphing calculator.

(a) $\int (5x^{3}+2)\ \mathrm{d}x=$

(b) $\int_{0}^{1} (4+5x)^{2}\ \mathrm{d}x=$

(c) $\int (2+x^{2})^{3}\ \mathrm{d}x=\int \left[2^{3}+3(2^{2})x^{2}+\ 3(2)(x^{2})^{2}+(x^{2})^{3}\right]\ \mathrm{d}x$

$=\int (8+12x^{2}+6x^{4}+x^{6})\ \mathrm{d}x$

$=$

(d) $\int \dfrac{2+x^{2}}{x}\ \mathrm{d}x=$

(e) $\int \dfrac{2}{2-3x}\ \mathrm{d}x=$

2.2 Integrals of the Form $\int \mathrm{f}'(x)[\mathrm{f}(x)]^n \, \mathrm{d}x$ and $\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)} \, \mathrm{d}x$

Recall:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left([\mathrm{f}(x)]^{n+1}\right) = (n+1)[\mathrm{f}(x)]^n \mathrm{f}'(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{[\mathrm{f}(x)]^{n+1}}{n+1}\right) = [\mathrm{f}(x)]^n \mathrm{f}'(x)$$

Hence,
$$\boxed{\int \mathrm{f}'(x)[\mathrm{f}(x)]^n \, \mathrm{d}x = \frac{[\mathrm{f}(x)]^{n+1}}{n+1} + c}$$

And recall that,
$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln \mathrm{f}(x)] = \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$$

Hence,
$$\boxed{\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)} \, \mathrm{d}x = \ln|\mathrm{f}(x)| + c}$$

Summarizing the above results,

1.	$\int \mathrm{f}'(x)[\mathrm{f}(x)]^n \, \mathrm{d}x = \frac{[\mathrm{f}(x)]^{n+1}}{n+1} + c$, where $n \neq -1$
2.	$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)} \, \mathrm{d}x = \ln|\mathrm{f}(x)| + c$

Example 2

Find the following integrals.

(a) $\int 4x(2x^2-5)^{10} \, \mathrm{d}x =$

(b) $\int \frac{8x+4}{\sqrt{x^2+x-1}} \, \mathrm{d}x =$

(c) $\int \frac{3x^2}{7x^3+2} \, \mathrm{d}x =$

(d) $\int \frac{x^2+1}{x^3+3x+1} \, \mathrm{d}x =$

2.3 Integrals Involving Exponential Functions

Recall:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) = \mathrm{e}^x$$

Hence, $\boxed{\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x + c}$

In general, $$\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{e}^{\mathrm{f}(x)}] = \mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}$$

Hence, $\boxed{\int \mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)} \, \mathrm{d}x = \mathrm{e}^{\mathrm{f}(x)} + c}$

		In general
1.	$\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x + c$	$\int \mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)} \, \mathrm{d}x = \mathrm{e}^{\mathrm{f}(x)} + c$

Example 3

Find the following integrals.

(a) $\int \mathrm{e}^{\frac{1}{2}x} \, \mathrm{d}x =$

(b) $\int 2x\mathrm{e}^{4x^2-1} \, \mathrm{d}x =$

(c) $\int \frac{10}{\mathrm{e}^{ax}} \, \mathrm{d}x$, where a is a constant.

$=$

2.4 Integrals Involving Trigonometric Functions

We can further categorise integrals involving trigonometric functions into

(i) Basic trigonometric functions (e.g. $\sin x$, $\cos x$, $\tan x$, etc)

(ii) Square of trigonometric functions (e.g. $\sin^2 x$, $\cos^2 x$, $\tan^2 x$, etc) or

(iii) Product of trigonometric functions (e.g. $\sec x \tan x$, $\sin 2x \cos x$, etc)

2.4.1 Integrals Involving Basic Trigonometric Functions

Recall from 'O' Levels:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x \qquad \Rightarrow \int \sin x \, \mathrm{d}x = -\cos x + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x \qquad \Rightarrow \int \cos x \, \mathrm{d}x = \sin x + c$$

Using the result $\boxed{\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)} \, \mathrm{d}x = \ln|\mathrm{f}(x)| + c}$ **from Section 2.2, we can derive the integral:**

$$\int \tan x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \, \mathrm{d}x = -\int \frac{-\sin x}{\cos x} \, \mathrm{d}x = -\ln|\cos x| + c = \ln|\sec x| + c$$

Summarizing the above results,

		In general
1.	$\int \sin x \, \mathrm{d}x = -\cos x + c$	$\int \sin(ax+b) \, \mathrm{d}x = -\frac{1}{a}\cos(ax+b) + c$
2.	$\int \cos x \, \mathrm{d}x = \sin x + c$	$\int \cos(ax+b) \, \mathrm{d}x = \frac{1}{a}\sin(ax+b) + c$
3.*	$\int \tan x \, \mathrm{d}x = \ln|\sec x| + c$	$\int \tan(ax+b) \, \mathrm{d}x = \frac{1}{a}\ln|\sec(ax+b)| + c$
4.*	$\int \operatorname{cosec} x \, \mathrm{d}x = -\ln|\operatorname{cosec} x + \cot x| + c$	$\int \operatorname{cosec}(ax+b) \, \mathrm{d}x = -\frac{1}{a}\ln|\operatorname{cosec}(ax+b) + \cot(ax+b)| + c$
5.*	$\int \sec x \, \mathrm{d}x = \ln|\sec x + \tan x| + c$	$\int \sec(ax+b) \, \mathrm{d}x = \frac{1}{a}\ln|\sec(ax+b) + \tan(ax+b)| + c$
6.*	$\int \cot x \, \mathrm{d}x = \ln|\sin x| + c$	$\int \cot(ax+b) \, \mathrm{d}x = \frac{1}{a}\ln|\sin(ax+b)| + c$

* Given in MF26.

Example 4

Find the following integrals.

(a) $\int -5\sin x \, dx =$

(b) $\int 3\cos(5x-1) \, dx =$

(c) $\int \frac{3}{\sin(2-x)} \, dx =$

(d) $\int [\sec 2x + \cot(3x-2)] \, dx =$

2.4.2 Integrals Involving Square of Trigonometric Functions

To find the integrals of the square of some trigonometric functions, we may need to use **trigonometric identities**.

		Trigonometric identities used
1.	$\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} \, dx$	$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$
2.	$\int \cos^2 x \, dx = \int \frac{1+\cos 2x}{2} \, dx$	
3.	$\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx$	$\tan^2 A + 1 = \sec^2 A$
4.	$\int \cot^2 x \, dx = \int (\operatorname{cosec}^2 x - 1) \, dx$	$1 + \cot^2 A = \operatorname{cosec}^2 A$

Some integrals of the square of trigonometric functions can be found by **recalling**,

$\frac{d}{dx}(\tan x) = \sec^2 x \quad \Rightarrow \int \sec^2 x \, dx = \tan x + c$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \quad \Rightarrow \int \operatorname{cosec}^2 x \, dx = -\cot x + c$

5.	$\int \sec^2 x \, dx = \tan x + c$
6.	$\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

Example 5

Find the following integrals.

(a) $\int \sin^2\left(\frac{x}{2}\right)\ \mathrm{d}x =$

(b) $\int [\cot^2(4x+1)+\tan(2x-3)+5]\ \mathrm{d}x =$

(c) $\int \frac{2}{\cos^2 3x}\ \mathrm{d}x =$

2.4.3 Integrals Involving Product of two Trigonometric Functions

Recall the following derivatives found in MF 26:

$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x \qquad \Rightarrow \int \sec x \tan x\ \mathrm{d}x = \sec x + c$

$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x \qquad \Rightarrow \int \operatorname{cosec} x \cot x\ \mathrm{d}x = -\operatorname{cosec} x + c$

In general

1.	$\int \sec x \tan x\ \mathrm{d}x = \sec x + c$	$\int \sec(ax+b)\tan(ax+b)\ \mathrm{d}x = \frac{1}{a}\sec(ax+b) + c$
2.	$\int \operatorname{cosec} x \cot x\ \mathrm{d}x = -\operatorname{cosec} x + c$	$\int \operatorname{cosec}(ax+b)\cot(ax+b)\ \mathrm{d}x = -\frac{1}{a}\operatorname{cosec}(ax+b) + c$

Other integrals involving a product of two trigonometric functions would require the use of the **Factor Formulae** found in MF 26:

$$\sin P + \sin Q \equiv 2\sin\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$$
$$\sin P - \sin Q \equiv 2\cos\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$$
$$\cos P + \cos Q \equiv 2\cos\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$$
$$\cos P - \cos Q \equiv -2\sin\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$$

The factor formulae above may be used to transform the **product** of two trigonometric functions into the **sum** of trigonometric functions of multiple angles. This is to facilitate integration of the product of two trigonometric functions.

For derivation of the Factor Formulae, refer to the Appendix A.

There are 2 ways that the factor formulae can be used:

(i) Expressing a **sum** of two trigonometric functions into a **product** of two trigonometric functions:

Eg: $\cos 5x - \cos 3x = -2\sin\left(\frac{5x+3x}{2}\right)\sin\left(\frac{5x-3x}{2}\right) = -2\sin 4x \sin x$

(ii) Expressing a **product** of two trigonometric functions into a **sum** of two trigonometric functions as shown in Example 6 below:

Example 6

Express $\sin 5x \cos x$ as a sum of two trigonometric functions.

Solution

From MF 26: $\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

$$\Rightarrow \sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right) = \frac{1}{2}\left(\sin P + \sin Q\right)$$

Comparing $\sin 5x \cos x$ with $\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$,

$\frac{P+Q}{2} = 5x$ ----------(1)

and $\frac{P-Q}{2} = x$ ----------(2).

Hence, $\sin 5x \cos x =$

Example 7

Find the following integrals.

(a) $\int \cos 6x \cos 2x \, \mathrm{d}x =$

(b) $\int 8\sin 3x \cos 5x \, \mathrm{d}x =$

Example 8

Find the following integrals.

(a) $\int (\tan 2x + \sec 2x)^2 \, \mathrm{d}x = \int (\tan^2 2x + \sec^2 2x + 2\tan 2x \sec 2x) \, \mathrm{d}x$

$= \int (\sec^2 2x - 1 + \sec^2 2x + 2\tan 2x \sec 2x) \, \mathrm{d}x$

$= \int (2\sec^2 2x + 2\tan 2x \sec 2x - 1) \, \mathrm{d}x$

$=$

(b) $\int \cos x \sin^5 x \, \mathrm{d}x = \int \cos x (\sin x)^5 \, \mathrm{d}x$

$=$

2.5 Integrals Involving Fractions

When integrating a fraction, it is essential to check if the integrand is a proper or improper fraction first. If it is an improper fraction, it must be converted into the form $Q(x)+\frac{N(x)}{D(x)}$, where $\frac{N(x)}{D(x)}$ is a proper fraction before integrating any term.

There are several strategies to find integrals involving fractions. These include:
(a) integration using standard results,
(b) integration by partial fractions.

Before finding the integral, it will be good to ask the following questions:

- Can the integrand be expressed in the form $\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$? (Refer to Section 2.2.)
- Can the integrand be simplified to one of the given forms in MF26?
- Is the denominator of the integrand factorisable?

The answers to the above questions will determine which strategy is to be used.

2.5.1 Integrals Involving Standard Results from MF 26

Recall the following derivatives found in MF 26:

$\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1} x)=\frac{1}{1+x^2}$ $\Rightarrow \int \frac{1}{1+x^2}\,\mathrm{d}x=\tan^{-1} x+c$

$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}}$ $\Rightarrow \int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x=\sin^{-1} x+c$

1.*	$\int \frac{1}{x^2+a^2}\,\mathrm{d}x=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$
2.*	$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x=\sin^{-1}\left(\frac{x}{a}\right)+c$
3.*	$\int \frac{1}{x^2-a^2}\,\mathrm{d}x=\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert+c$
4.*	$\int \frac{1}{a^2-x^2}\,\mathrm{d}x=\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert+c$

* Given in MF26.

Note that the integrands are of the form $\frac{\text{Constant}}{\text{Quadratic Expression}}$ or $\frac{\text{Constant}}{\sqrt{\text{Quadratic Expression}}}$.

Example 9

Find the following integrals.

(a) $\int \frac{7}{x^2+4}\ \mathrm{d}x =$

(b) $\int \frac{2}{\sqrt{1-16x^2}}\ \mathrm{d}x =$

2.5.2 Integration by Partial Fractions

Use partial fractions when the denominator may be factorized. After expressing in partial fractions, integrate each fraction separately using standard results discussed in Section 2.5.1.

Note that MF26 provides the various cases of the **partial fractions decomposition**:

Non-repeated linear factors:

$$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

Repeated linear factors:

$$\frac{px^2+qx+r}{(ax+b)(cx+d)^2} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$$

Non-repeated quadratic factor:

$$\frac{px^2+qx+r}{(ax+b)(x^2+c^2)} = \frac{A}{(ax+b)} + \frac{Bx+C}{(x^2+c^2)}$$

Example 10

Find the integral $\int \frac{x}{x^2-5x+6}\,\mathrm{d}x$.

Solution:

$$\int \frac{x}{x^2-5x+6}\,\mathrm{d}x = \int \frac{x}{(x-3)(x-2)}\,\mathrm{d}x$$

Other examples with Integrals involving Fractions

Example 11

Show that $4x+11 = a(2x+4)+b$, where a and b are constants to be determined.

Hence find $\int \frac{4x+11}{x^2+4x+8}\,\mathrm{d}x$.

Solution:

$$\begin{aligned} 4x+11 &= a(2x+4)+b \\ &= 2ax+4a+b \end{aligned}$$

Comparing coefficients on LHS and RHS,

$a = 2,\ b = 3$

$\therefore 4x+11 = 2(2x+4)+3$

$$\int \frac{4x+11}{x^2+4x+8}\,\mathrm{d}x =$$

Example 12

Show that $2x-1=a(2-2x)+b$, where a and b are constants to be determined.

Hence find $\int \frac{2x-1}{\sqrt{3+2x-x^2}}\,\mathrm{d}x$.

Soln:

$2x-1=a(2-2x)+b$

$\quad =-2ax+2a+b$

Comparing coefficients on LHS and RHS,

$a=-1,\ b=1$

$\therefore 2x-1=1-(2-2x)$

$\int \frac{2x-1}{\sqrt{3+2x-x^2}}\,\mathrm{d}x =$

$$=\int \frac{1}{\sqrt{3+2x-x^2}}\,\mathrm{d}x-\int \frac{2-2x}{\sqrt{3+2x-x^2}}\,\mathrm{d}x$$

$$=\int \frac{1}{\sqrt{4-(x-1)^2}}\,\mathrm{d}x-\int (2-2x)\left(3+2x-x^2\right)^{-\frac{1}{2}}\,\mathrm{d}x$$

$$=$$

3. Integration by Substitution

The method of substitution is used to **simplify** an integral to one that can be integrated using some basic techniques and formulae.

In the process of substitution, a **new** variable, say u, will be introduced. All the terms in x, dx and limits for x (if given) will be changed to terms in u, du and limits for u respectively,

$$\text{i.e. } \int_{x_1}^{x_2} \mathrm{f}(x)\, \mathrm{d}x \to \int_{u_1}^{u_2} \mathrm{g}(u)\, \mathrm{d}u .$$

For the H2 Math syllabus, the substitution will be given in the question.

Remember to express the final answer in the original variable for the case of an indefinite integral.

Example 13

Find the following integrals using the given substitution in brackets.

(a) $\int x(2-x)^5 \, \mathrm{d}x \qquad (u = 2-x)$

Let $u = 2 - x$

$=$

$= \int (u^6 - 2u^5)\, \mathrm{d}u$

$= \dfrac{u^7}{7} - \dfrac{2u^6}{6} + c$

$= \dfrac{(2-x)^7}{7} - \dfrac{(2-x)^6}{3} + c$

(b) $\int \dfrac{\mathrm{e}^x}{4+\mathrm{e}^{2x}} \, \mathrm{d}x \qquad (u = \mathrm{e}^x)$

Let $u = \mathrm{e}^x$

$=$

$= \dfrac{1}{2}\tan^{-1}\left(\dfrac{u}{2}\right) + c$

$= \dfrac{1}{2}\tan^{-1}\left(\dfrac{\mathrm{e}^x}{2}\right) + c$

(c) $\int_0^1 x^2\sqrt{1-x}\ \mathrm{d}x \qquad (u=\sqrt{1-x})$

Let $u=\sqrt{1-x}$

$=$

$=\int_1^0 (-2u^2+4u^4-2u^6)\ \mathrm{d}u$

$=$

When $x=1$,

$=2\left[\frac{u^3}{3}-\frac{2u^5}{5}+\frac{u^7}{7}\right]_0^1$

When $x=0$,

$=2\left(\frac{1}{3}-\frac{2}{5}+\frac{1}{7}\right)$

$=\frac{16}{105}$

Example 14

Find the following integrals using the given substitution in brackets, leaving your answers in exact form.

(a) $\int_0^{\frac{\pi}{6}} \frac{1}{\cos^2 x+3\sin^2 x}\ \mathrm{d}x \qquad (\theta=\tan x)$

Let $\theta=\tan x$

$=$

When $x=\frac{\pi}{6}$,

$=$

When $x=0$,

$=$

$=\frac{1}{\sqrt{3}}\left[\tan^{-1}(\sqrt{3}\theta)\right]_0^{\frac{1}{\sqrt{3}}}$

$=\frac{\pi}{4\sqrt{3}}$

(b) $\int \frac{1}{(4+x^2)^2}\,\mathrm{d}x \qquad (x = 2\tan\theta)$

Let $x = 2\tan\theta$

Then $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sec^2\theta$

$$= \int \frac{1}{(4+4\tan^2\theta)^2}(2\sec^2\theta)\,\mathrm{d}\theta$$

$$= \int \frac{1}{16(\sec^2\theta)^2}(2\sec^2\theta)\,\mathrm{d}\theta$$

$$= \frac{1}{8}\int \frac{1}{\sec^2\theta}\,\mathrm{d}\theta$$

$$= \frac{1}{8}\int \cos^2\theta\,\mathrm{d}\theta$$

$$= \frac{1}{8}\int \frac{\cos 2\theta + 1}{2}\,\mathrm{d}\theta$$

$$= \frac{1}{16}\left(\frac{\sin 2\theta}{2} + \theta\right) + c$$

$$= \frac{1}{16}\left(\sin\theta\cos\theta + \theta\right) + c$$

$=$

$$= \frac{x}{8(x^2+4)} + \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + c$$

4. **<u>Integration by Parts</u>**

From product rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}, \text{ where } u \text{ and } v \text{ are functions of } x.$$

$$\Rightarrow \quad uv = \int u\frac{\mathrm{d}v}{\mathrm{d}x}\ \mathrm{d}x + \int v\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x.$$

By making the expression, $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\ \mathrm{d}x$, the subject, we have the "by-parts formula":

$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\ \mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x$, where $\frac{\mathrm{d}v}{\mathrm{d}x}$ is an integrable function.
With limits: $\int_a^b u\frac{\mathrm{d}v}{\mathrm{d}x}\ \mathrm{d}x = \left[uv\right]_a^b - \int_a^b v\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x$

The method of integration by parts is used for

(i) a single function (logarithmic function and inverse trigonometric function) that cannot be integrated using standard formulas,
e.g. $\int \ln x\ \mathrm{d}x$, $\int \sin^{-1} x\ \mathrm{d}x$, $\int \tan^{-1} x\ \mathrm{d}x$, etc,

(ii) a product of different types of functions,
e.g. $\int x^n \ln x\ \mathrm{d}x$, $\int x^n \sin x\ \mathrm{d}x$, $\int x^n \tan^{-1} x\ \mathrm{d}x$, $\int x^n \mathrm{e}^x\ \mathrm{d}x$, $\int \mathrm{e}^x \cos x\ \mathrm{d}x$, etc.

<u>Remarks</u>:

(1) The choice of u (the <u>function to be differentiated</u>) depends on the types of function given in the integral and generally follows the order of the letters appearing in the code '**LIATE**', where

L - Logarithmic function, e.g. $\ln x$;
I - Inverse trigonometric function, e.g. $\sin^{-1} x$, $\tan^{-1} x$;
A - Algebraic function, e.g. x^n;
T - Trigonometric function, e.g. $\sin nx$, $\cos nx$;
E - Exponential function, e.g. e^{nx}.

L I A T E →
differentiate integrate

(2) The method of integration by parts is applied twice on certain integrals,
e.g. $\int (\ln x)^2\ \mathrm{d}x$, $\int x^2\mathrm{e}^x\ \mathrm{d}x$, $\int x^2 \sin x\ \mathrm{d}x$, $\int \mathrm{e}^x \cos x\ \mathrm{d}x$, etc.

In particular, for integrals of the type $\int \mathrm{ET}\ \mathrm{d}x$, we will get into a 'loop' after applying integration by parts the second time.

<u>Example 15</u>

Find the following integrals:

(a) $\int x\ln x\,\mathrm{d}x =$

$=$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$

$u =$ $\quad \frac{\mathrm{d}v}{\mathrm{d}x} =$

$\frac{\mathrm{d}u}{\mathrm{d}x} =$ $\quad v =$

(b) $\int x^2\cos 2x\,\mathrm{d}x$

$=$

$u =$ $\quad \frac{\mathrm{d}v}{\mathrm{d}x} =$

$\frac{\mathrm{d}u}{\mathrm{d}x} =$ $\quad v =$

$=$

$u =$ $\quad \frac{\mathrm{d}v}{\mathrm{d}x} =$

$= \frac{x^2}{2}\sin 2x + \frac{x}{2}\cos 2x - \frac{1}{4}\sin 2x + c$

$\frac{\mathrm{d}u}{\mathrm{d}x} =$ $\quad v =$

(c) $\int \tan^{-1}x\,\mathrm{d}x =$

$=$

$= x\tan^{-1}x - \frac{1}{2}\ln\left(1+x^2\right) + c$

$u =$ $\quad \frac{\mathrm{d}v}{\mathrm{d}x} =$

$\frac{\mathrm{d}u}{\mathrm{d}x} =$ $\quad v =$

(d) $\int e^{2x}\sin 2x\,\mathrm{d}x = \frac{e^{2x}}{2}\sin 2x - \int e^{2x}\cos 2x\,\mathrm{d}x$

$u = \sin 2x$ $\quad \frac{\mathrm{d}v}{\mathrm{d}x} = e^{2x}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 2\cos 2x$ $\quad v = \frac{1}{2}e^{2x}$

$= \frac{e^{2x}}{2}\sin 2x - \left(\frac{e^{2x}}{2}\cos 2x + \int e^{2x}\sin 2x\,\mathrm{d}x\right)$

$u = \cos 2x$ $\quad \frac{\mathrm{d}v}{\mathrm{d}x} = e^{2x}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = -2\sin 2x$ $\quad v = \frac{1}{2}e^{2x}$

Example 16

Evaluate the following integrals:

(a) $\int_1^{e} x^5 \ln x \, dx =$

$u = \ln x \qquad \frac{dv}{dx} = x^5$

$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{x^6}{6}$

$$= \frac{e^6}{6} - \frac{1}{6}\left[\frac{x^6}{6}\right]_1^{e}$$

$$= \frac{e^6}{6} - \frac{1}{6}\left(\frac{e^6 - 1}{6}\right)$$

$$= \frac{1}{36}(5e^6 + 1)$$

(b) $\int_0^{\frac{\pi}{2}} x \cos^2 x \, dx$

$=$

$u = x \qquad \frac{dv}{dx} = \cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\frac{du}{dx} = 1 \qquad v = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)$

$$= \frac{\pi^2}{8} - \frac{1}{2}\left[\frac{x^2}{2} - \frac{\cos 2x}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^2}{8} - \frac{1}{2}\left(\frac{\pi^2}{8} + \frac{1}{4} + \frac{1}{4}\right)$$

$$= \frac{\pi^2}{16} - \frac{1}{4}$$

APPENDIX A

Derivation of the Factor Formulae

From the Addition Formulae, we have

$\sin(A+B) = \sin A \cos B + \cos A \sin B$ -------------- (1)

$\sin(A-B) = \sin A \cos B - \cos A \sin B$ -------------- (2)

(1) + (2): $\sin(A+B) + \sin(A-B) = 2\sin A \cos B$ -------------- (3)

(1) – (2): $\sin(A+B) - \sin(A-B) = 2\cos A \sin B$ -------------- (4)

Let $P = A + B$ and $Q = A - B$, we have $A = \dfrac{P+Q}{2}$ and $B = \dfrac{P-Q}{2}$,

So we obtain the factor formulae:

$$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$$
$$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

Similarly,

$\cos(A+B) = \cos A \cos B - \sin A \sin B$ -------------- (5)

$\cos(A-B) = \cos A \cos B + \sin A \sin B$ -------------- (6)

(3) + (4): $\cos(A+B) + \cos(A-B) = 2\cos A \cos B$ -------------- (7)

(3) – (4): $\cos(A+B) - \cos(A-B) = -2\sin A \sin B$ -------------- (8)

By letting $P = A + B$, $Q = A - B$, we get

$$\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$$
$$\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

From equations (3), (4), (7) and (8), we can also get the following factor formulae:

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$
$$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$$
$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$
$$\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$$