RAFFLES INSTITUTION

2022 Year 6

H3 9820 Lecture Test 3

Time allocated: 2 hours **Total Marks: 60**

Instructions: Write your name and CT group on all the work you hand in.
Answer **all** questions.

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1 A clothes shop sells a particular make of T-shirt in four different colours. The shopkeeper has a large number of T-shirts of each colour.

(a) A customer wishes to buy eight T-shirts.

(i) In how many ways can he do this? [2]

(ii) In how many ways can he do this if he buys at least one of each colour? [2]

(b) The shopkeeper places eight T-shirts in a line.

(i) In how many ways can she do this? [1]

(ii) In how many ways can she do this if no two T-shirts of the same colour are to be next to each other? [2]

(iii) Use the principle of inclusion and exclusion to find the number of ways in which she can do this if she has to use at least one T-shirt of each colour but with no other restriction. [4]

(c) The shopkeeper is left with r T-shirts of one colour, where $r > 4$ and another 2 T-shirts of another colour. She wishes to store them in 2 identical boxes so that no box is empty. In how many ways can she do this? [4]

1 (a) **(i)** **[2]**	Let x_1, x_2, x_3, x_4 be the respective number of T-shirts the customer buys in the 4 different colours. Then the problem is equivalent to the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 8$, with $x_1, x_2, x_3, x_4 \geq 0$. (Identical objects into distinct boxes) The number of ways is thus $\binom{11}{3} = 165$.

(a)(ii) **[2]**	Let x_1, x_2, x_3, x_4 be the respective number of T-shirts the customer buys in the 4 different colours. Then the problem is equivalent to $x_1 + x_2 + x_3 + x_4 = 8$, with $x_1, x_2, x_3, x_4 \geq 1$. Let $y_i = x_i - 1$ (i.e. ensure he has one of each different colour). Then we need to find the number of integer solutions to $y_1 + y_2 + y_3 + y_4 = 4$ with $y_1, y_2, y_3, y_4 \geq 0$. The number of ways is thus $\binom{7}{3} = 35$.
(b)(i) **[1]**	The number of ways is $4^8 = 65536$.
(b)(ii) **[2]**	There are 4 ways to choose the first shirt, and subsequently, 3 ways for the next shirt so that no two shirts are of the same colour. The number of ways is thus $4(3)^7 = 8748$.
(b)(iii) **[4]**	Let A_i denote the event in each colour i is not used. Then required answer = $\left\|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}\right\| = 4^8 - \left\|A_1 \cup A_2 \cup A_3 \cup A_4\right\|$ We have $\left\|A_1 \cup A_2 \cup A_3 \cup A_4\right\| = \left\|A_1\right\| + \left\|A_2\right\| + \left\|A_3\right\| + \left\|A_4\right\|$ $-\left\|A_1 \cap A_2\right\| - \left\|A_1 \cap A_3\right\| - \ldots - \left\|A_3 \cap A_4\right\|$ $+\left\|A_1 \cap A_2 \cap A_3\right\| + \left\|A_1 \cap A_2 \cap A_4\right\| \ldots + \left\|A_2 \cap A_3 \cap A_4\right\|$ $-\left\|A_1 \cap A_2 \cap A_3 \cap A_4\right\|$ $= 4^8 - 4(3)^8 + 6(2)^8 - 4 = 40824$.

(c) **[4]**	Consider the 2 T-shirts of one colour first. They can be split into the 2 boxes in the following manner: (0, 2) or (1, 1). Let the number of T-shirts contained in each of the two boxes be a and b respectively, where $a+b=r$. Case 1: 1 and 1. Here the 2 boxes are essentially still identical. If r is even, since $a \le \frac{1}{2}(a+b)=\frac{r}{2}$, a can only take a value from 0, 1, 2, …, $\frac{r}{2}$ when r is even. There are $\frac{r}{2}+1$ ways. $\frac{r}{2}$ is not an integer when r is odd. So $a \le \frac{r}{2} \Rightarrow a \le \frac{r-1}{2}$. Similarly, there are $\frac{r-1}{2}+1=\frac{r+1}{2}$ ways if r is odd. Case 2: 0 and 2. Here the 2 boxes are now distinct. Let Box A be the one with 0 of the other, and Box B be the one with 2. Box B can contain 0, 1, 2, …, $r-1$ (cannot contain all r or else Box A is empty) of the shirts. So there are r ways. Hence in total, if r is even, there are $\frac{r}{2}+1+r=\frac{3r+2}{2}$ ways. If r is odd, there are $\frac{r+1}{2}+r=\frac{3r+1}{2}$ ways.

2 Let $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{\left(1+2^x\right)\sin x}\ \mathrm{d}x$.

(a) Show that $I_n = \int_0^{\pi} \frac{\sin nx}{\sin x}\ \mathrm{d}x$. [4]

(b) Hence show that for $n \geq 2$, $I_n = I_{n-2}$. [2]

(c) Hence evaluate I_n for all nonnegative integers n. [3]

2 **(a)** **[4]**	$I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{\left(1+2^x\right)\sin x}\ \mathrm{d}x = \int_{-\pi}^{0} \frac{\sin nx}{\left(1+2^x\right)\sin x}\ \mathrm{d}x + \int_{0}^{\pi} \frac{\sin nx}{\left(1+2^x\right)\sin x}\ \mathrm{d}x$ Using the substitution $t=-x$ in the first integral, we get $I_n = \int_{\pi}^{0} \frac{\sin(-nt)}{\left(1+2^{-t}\right)\sin(-t)}\ (-\mathrm{d}t) + \int_{0}^{\pi} \frac{\sin nx}{\left(1+2^x\right)\sin x}\ \mathrm{d}x$ $= \int_{0}^{\pi} \frac{\sin(nt)}{\left(1+2^{-t}\right)\sin(t)}\ \mathrm{d}t + \int_{0}^{\pi} \frac{\sin nx}{\left(1+2^x\right)\sin x}\ \mathrm{d}x$ (since sin is an odd function) $= \int_{0}^{\pi} \frac{2^t\sin(nt)}{\left(1+2^{t}\right)\sin(t)}\ \mathrm{d}t + \int_{0}^{\pi} \frac{\sin nx}{\left(1+2^x\right)\sin x}\ \mathrm{d}x$ $= \int_{0}^{\pi} \frac{\left(1+2^x\right)\sin nx}{\left(1+2^x\right)\sin x}\ \mathrm{d}x$ $= \int_{0}^{\pi} \frac{\sin nx}{\sin x}\ \mathrm{d}x$
(b) **[2]**	$I_n - I_{n-2} = \int_{0}^{\pi} \frac{\sin nx - \sin(n-2)x}{\sin x}\ \mathrm{d}x$ $= \int_{0}^{\pi} \frac{2\cos(n-1)x\sin x}{\sin x}\ \mathrm{d}x$ $= \int_{0}^{\pi} 2\cos(n-1)x\ \mathrm{d}x$ $= \frac{2}{n-1}\left[\sin(n-1)x\right]_0^{\pi} = 0 \Rightarrow I_n = I_{n-2}$
(c) **[3]**	Thus we split into the odd and even n. We have $I_1 = \int_0^{\pi} \frac{\sin x}{\sin x}\ \mathrm{d}x = \pi$ and $I_0 = 0$. Hence $I_n = \begin{cases} \pi & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$.

3 Let $b_0, b_1, b_2, \ldots$ be a sequence of positive real numbers such that $b_0 = 1$, $b_n = 2 + \sqrt{b_{n-1}} - 2\sqrt{1+\sqrt{b_{n-1}}}$.

Let $a_n = 1 + \sqrt{b_n}$ for $n \geq 0$.

(a) Show that $a_n = \sqrt{a_{n-1}}$ for $n \geq 1$. [2]

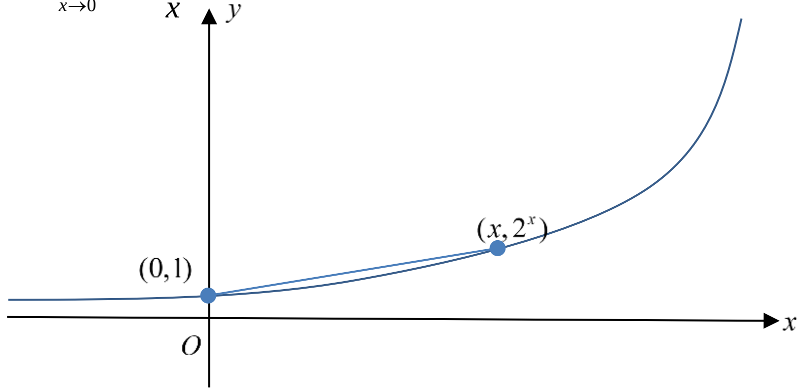
(b) Hence express a_n in terms of *n*. [2]

(c) Show that $\sum_{n=1}^{N} b_n 2^n = (a_0 - 1)2^1 - (a_N - 1)2^{N+1}$. [3]

(d) Use a sketch to explain why $\lim_{x \to 0} \frac{2^x - 1}{x} = \ln 2$. [2]

(e) Hence calculate $\sum_{n=1}^{\infty} b_n 2^n$. [2]

3 **(a)** **[2]**	Note that $a_n = 1 + \sqrt{b_n} > 1$. We also have $a_n = 1 + \sqrt{1 + \left(1+\sqrt{b_{n-1}}\right) - 2\sqrt{1+\sqrt{b_{n-1}}}}$ $= 1 + \sqrt{1 + a_{n-1} - 2\sqrt{a_{n-1}}}$ $= 1 + \sqrt{\left(1 - \sqrt{a_{n-1}}\right)^2}$ $= 1 + \left\lvert 1 - \sqrt{a_{n-1}} \right\rvert$ $= 1 + \sqrt{a_{n-1}} - 1$ since $a_n > 1$ $= \sqrt{a_{n-1}}$

(b) **[2]**	Hence $$a_n=(a_{n-1})^{\frac{1}{2}}$$ $$=\left((a_{n-2})^{\frac{1}{2}}\right)^{\frac{1}{2}}$$ $$=(a_{n-2})^{\frac{1}{2^2}}$$ $$=\ldots$$ $$=(a_0)^{\frac{1}{2^n}}=2^{2^{-n}}$$
(c) **[3]**	$$\sum_{n=1}^{N} b_n 2^n=\sum_{n=1}^{N}(a_n-1)^2\,2^n$$ $$=\sum_{n=1}^{N}\left(a_n^2 2^n-a_n 2^{n+1}+2^n\right)$$ $$=\sum_{n=1}^{N}\left((a_{n-1}-1)2^n-(a_n-1)2^{n+1}\right)$$ $$=\sum_{n=1}^{N}(a_{n-1}-1)2^n-\sum_{n=2}^{N+1}(a_{n-1}-1)2^n$$ $$=(a_0-1)2^1-(a_N-1)2^{N+1}.$$
(d) **[2]**	Consider the function $f(x)=2^x$ and the gradient of the function at $x=0$ .We see that $\lim_{x\to 0}\frac{2^x-1}{x}=\lim_{x\to 0}\frac{f(x)-f(0)}{x}=f'(0)=\ln 2$. 
(e) **[2]**	Hence $$\sum_{n=1}^{\infty} b_n 2^n=\lim_{n\to\infty}\sum_{n=1}^{N} b_n 2^n$$ $$=\lim_{n\to\infty}(a_0-1)2^1-(a_N-1)2^{N+1}$$ $$=2-2\lim_{n\to\infty}\left(\frac{2^{2^{-N}}-1}{2^{-N}}\right)$$ $$=2-2\lim_{x\to 0}\left(\frac{2^x-1}{x}\right)=2-2\ln 2.$$

4 Let $S_n = \{1, 2, ..., n\}$. For any subset X of S_n, define the capacity of X, $c(X)$ to be the sum of all the elements of X. If the capacity of X is odd, we say that X is an odd subset of S_n and similarly, if X is even, we say that X is an even subset of S_n.

For example, if $n = 3$, then $c(S_3) = 1 + 2 + 3 = 6$, $c(\{1, 2\}) = 1 + 2 = 3$, $c(\{\ \}) = 0$ and thus S_3 is an even subset of S_3, and $\{1, 2\}$ is an odd subset of S_3 and the empty set is an even subset of S_3.

Let A be the set of all subsets of S_n that do not contain 1, and B be the set of all subsets of S_n that contains 1.

For example, if $n = 3$, $A = \{\{\ \}, \{2\}, \{3\}, \{2, 3\}\}$ and $B = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$.

(a) Show using a bijection between A and B that the number of odd subsets and even subsets of S_n are the same. [3]

(b) If $n \geq 3$, show that the number of odd subsets and even subsets in A are also the same. [2]

(c) Hence show that if $n \geq 3$, the sum of all capacities of odd subsets of S_n is equal to the sum of all capacities of even subsets of S_n. [2]

(d) Determine the sum of all capacities of odd subsets. [3]

(a) **[3]**	Consider the map $\mathrm{f} : A \to B$, defined by $\mathrm{f}(X) = X \cup \{1\}$. If $\mathrm{f}(X) = \mathrm{f}(Y) \Rightarrow X \cup \{1\} = Y \cup \{1\} \Rightarrow X = Y$ thus f is injective. For any set $Y \in B$, removing 1 from it clearly gives an element of *A*. Hence f is bijective. Hence if the subset *X* is odd (resp. even), then $X \cup \{1\}$ is even (resp. odd). Hence the number of odd subsets and even subsets of S_n are the same.
(b) **[2]**	We note that $\|A\| = \|B\| = 2^{n-1}$. Consider the subsets of *A* that do not contain 3 and those that contain 3. Similar to the bijection in (a), we can establish a bijection between these 2 sets. Note that $n \geq 3$ and thus the image $X \cup \{3\} \in S_n$. This means that the number of odd and even subsets of *A* are also the same, and there are 2^{n-2} of them.

(c) **[2]**	For any odd subset C of A, its capacity is 1 less than the corresponding even subset $C \cup \{1\}$ of B. Hence the sum of all the capacities of the odd subsets of A is 2^{n-2} less than the sum of all the capacities of the even subsets of B. But similarly, all the even subsets of A have a total capacity of 2^{n-2} lesser than the sum of all the capacities of the odd subsets of B. Therefore the sum of all capacities of odd subsets of S_n is equal to the sum of all capacities of even subsets of S_n.
(d) **[3]**	We note that each element 1, 2, …, n appears in all the subsets of S_n a total of 2^{n-1} times. Hence the total capacity of all the subsets of S_n is given by $2^{n-1}(1+2+...+n)=2^{n-2}n(n+1)$. From (c), we deduce that the sum of all capacities of odd subsets is $2^{n-3}n(n+1)$.

5 Let $\{x_1, x_2, ..., x_n\}$ and $\{y_1, y_2, ..., y_n\}$ be two sequences of real numbers. We say that the sequence $\{x_1, x_2, ..., x_n\}$ **majorizes** the sequence $\{y_1, y_2, ..., y_n\}$, if the following conditions are fulfilled:

- $x_1 \ge x_2 \ge ... \ge x_n$;
- $y_1 \ge y_2 \ge ... \ge y_n$;
- $x_1 + x_2 + ... + x_n = y_1 + y_2 + ... + y_n$;
- $x_1 + x_2 + ... + x_k \ge y_1 + y_2 + ... + y_k$ for all $1 \le k \le n-1$.

For example, {3, 0, 0} majorizes {2, 1, 0}, and {2, 1, 0} majorizes {1, 1, 1}.

Let f be a convex function defined over the real numbers.

(a) Use a sketch to explain why if $x < y < z$ then

$$\frac{\mathrm{f}(z)-\mathrm{f}(x)}{z-x} \le \frac{\mathrm{f}(z)-\mathrm{f}(y)}{z-y}.$$

[2]

Let $\{a_1, a_2, ..., a_n\}$ and $\{b_1, b_2, ..., b_n\}$ be two sequences of real numbers such that $\{a_1, a_2, ..., a_n\}$ **majorizes** the sequence $\{b_1, b_2, ..., b_n\}$, and let $c_i = \dfrac{\mathrm{f}(b_i)-\mathrm{f}(a_i)}{b_i - a_i}$.

Let $A_k = \sum_{i=1}^{k} a_i$ and $B_k = \sum_{i=1}^{k} b_i$.

(b) **(i)** Explain why $c_{i+1} \le c_i$. [2]

(ii) Show that $\sum_{i=1}^{n}\left(\mathrm{f}(a_i)-\mathrm{f}(b_i)\right) = \sum_{i=1}^{n-1}\left(c_i - c_{i+1}\right)\left(A_i - B_i\right)$ and hence deduce that $\mathrm{f}(a_1)+\mathrm{f}(a_2)+...+\mathrm{f}(a_n) \ge \mathrm{f}(b_1)+\mathrm{f}(b_2)+...+\mathrm{f}(b_n)$. [4]

(c) Let a, b, c be positive real numbers. Show that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \leq \frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}.$$

[3]

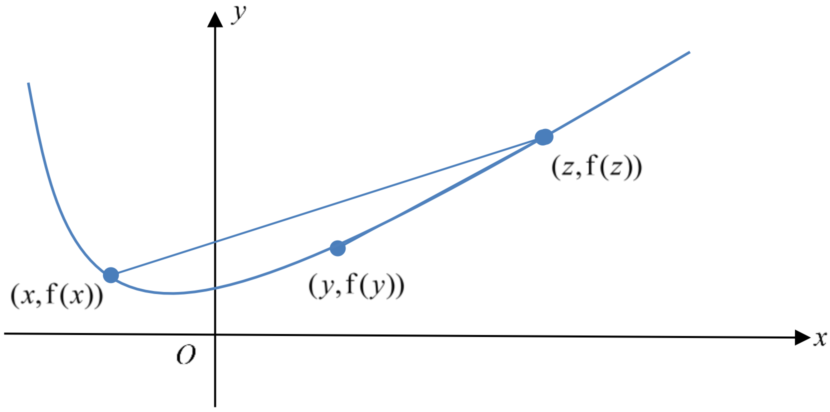
(d) Let $x_1, x_2, \ldots, x_n \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$. Show that

$$\cos(2x_1 - x_2) + \cos(2x_2 - x_3) + \ldots + \cos(2x_n - x_1) \leq \cos x_1 + \cos x_2 + \ldots + \cos x_n.$$

[4]

<table>
<tr>
<td>5

(a)

[2]</td>
<td>As shown below, given a convex function f and 3 points on the curve with $x < y < z$,
$\dfrac{\mathrm{f}(z)-\mathrm{f}(x)}{z-x} \leq \dfrac{\mathrm{f}(z)-\mathrm{f}(y)}{z-y}$.

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<tr>
<td>(b)

(i)

[2]</td>
<td>We first note that result in (a) still holds if $x \leq y$.

$c_{i+1} = \dfrac{\mathrm{f}(b_{i+1})-\mathrm{f}(a_{i+1})}{b_{i+1}-a_{i+1}}$
$\leq \dfrac{\mathrm{f}(b_{i+1})-\mathrm{f}(a_i)}{b_{i+1}-a_i}$ since $a_i \geq a_{i+1}$ and using (a)
$\leq \dfrac{\mathrm{f}(b_i)-\mathrm{f}(a_i)}{b_i-a_i}$ since $b_i \geq b_{i+1}$ and using (a)
$= c_i$</td>
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</table>

(b) **(ii)** **[4]**	$\sum_{i=1}^{n}\left(\mathrm{f}(a_i)-\mathrm{f}(b_i)\right)$ $=\sum_{i=1}^{n}c_i\left(a_i-b_i\right)$ $=\sum_{i=1}^{n}c_i\left(A_i-A_{i-1}-B_i+B_{i-1}\right)$ $=\sum_{i=1}^{n}c_i\left(A_i-B_i\right)-\sum_{i=1}^{n}c_i\left(A_{i-1}-B_{i-1}\right)$ $=c_n\left(A_n-B_n\right)+\sum_{i=1}^{n-1}c_i\left(A_i-B_i\right)-\sum_{i=0}^{n-1}c_{i+1}\left(A_i-B_i\right)$ $=\sum_{i=1}^{n-1}\left(c_i-c_{i+1}\right)\left(A_i-B_i\right)$ since $A_n=B_n$ and $A_0=B_0=0.$ From (b)(i), we know that $c_{i+1}\le c_i$ and the condition of majorization that $A_i\ge B_i$. Therefore, $\sum_{i=1}^{n}\left(\mathrm{f}(a_i)-\mathrm{f}(b_i)\right)\ge 0$ as desired.
(c) **[3]**	The function $\mathrm{f}(x)=\frac{1}{x}$ is convex over the positive reals. By symmetry, we may assume WLOG that $a\ge b\ge c$. Then $\{2a,2b,2c\}$ majorizes $\{a+b,a+c,b+c\}$. Hence from (b)(ii), $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\le\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}$.
(d) **[4]**	We note that $x_1,x_2,...,x_n\in\left[-\frac{\pi}{6},\frac{\pi}{6}\right]$ implies that $2x_i-x_{i+1}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. The function cosine is concave on this interval, and hence the inequality in (b)(ii) has its sign reversed. However, the inequality is no longer symmetric (it is cyclic). If $y_1\ge y_2\ge...\ge y_n$ are $x_1,x_2,...,x_n\in\left[-\frac{\pi}{6},\frac{\pi}{6}\right]$ rearranged in order, and $z_1,z_2,...,z_n$ are $x_1,x_2,...,x_n\in\left[-\frac{\pi}{6},\frac{\pi}{6}\right]$ rearranged such that $2z_1-z_2\ge 2z_2-z_3\ge...\ge 2z_n-z_1$ then the sequence $\left\{y_1,y_2,...,y_n\right\}$ is majorized by $\left\{2z_1-z_2,2z_2-z_3,...,2z_n-z_1\right\}$. We can see this is true since $2z_1-z_2+2z_2-z_3+...+2z_k-z_{k+1}\ge y_1+y_2+...+y_k$ $\Leftrightarrow\ 2y_1-y_2+2y_2-y_3+...+2y_k-y_{k+1}\ge y_1+y_2+...+y_k$ $\Leftrightarrow\ \left(y_1+y_2+...+y_k\right)-\left(y_2+y_3+...+y_{k+1}\right)\ge 0\Leftrightarrow\ y_1\ge y_{k+1}$ The inequality sign is reversed due to cosine being concave (or –cosine is convex) , and we have $\cos\left(2x_1-x_2\right)+\cos\left(2x_2-x_3\right)+...+\cos\left(2x_n-x_1\right)\le\cos x_1+\cos x_2+...+\cos x_n$.

[END OF PAPER]