RAFFLES INSTITUTION
H2 Mathematics 9758
2023 Year 6 Term 3 Revision 7a (Summary and Tutorial)

Topic: Vectors 2 (Lines and Planes)

<u>Summary for Lines and Planes</u>

Line: $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b},\ \lambda \in \mathbb{R}$ where

$\mathbf{r}$ is the position vector of a general point on the line,
$\mathbf{a}$ is the position vector of a known point on the line and
$\mathbf{b}$ is the vector that indicate the direction parallel to the line (known as the *direction vector*)

Note : $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b},\ \lambda \in \mathbb{R}$ is not **UNIQUE**.

Different Forms of Equations of a Line

$$\mathbf{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},\ \lambda \in \mathbb{R} \qquad \text{– } \textbf{\textit{Vector Form}}$$

$$\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},\ \lambda \in \mathbb{R}$$

$$\Leftrightarrow x = a_1 + \lambda\, b_1, y = a_2 + \lambda\, b_2, z = a_3 + \lambda\, b_3 \qquad \text{– } \textbf{\textit{Parametric Form}}$$

$$\Leftrightarrow \frac{x - a_1}{b_1} = \lambda\,,\ \frac{y - a_2}{b_2} = \lambda\,,\ \frac{z - a_3}{b_3} = \lambda$$

$$\Leftrightarrow \frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} \qquad \text{– } \textbf{\textit{Cartesian Form}}$$

Plane: $\mathbf{r} \,.\, \mathbf{n} = d$ where

$\mathbf{r}$ is the position vector of a general point on the plane,
$\mathbf{n}$ is the normal vector to the plane (a vector perpendicular to the plane)
d is a scalar constant.

Notes:
The position vector of any point A on the plane, $\mathbf{a}$ will always give the result, $\mathbf{a} \,.\, \mathbf{n} = d$

The vector equation of the plane can always be reduced to the form $\mathbf{r} \,.\, \hat{\mathbf{n}} = \dfrac{d}{|\mathbf{n}|} = D$. In this form, the normal vector is reduced to a unit vector and $|D|$ is the ***shortest distance of the plane from the origin.***

Different Forms of Equations of a Plane

Scalar Product form: $\mathbf{r}.\mathbf{n} = d$

Cartesian form: $n_1x + n_2\,y + n_3\,z = d$ where $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$

Vector form: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{m}_1 + \mu\mathbf{m}_2, \quad \lambda, \mu \in \mathbb{R}$

a is the position vector of a point on the plane,
$\mathbf{m}_1$ and $\mathbf{m}_2$ are non-parallel vectors that are parallel to the plane. Thus, $\mathbf{m}_1 \times \mathbf{m}_2$ will be a normal to the plane.

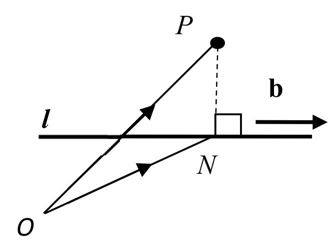
We will now summarize the various relationships involving points, lines and planes.

Involving Points and Lines

- To check whether the point P with position vector $\mathbf{p}$ lies on the given line l: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$

  – Let $\mathbf{p} = \mathbf{a} + \lambda\mathbf{b}$ and find a possible value for λ.
  – If there is a unique value for λ from the three possible equations, then the point P lies on the line.

- To find the position vector of the *foot of the perpendicular* from a point P with position vector $\mathbf{p}$ to a given line l : $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$

  Let N be the foot of the perpendicular from the point P to the line, l. Make use of the fact that $\overrightarrow{PN}$ is perpendicular to l

  Step 1: Since N lies on l , $\overrightarrow{ON} = \mathbf{a} + \lambda\,\mathbf{b}$ for a unique value of λ.

  Step 2: Find $\overrightarrow{PN} = \overrightarrow{ON} - \overrightarrow{OP}$ in terms of λ, where $\overrightarrow{OP} = \mathbf{p}$ is given.

  Step 3: Solve for the value of λ using the fact that $\overrightarrow{PN} \cdot \mathbf{b} = 0$
  Step 4: Using the value of λ found above, the position vector of N is given by $\mathbf{a} + \lambda\mathbf{b}$.

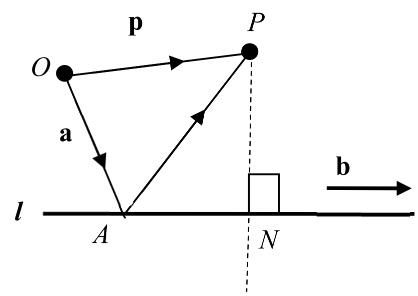
- To find the perpendicular distance (shortest distance) from a point, P with position vector, $\mathbf{p}$, to a given line l : $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$

  Let N be the foot of the perpendicular from the point P to the line, l
  Method 1
  Step 1 : Find the position vector of the foot of the perpendicular from the point P to the given line l .

  Step 2 : Perpendicular distance from the point P to the given line l is $|\overrightarrow{PN}|$.

Method 2 (Cross Product)

Perpendicular distance from the point P to the given line l is $\left|\overrightarrow{PN}\right| = \left|\overrightarrow{AP} \times \hat{\mathbf{b}}\right|$



Method 3 (Dot Product)

Step 1 : Find $\left|\overrightarrow{AN}\right| = \left|\overrightarrow{AP} \bullet \hat{\mathbf{b}}\right|$

Step 2 : Using Pythagoras' Theorem,
Perpendicular distance from the point P to the given line l is $\left|\overrightarrow{PN}\right| = \sqrt{\left|\overrightarrow{AP}\right|^2 - \left|\overrightarrow{AN}\right|^2}$

❖ To find the position vector of the reflection of a point, P, about a line l : $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$

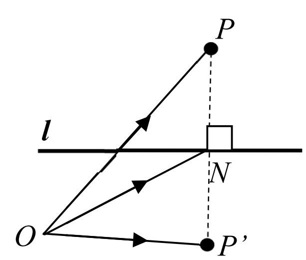
Let P' be the reflection of P about l and that N be foot of the perpendicular from P to l .

Method

– Find the vector $\overrightarrow{PN}$ (**Note that** $\overrightarrow{PN} = \overrightarrow{NP'}$)

– Position vector of P' $= \overrightarrow{OP'} = \overrightarrow{OP} + \overrightarrow{PN} + \overrightarrow{NP'} = \overrightarrow{OP} + 2\overrightarrow{PN}$

(**OR** use Ratio Theorem: $\overrightarrow{ON} = \dfrac{\overrightarrow{OP} + \overrightarrow{OP'}}{2}$)



Involving Lines

❖ To check if the lines $l_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = \mathbf{c} + \mu\mathbf{d},\ \mu \in \mathbb{R}$ are parallel

If **b** // **d** (i.e. $\mathbf{b} = k\,\mathbf{d}$), then l_1 // l_2

❖ To find the acute angle between 2 lines, $l_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = \mathbf{c} + \mu\mathbf{d},\ \mu \in \mathbb{R}$

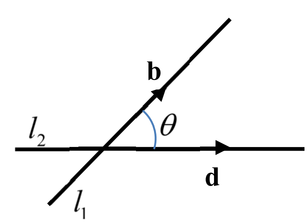
The acute angle between the lines is determined by the two direction vectors **b** and **d**.

Method

Make use of scalar product of the two direction vectors **b** and **d**.

– $|\,\mathbf{b} \cdot \mathbf{d}\,| = |\mathbf{b}|\,|\mathbf{d}|\cos\theta$ where θ is the acute angle between the 2 direction vectors, **b** and **d**.

Therefore $\cos\theta = \dfrac{|\mathbf{b}.\mathbf{d}|}{|\mathbf{b}||\mathbf{d}|}$



- To find the position vector of the point of intersection of the two lines $l_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = \mathbf{c} + \mu\mathbf{d},\ \mu \in \mathbb{R}$

  Let P be the point of intersection of the 2 lines l_1 and l_2, and $\mathbf{p}$ be the position vector of P. Then P lies on both lines and $\mathbf{p} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{p} = \mathbf{c} + \mu\mathbf{d}$ for some values of λ and μ.

  Method
  Step 1 : Since the lines intersect, $\mathbf{p} = \mathbf{a} + \lambda\mathbf{b} = \mathbf{c} + \mu\mathbf{d}$
  Step 2 : From Step 1, we will have 3 linear equations in terms of λ and μ.
  Solve for the values of λ and μ
  – If there exist unique values for λ and μ, then the position vector of the intersection point is given by $\mathbf{a} + \lambda\mathbf{b}$ or $\mathbf{c} + \mu\mathbf{d}$.
  – If there are no unique values for λ and μ, then the 2 lines are non-intersecting lines. (They can be either parallel or skew lines)

- To check if the lines $l_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = \mathbf{c} + \mu\mathbf{d},\ \mu \in \mathbb{R}$ are skew lines
  (Skew lines are non–parallel and non–intersecting lines.)

  Assume that P, the point of intersection of the 2 lines l_1 and l_2, and $\mathbf{p}$ be the position vector of P exists show that it does not exist.
  Then P lies on both lines and $\mathbf{p} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{p} = \mathbf{c} + \mu\mathbf{d}$ for some values of λ and μ.

  Method
  Step 1 : Show that l_1 and l_2 are not parallel ($\mathbf{b} \neq k\,\mathbf{d}$)
  Step 2 : Then assume $\mathbf{a} + \lambda\mathbf{b} = \mathbf{c} + \mu\mathbf{d}$ and show that there is **NO** unique values for λ and μ that satisfy the equations formed.

- To find the shortest distance between two parallel lines $l_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = \mathbf{c} + \mu\mathbf{b},\ \mu \in \mathbb{R}$.

  The shortest distance between two parallel lines can be found by taking any point, says A on $l_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ and find the shortest distance between point A and $l_2 : \mathbf{r} = \mathbf{c} + \mu\mathbf{b},\ \mu \in \mathbb{R}$.

  OR

  The shortest distance between two parallel lines can be found by taking any point, says C on $l_2 : \mathbf{r} = \mathbf{c} + \mu\mathbf{b},\ \mu \in \mathbb{R}$ and find the shortest distance between point C and $l_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$.

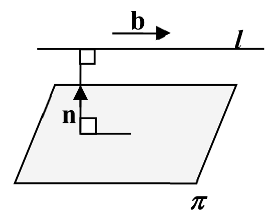
  (Refer above for the shortest distance between a point and a line)

❖ To check whether the point P with position vector **p** lies on the given plane π : $\mathbf{r}\,.\,\mathbf{n} = d$

Method
– If $\mathbf{p}.\,\mathbf{n} = d$, then the point P lies on the plane.
If $\mathbf{p}.\,\mathbf{n} \neq d$, then the point P does not lie on the plane.

❖ To check if the line $l:\mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ is parallel to a given plane π : $\mathbf{r}\,.\,\mathbf{n} = d$

Method
– If $\mathbf{b}.\,\mathbf{n} = 0$, then the line is parallel to the plane.
(perpendicular to the normal)



❖ To check if the line $l:\mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ lies in a given plane π: $\mathbf{r}\,.\,\mathbf{n} = d$

Method 1
– Show that $(\mathbf{a} + \lambda\mathbf{b}).\,\mathbf{n} = d$ for all values of λ, then the line lies in the plane.

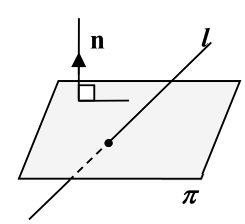
Method 2
Step 1 : Show that $\mathbf{b}.\,\mathbf{n} = 0$, then the line is parallel to the plane.

Step 2 : Show that $\mathbf{a}\,.\,\mathbf{n} = d$, then the point with position vector, **a** lies on the plane.
Then conclude the line lies in the plane.

❖ To find the position vector of point of intersection between the line $l:\mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ and the plane π : $\mathbf{r}\,.\,\mathbf{n} = d$

Let P with position vector **p** be the point of intersection between the line l and the plane π
Then $\mathbf{p} = \mathbf{a} + \lambda\mathbf{b}$, for some $\lambda \in \mathbb{R}$ and $\mathbf{p}.\,\mathbf{n} = d$

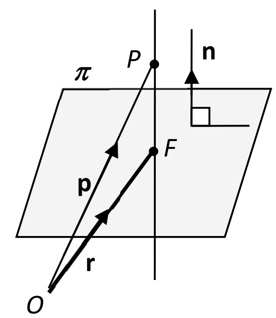
Method
– Since $\mathbf{p} = \mathbf{a} + \lambda\mathbf{b}$, for some $\lambda \in \mathbb{R}$ and $\mathbf{p}.\,\mathbf{n} = d$, then $(\mathbf{a} + \lambda\mathbf{b}).\,\mathbf{n} = d$.
– Solve for λ and use this value to find the position vector of the point of intersection is given by $\mathbf{p} = \mathbf{a} + \lambda\mathbf{b}$



❖ To find the position vector of the foot of the perpendicular from the point P to the plane π: $\mathbf{r}\,.\,\mathbf{n} = d$

Let F be the foot of the perpendicular from the point P to the plane π : $\mathbf{r}\,.\,\mathbf{n} = d$.

Method



Step 1: Find vector equation of the line that is perpendicular to the plane (hence direction vector of this line is **n)** and passing through $P \Rightarrow$ $\mathbf{r} = \mathbf{p} + \lambda\mathbf{n}$, $\lambda \in \mathbb{R}$

Step 2: The point of intersection between line, $\mathbf{r} = \mathbf{p} + \lambda\mathbf{n}$, $\lambda \in \mathbb{R}$ and the plane π: $\mathbf{r}\,.\,\mathbf{n} = d$ is the foot the perpendicular from the point P to the plane.
Consider $(\mathbf{p} + \lambda\mathbf{n}).\,\mathbf{n} = d$ and solve for λ to use this value to find the position vector of the point given by $\mathbf{p} + \lambda\mathbf{n}$ **.**

(Same as above for the point of intersection between a line and a plane)

❖ To find the distance between point P with position vector **p** and the plane π : $\mathbf{r}\,.\,\mathbf{n} = d$

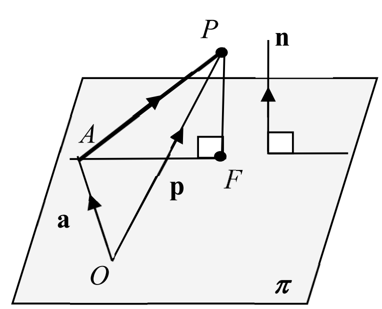
Let F be the foot of the perpendicular from the point P to the plane π : $\mathbf{r}\,.\,\mathbf{n} = d$.

Method 1

Step 1: Find the position vector of the foot of the perpendicular from the point P to the plane π : $\mathbf{r}\,.\,\mathbf{n} = d$ **.**

Step 2: The distance between the point and the plane is given by $|\overrightarrow{PF}|$.
(Refer to diagram above)

Method 2



Step 1: Find any point, says A on the plane with position vector **a**.

Step 2: Find $\overrightarrow{AP} = \mathbf{p} - \mathbf{a}$

The shortest distance between the point P and the plane is given by $\dfrac{|(\mathbf{p} - \mathbf{a}).\mathbf{n}|}{|\mathbf{n}|}$, that is length of projection of AP on the normal vector of the plane.

Method 3

The distance between the point and the plane is given by $\dfrac{|\mathbf{p}.\mathbf{n} - d|}{|\mathbf{n}|}$.

Note from method 2: $\dfrac{|(\mathbf{p} - \mathbf{a}).\mathbf{n}|}{|\mathbf{n}|} = \dfrac{|\mathbf{p}.\mathbf{n} - \mathbf{a}.\mathbf{n}|}{|\mathbf{n}|} = \dfrac{|\mathbf{p}.\mathbf{n} - d|}{|\mathbf{n}|}$.

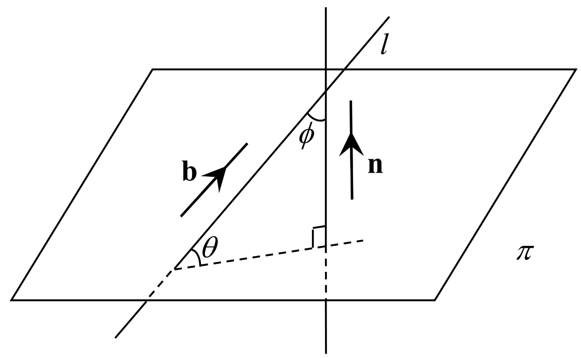
❖ To find the acute angle between the line $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ and the plane $\pi : \mathbf{r} \cdot \mathbf{n} = d$

Let the acute angle between **n** and **b** be ϕ.

Method 1

Step 1: Find the acute angle between **n** and **b** using $\cos\phi = \dfrac{|\mathbf{n}.\mathbf{b}|}{|\mathbf{n}||\mathbf{b}|}$

Step 2: The acute angle between the line and the plane is $90° - \phi$.



Let the acute angle between the line and the plane be θ. Then $\phi = 90° - \theta$.

Method 2

$-\ \sin\theta = \dfrac{|\mathbf{n}.\mathbf{b}|}{|\mathbf{n}||\mathbf{b}|}$

(since $\cos\phi = \cos(90° - \theta) = \sin\theta$)

- ***Involving Planes***

❖ To check if the two planes $\pi_1 : \mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\pi_2 : \mathbf{r} \cdot \mathbf{n}_2 = d_2$ are parallel

Method

– If $\mathbf{n}_1 \ //\ \mathbf{n}_2$ (i.e. $\mathbf{n}_1 = k\ \mathbf{n}_2$), then $\pi_1 // \pi_2$

❖ To find the distance between two parallel planes $\pi_1 : \mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\pi_2 : \mathbf{r} \cdot \mathbf{n}_2 = d_2$

Method

Step 1 : Find any point, says A with position vector, **a** on the plane π_1.

Step 2 : Find the shortest distance between A and π_2 which is $\dfrac{|\mathbf{a}.\mathbf{n} - d|}{|\mathbf{n}|}$

(Refer to Distance between point and the plane $\pi : \mathbf{r} \cdot \mathbf{n} = d$)

❖ To find the acute angle between two planes $\pi_1 : \mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\pi_2 : \mathbf{r} \cdot \mathbf{n}_2 = d_2$

Let the acute angle between the two planes be θ.

Method

– To find the acute angle between the two planes , we make use of $\cos\theta = \dfrac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$

❖ To find vector equation of the line of intersection between two planes π_1: $\mathbf{r}\,.\,\mathbf{n}_1 = d_1$ and π_2 : $\mathbf{r}\,.\,\mathbf{n}_2 = d_2$

Method 1 (using GC)
Step 1: Find the cartesian equations for the planes

Step 2: Using GC to find the line of intersection

Method 2 (if GC is not allowed)
Step 1: Find the cartesian equations for the planes

Step 2: Let one of the variables says $x = 0$. Then solve for y and z using the cartesian equations from **Step 1** to get a common point, A with position vector, $\mathbf{a}$.

Step 3: Compute $\mathbf{b} = \mathbf{n}_1 \times \mathbf{n}_2$ which is the direction vector of the line of intersection. Then the equation of the line of intersection is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$

Method 3 (if GC is not allowed)
Step 1: Find the cartesian equations for the planes

Step 2: Find two common points using the cartesian equations from **Step 1** by first letting $x = 0$ and solve for y and z and then letting $y = 0$ and solve for x and z to get another common point

Step 3: Use the two points to find the equation of the line of intersection .

Example to illustrate Method 2 and 3:
Find the equation of the line of intersection of the two planes whose equations are $\mathbf{r}\,.\begin{pmatrix}1\\1\\-3\end{pmatrix} = 6$ and $\mathbf{r}\,.\begin{pmatrix}1\\0\\2\end{pmatrix} = 1$ respectively

Method 2

$$\mathbf{r}\,.\begin{pmatrix}1\\1\\-3\end{pmatrix} = 6 \Rightarrow x + y - 3z = 6 \quad \text{-------} \quad (1)$$

and $$\mathbf{r}\,.\begin{pmatrix}1\\0\\2\end{pmatrix} = 1 \Rightarrow x + 2z = 1 \quad \text{-------}(2)$$

Let $z = 0$ and solve, we have $x = 1, y = 5$ and $z = 0$

$$\begin{pmatrix}1\\1\\-3\end{pmatrix} \times \begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}2\\-5\\-1\end{pmatrix}$$

$\Rightarrow$ the required vector equation is $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}, \ \lambda \in \mathbb{R}$

<u>Method 3</u>

$\mathbf{r} \,.\, \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} = 6 \Rightarrow x + y - 3z = 6$ ------- (1) and $\quad \mathbf{r} \,.\, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 1 \Rightarrow x + 2z = 1$ -------(2)

Let $z = 0$ and solve, we have $x = 1, y = 5$ and $z = 0$

Let $x = 0$ and solve, we have $x = 0, y = \frac{15}{2}$ and $z = \frac{1}{2}$

$$\begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{15}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{5}{2} \\ -\frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$$

$\Rightarrow$ the required vector equation is $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}, \ \lambda \in \mathbb{R}$

Revision Tutorial Questions

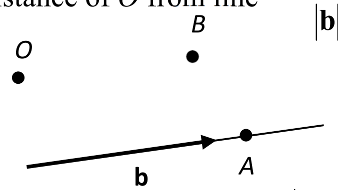
1 Source of Question: HCI 2016 JC2 CTP1Q10(a)

Three distinct points *O*, *A* and *B* are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(i) Let $\overrightarrow{OP} = \mathbf{r}$ and $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$. By expressing $\mathbf{r}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, show that the locus of *P* is a line parallel to $\mathbf{b}$. [2]

(ii) In terms of $\mathbf{a}$ and $\mathbf{b}$, write down the distance from *O* to this line.
Hence write down the distance from *B* to this line. [2]

(iii) Given that the line passes through *O,* what can you say about the relationship between $\mathbf{a}$ and $\mathbf{b}$? [1]

1(a)(i)	$\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b} \Rightarrow (\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0} \Rightarrow \mathbf{r} - \mathbf{a} = \lambda\mathbf{b} \Rightarrow \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ The locus of *P* is a line passing through the point *A* and parallel to **b**.
(a)(ii)	Distance of *O* from line = $\dfrac{\lvert\mathbf{a} \times \mathbf{b}\rvert}{\lvert\mathbf{b}\rvert}$  Distance of *B* from line = $\dfrac{\lvert\mathbf{a} \times \mathbf{b}\rvert}{\lvert\mathbf{b}\rvert}$ since **b** is parallel to the line

(a)(iii)	Since the line passes through *O* and *A*,
	$\Rightarrow \overrightarrow{OA} // \text{line}$
	$\Rightarrow \mathbf{a} // \mathbf{b}$, or $\mathbf{a} = k\mathbf{b}$

2 Source of Question: CJC 2016 JC2 CTQ2

In the rhombus *OABC* where *O* is the origin, the position vectors of the points *A*, *B* and *C* are **a**, **b** and **c** respectively.

(i) Show that $(\mathbf{a} - \mathbf{c}) \cdot (\mathbf{a} + \mathbf{c}) = 0$. [2]

Another point *D* on *AB* produced is such that *AB*:*AD* = 1:4 and the lines *OD* and *BC* intersect at the point *E*.

(ii) **(a)** Find $\overrightarrow{OD}$ in terms of $\mathbf{a}$ and **c**, [1]

(b) By considering the lines of *OD* and *BC*, find $\overrightarrow{OE}$ in terms of $\mathbf{a}$ and **c**. [4]

Hence, find the value of $\dfrac{BE}{EC}$. [1]

2(i)	$(\mathbf{a}-\mathbf{c})\cdot(\mathbf{a}+\mathbf{c})=\mathbf{a}\cdot\mathbf{a}-\mathbf{c}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{c}-\mathbf{c}\cdot\mathbf{c}=\lvert\mathbf{a}\rvert^2-\lvert\mathbf{c}\rvert^2=0 \quad (\because \lvert\mathbf{a}\rvert=\lvert\mathbf{c}\rvert)$ <u>Alternative solution:</u> The diagonals of a rhombus is perpendicular. $\therefore \overrightarrow{AC}\perp\overrightarrow{OB}\Leftrightarrow(\mathbf{c}-\mathbf{a})\cdot(\mathbf{b})=0\Leftrightarrow(\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}+\mathbf{a})=0\Leftrightarrow(\mathbf{a}-\mathbf{c})\cdot(\mathbf{a}+\mathbf{c})=0$
(ii)	(a) $\overrightarrow{OD}=\overrightarrow{OA}+4\overrightarrow{AB}\ =\mathbf{a}+4\mathbf{c}$
	(b) $\overrightarrow{OE}=\mu\overrightarrow{OD}\ =\mu\mathbf{a}+4\mu\mathbf{c}$ $\overrightarrow{OE}\quad=\overrightarrow{OB}+\lambda\overrightarrow{BC}=\lambda\overrightarrow{OC}+(1-\lambda)\overrightarrow{OB}$ $\quad=\lambda\mathbf{c}+(1-\lambda)\mathbf{b}=\lambda\mathbf{c}+(1-\lambda)(\mathbf{a}+\mathbf{c})=(1-\lambda)\mathbf{a}+\mathbf{c}$ Comparing the coefficients of vectors **a** and **c**, $4\mu=1\Leftrightarrow\mu=\frac{1}{4}$ $\qquad\qquad$ $\mu=1-\lambda\Leftrightarrow\lambda=\frac{3}{4}$ Therefore, $\overrightarrow{OE}=\frac{1}{4}\mathbf{a}+\mathbf{c}$
(iii)	$\dfrac{BE}{EC}=\dfrac{\frac{3}{4}}{\frac{1}{4}}=\dfrac{3}{1}$

3 **Source of Question: MI 2015 Prelim P1Q7**

A line l passes through the points A and B with coordinates $(0,\ -2,\ 2)$ and $(1,\ 0,\ 1)$ respectively.

(i) Find the acute angle between $\overrightarrow{OA}$ and the line l, where O is the origin. [2]

(ii) Hence, find the shortest distance from O to the line l, leaving your answer in exact form. [1]

(i)(ii)	$\overrightarrow{AB}=\begin{pmatrix}1\\0\\1\end{pmatrix}-\begin{pmatrix}0\\-2\\2\end{pmatrix}=\begin{pmatrix}1\\2\\-1\end{pmatrix}$ $\cos\theta=\dfrac{\left\lvert\begin{pmatrix}1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\-2\\2\end{pmatrix}\right\rvert}{(\sqrt{6})(\sqrt{8})} \quad\Rightarrow\quad \theta=30°$ Thus, acute angle between $\overrightarrow{OA}$ and the line $l=30°$ Shortest distance $=\sqrt{8}\sin 30°=\sqrt{2}$ units

4 Source of Question: NJC 2016 JC2 CTQ9

A line l and a plane p have equations $\mathbf{r}=\begin{pmatrix}3\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\6\end{pmatrix}$ and $\mathbf{r}\cdot\begin{pmatrix}3\\2\\-2\end{pmatrix}=7$ respectively, where λ is a real parameter.

(i) Show, with working, that l lies completely on p. [2]

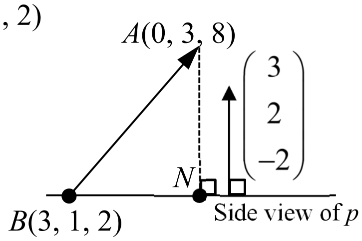
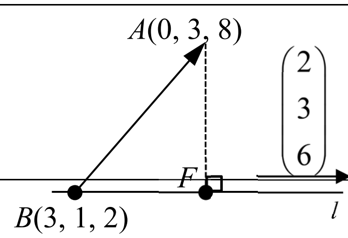
The point A has coordinates (0, 3, 8). Find, in exact form, the shortest distance between

(ii) A and p, [2]

(iii) A and l. [3]

Using the results obtained in parts **(ii)** and **(iii)**, find the shortest distance between l and the foot of perpendicular of A onto p. [2]

(i)	<u>**Method 1**</u> Since $\begin{pmatrix}3\\1\\2\end{pmatrix}\cdot\begin{pmatrix}3\\2\\-2\end{pmatrix}=9+2-4=7,$ one point on l lies on p. Since $\begin{pmatrix}2\\3\\6\end{pmatrix}\cdot\begin{pmatrix}3\\2\\-2\end{pmatrix}=6+6-12=0,$ the direction vector of l is perpendicular to the normal vector of p, and hence is parallel to p. Therefore l lies on p.

	<u>Method 2</u> $\begin{pmatrix} 3+2\lambda \\ 1+3\lambda \\ 2+6\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} = 3(3+2\lambda)+2(1+3\lambda)-2(2+6\lambda)$ $= 9+6\lambda+2+6\lambda-4-12\lambda$ $= 7$ Since $\begin{pmatrix} 1+2\lambda \\ 1+3\lambda \\ -1+6\lambda \end{pmatrix}$ satisfies the equation of p regardless of the value of λ, l lies on p.
(ii)	Let B denote the point (3, 1, 2) and N denote the foot of perpendicular of A onto p.  Then Shortest distance from A to p $= AN$ $= \left\| \dfrac{\overrightarrow{BA} \cdot \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}}{\sqrt{3^2+2^2+(-2)^2}} \right\| = \left\| \dfrac{\left(\begin{pmatrix} 0 \\ 3 \\ 8 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right) \cdot \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}}{\sqrt{9+4+4}} \right\|$ $= \left\| \dfrac{\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}}{\sqrt{17}} \right\| = \left\| \dfrac{-9+4-12}{\sqrt{17}} \right\| = \sqrt{17}$
(iii)	Let F denote the foot of perpendicular of A onto l.  Then Shortest distance from A to l

$= AF$

$$= \left| \frac{\overrightarrow{BA} \times \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}}{\sqrt{2^2 + 3^2 + 6^2}} \right| = \left| \frac{\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}}{\sqrt{4+9+36}} \right|$$

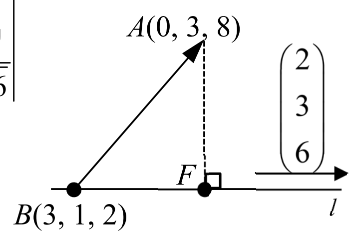
$$= \left| \frac{\begin{pmatrix} (2)(6)-(6)(3) \\ (6)(2)-(-3)(6) \\ (-3)(3)-(2)(2) \end{pmatrix}}{\sqrt{49}} \right| = \frac{1}{7} \left| \begin{pmatrix} -6 \\ 30 \\ -13 \end{pmatrix} \right|$$

$$= \frac{1}{7}\sqrt{(-6)^2 + 30^2 + (-13)^2} = \frac{1}{7}\sqrt{36+900+169} = \frac{\sqrt{1105}}{7}$$

Alternatively, BF

$$= \left| \frac{\overrightarrow{BA} \cdot \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}}{\sqrt{2^2 + 3^2 + 6^2}} \right| = \left| \frac{\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}}{\sqrt{4+9+36}} \right|$$

$$= \left| \frac{-6+6+36}{\sqrt{49}} \right| = \frac{36}{7}$$



By Pythagoras' Theorem,

$$AF = \sqrt{BA^2 - BF^2}$$

$$= \sqrt{\left((-3)^2 + 2^2 + 6^2\right) - \left(\frac{36}{7}\right)^2}$$

$$= \sqrt{49 - \frac{1296}{49}} = \frac{\sqrt{1105}}{7}$$

(last part)

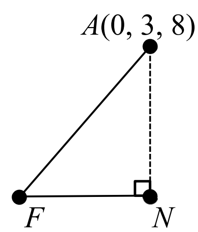
Shortest distance between l and the foot of perpendicular of A on p

$= FN$

$$= \sqrt{AF^2 - AN^2}$$

$$= \sqrt{\left(\frac{\sqrt{1105}}{7}\right)^2 - \left(\sqrt{17}\right)^2}$$

$$= \sqrt{\frac{1105}{49} - 17} = \frac{\sqrt{272}}{7} \text{ or } \frac{4}{7}\sqrt{17} \text{ or } 2.36 \text{ (to 3 s.f.)}$$



5 **Source of Question: MI 2015 Prelim P2 Q3**

Relative to a fixed origin O, the points A, B and C have position vectors $\alpha\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $4\mathbf{i} - 2\mathbf{j}$ and $-\mathbf{i} - 7\mathbf{j} + \beta\mathbf{k}$ respectively where α and β are constants.

Given that A, B and C are collinear, show that $\alpha = 5$ and $\beta = -10$. [2]

Hence state the ratio of the area of ΔOBA to the area of ΔOBC. [1]

Find the vector equation of the line l which passes through points A and B. [1]

The point P lies on OB such that $\overrightarrow{OP} = \frac{1}{3}\overrightarrow{PB}$. Q is a point on the line l such that the length of projection of $\overrightarrow{PQ}$ on the line OB is $\sqrt{5}$ units. Find the possible coordinates of Q. [7]

i	$\overrightarrow{AB} = k\overrightarrow{BC}$ $\begin{pmatrix} 4-\alpha \\ -1 \\ -2 \end{pmatrix} = k\begin{pmatrix} -5 \\ -5 \\ \beta \end{pmatrix}$ $k = \frac{1}{5}$ $\therefore\ \alpha = 5$ and $\beta = -10$
ii	Since both triangles share same base with points A, B, C, so ratio is 1 : 5.
iii	$\boldsymbol{r} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$

3iv	Since $\overrightarrow{OP}=\frac{1}{3}\overrightarrow{PB}$, $\overrightarrow{OP}=\frac{1}{4}\overrightarrow{OB}=\frac{1}{4}\begin{pmatrix}4\\-2\\0\end{pmatrix}=\begin{pmatrix}1\\-0.5\\0\end{pmatrix}$ $\overrightarrow{OQ}=\begin{pmatrix}4+\lambda\\-2+\lambda\\2\lambda\end{pmatrix}$ $\overrightarrow{PQ}=\begin{pmatrix}4+\lambda-1\\-2+\lambda+0.5\\2\lambda\end{pmatrix}=\begin{pmatrix}3+\lambda\\-1.5+\lambda\\2\lambda\end{pmatrix}$ $\therefore\ \frac{\left\|\overrightarrow{PQ}\bullet\overrightarrow{OB}\right\|}{\left\|\overrightarrow{OB}\right\|}=\sqrt{5}$ $\|15+2\lambda\|=10$ So $\lambda=-\frac{5}{2}$ or $\lambda=-\frac{25}{2}$ For $\lambda=-\frac{5}{2}$, $\overrightarrow{OQ}=\begin{pmatrix}1.5\\-4.5\\-5\end{pmatrix}$ and for $\lambda=-\frac{25}{2}$, $\overrightarrow{OQ}=\begin{pmatrix}-8.5\\-14.5\\-25\end{pmatrix}$ Coordinates of Q are (1.5, -4.5, -5) and (-8.5, -14.5, -25).

6 Source of Question: RI 2015 Prelim P1Q6

The line l_1 has equation

$$\mathbf{r}=(3\mathbf{i}+2\mathbf{j}+\mathbf{k})+\lambda(\mathbf{i}+\mathbf{j}+\mathbf{k}),\ \lambda\in\mathbb{R},$$

and the line l_2 passes through the origin O and point A, whose position vector is given by $\mathbf{a}=2\mathbf{i}+\mathbf{j}$. Find the acute angle between

(i) l_1 and l_2, **[2]**

(ii) l_1 and the xy-plane. **[2]**

A point B on l_1 is such that $OB=\sqrt{5}$.

(iii) Find the possible position vectors of B. **[4]**

6(i) **2m**	Let θ be the acute angle between $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\1\\0\end{pmatrix}$. $\cos\theta = \left\lvert \dfrac{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\0\end{pmatrix}}{\sqrt{1^2+1^2+1^2}\sqrt{2^2+1^2+0^2}} \right\rvert = \dfrac{3}{\sqrt{15}}$ $\theta = 39.2^\circ$ (1 d.p.)
(ii) **2m**	A normal vector to the xy-plane is $\begin{pmatrix}0\\0\\1\end{pmatrix}$. Let ϕ be the angle between $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$. $\cos\phi = \dfrac{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}{\sqrt{1^2+1^2+1^2}\sqrt{0^2+0^2+1^2}} = \dfrac{1}{\sqrt{3}} \Rightarrow \phi = 54.736^\circ$ (3 d.p) Hence the acute angle between l and the xy-plane is $90^\circ - 54.736^\circ = 35.3^\circ$ (1 d.p)
(iii) **4m**	Since B lies on l_1, $\overrightarrow{OB} = \begin{pmatrix}3\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$, for some $\lambda \in \mathbb{R}$ Given $\left\lvert\overrightarrow{OB}\right\rvert = \sqrt{5}$, $(3+\lambda)^2 + (2+\lambda)^2 + (1+\lambda)^2 = 5$ $3\lambda^2 + 12\lambda + 9 = 0$ $\lambda^2 + 4\lambda + 3 = 0$ $(\lambda+3)(\lambda+1) = 0$ $\therefore \lambda = -3$ or $\lambda = -1$ When $\lambda = -3$, $\overrightarrow{OB} = \begin{pmatrix}3\\2\\1\end{pmatrix} - 3\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\-1\\-2\end{pmatrix}$ When $\lambda = -1$, $\overrightarrow{OB} = \begin{pmatrix}3\\2\\1\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix}$

7 Source of Question: YJC 2015 PrelimP1Q12

Relative to the origin O, two points A and B have position vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ respectively.

(i) Find $\mathbf{a} \times \mathbf{b}$ and give the geometrical meaning of $|\mathbf{a} \times \mathbf{b}|$. [2]

Hence write down the area of triangle OAB. [1]

(ii) Find a vector equation of the line l passing through A and B. [2]

(iii) The perpendicular to l from the point C with position vector $-13\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ meets the line at the point M. Show that the position vector of M is $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. [3]

(iv) Find a cartesian equation of the plane containing O, A and B and the exact length of projection of $\overrightarrow{CM}$ onto this plane. [4]

(v) Find the acute angle between the line OC and the triangle OAB. [2]

i	$\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} -1 \\ 4+1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} = -\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram with two adjacent sides OA and OB or twice the area of ΔOAB Area of $\Delta OAB = \frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{1}{2}\sqrt{1^2 + 5^2 + 2^2}$ $= \frac{1}{2}\sqrt{30}$ sq units
ii	$\mathbf{a} - \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ Eqn. of line through pts A and B :

	$\mathbf{r}=\begin{pmatrix}2\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}$ where $\lambda\in\mathbb{R}$ Or $\mathbf{r}=\begin{pmatrix}1\\1\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}$ where $\lambda\in\mathbb{R}$
iii	$\overrightarrow{OC}=\begin{pmatrix}-13\\2\\3\end{pmatrix}$ $\overrightarrow{OM}=\begin{pmatrix}2\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}=\begin{pmatrix}2+\lambda\\-\lambda\\1+3\lambda\end{pmatrix}$ for some $\lambda\in\mathbb{R}$ $\overrightarrow{CM}=\overrightarrow{OM}-\overrightarrow{OC}=\begin{pmatrix}15+\lambda\\-\lambda-2\\3\lambda-2\end{pmatrix}$ $\overrightarrow{CM}\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}=0\Rightarrow 15+\lambda+\lambda+2+9\lambda-6=0$ $11\lambda=-11\Rightarrow\lambda=-1$ $\therefore\overrightarrow{OM}=\begin{pmatrix}1\\1\\-2\end{pmatrix}=\mathbf{i}+\mathbf{j}-2\mathbf{k}$ (Shown)
iv	$\mathbf{a}\times\mathbf{b}=\begin{pmatrix}-1\\5\\2\end{pmatrix}$ from (i) Plane OAB: $\mathbf{r}\cdot\begin{pmatrix}-1\\5\\2\end{pmatrix}=0$ (as origin is on the plane)

	Required equation is $x-5y-2z=0$
	$\overrightarrow{CM}=\begin{pmatrix}14\\-1\\-5\end{pmatrix}$ from (iii) Length of the projection of vector $\overrightarrow{CM}$ onto this plane $=\frac{1}{\sqrt{30}}\left\|\overrightarrow{CM}\times\begin{pmatrix}-1\\5\\2\end{pmatrix}\right\|$ $=\frac{1}{\sqrt{30}}\left\|\begin{pmatrix}14\\-1\\-5\end{pmatrix}\times\begin{pmatrix}-1\\5\\2\end{pmatrix}\right\|=\frac{23}{\sqrt{30}}\left\|\begin{pmatrix}1\\-1\\3\end{pmatrix}\right\|$ $=23\sqrt{\frac{11}{30}}$
v	Let θ be the angle between line OC and the normal of triangle OAB. $\cos\theta=\frac{\begin{pmatrix}-13\\2\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\5\\2\end{pmatrix}}{\sqrt{182}\sqrt{30}}=\frac{13+10+6}{\sqrt{5460}} \quad\Rightarrow\quad \theta=66.892°$ Therefore acute angle between line OC and triangle OAB . $=23.1°$

8 Source of Question: CJC 2015 Promo Q6 (a)

The equation of a line l is given by $\frac{x-5}{2}=\frac{y+1}{-2}=z+1$. A point *P*, not lying on l, has position vector $9\mathbf{i}+7\mathbf{j}-2\mathbf{k}$.

(i) Given that *Q* is a point on *l* such that $PQ=6\sqrt{3}$, find the possible coordinates of *Q*. **[4]**

(ii) Hence, or otherwise, find the position vector of the foot of perpendicular from *P* to *l*. **[2]**

(i)	Equation of l: $\mathbf{r}=\begin{pmatrix}5\\-1\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\1\end{pmatrix},\lambda\in\mathbb{R}$ Since Q lies on l, $\overrightarrow{OQ}=\begin{pmatrix}5\\-1\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\1\end{pmatrix}$, for some $\lambda\in\mathbb{R}$. $\overrightarrow{PQ}=\begin{pmatrix}5\\-1\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\1\end{pmatrix}-\begin{pmatrix}9\\7\\-2\end{pmatrix}=\begin{pmatrix}-4+2\lambda\\-8-2\lambda\\1+\lambda\end{pmatrix}$ So $PQ=\left\|\begin{pmatrix}-4+2\lambda\\-8-2\lambda\\1+\lambda\end{pmatrix}\right\|=6\sqrt{3}$ $(-4+2\lambda)^2+(-8-2\lambda)^2+(1+\lambda)^2=108$ $9\lambda^2+18\lambda-27=0$ $(\lambda+3)(\lambda-1)=0$ $\therefore\lambda=-3$ or $\lambda=1$ So $\overrightarrow{OQ}=\begin{pmatrix}5\\-1\\-1\end{pmatrix}-3\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}-1\\5\\-4\end{pmatrix}$ or $\overrightarrow{OQ}=\begin{pmatrix}5\\-1\\-1\end{pmatrix}+\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}7\\-3\\0\end{pmatrix}$ Possible coordinates of Q are $(-1,5,-4)$ and $(7,-3,0)$.
(ii)	Method 1: Let $Q_1(-1,5,-4)$ and $Q_2(7,-3,0)$ and let F be the foot of perpendicular from P to l. Since $\triangle PQ_1Q_2$ is isosceles, F is the midpoint of Q_1 and Q_2. So $\overrightarrow{OF}=\dfrac{\overrightarrow{OQ_1}+\overrightarrow{OQ_2}}{2}=\dfrac{1}{2}\left[\begin{pmatrix}-1\\5\\-4\end{pmatrix}+\begin{pmatrix}7\\-3\\0\end{pmatrix}\right]=\begin{pmatrix}3\\1\\-2\end{pmatrix}$

	Method 2: Let F be the foot of perpendicular from A to l. Then $\overrightarrow{OF}=\begin{pmatrix}5\\-1\\-1\end{pmatrix}+\alpha\begin{pmatrix}2\\-2\\1\end{pmatrix}$ for some $\alpha\in\mathbb{R}$. $\overrightarrow{PF}=\begin{pmatrix}5\\-1\\-1\end{pmatrix}+\alpha\begin{pmatrix}2\\-2\\1\end{pmatrix}-\begin{pmatrix}9\\7\\-2\end{pmatrix}=\begin{pmatrix}-4\\-8\\1\end{pmatrix}+\alpha\begin{pmatrix}2\\-2\\1\end{pmatrix}$ $\overrightarrow{PF}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}=0\Rightarrow\left[\begin{pmatrix}-4\\-8\\1\end{pmatrix}+\alpha\begin{pmatrix}2\\-2\\1\end{pmatrix}\right]\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}=0$ $\begin{pmatrix}-4\\-8\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}+\alpha\begin{pmatrix}2\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}=0$ $9+9\alpha=0\Rightarrow\alpha=-1$ So $\overrightarrow{OF}=\begin{pmatrix}5\\-1\\-1\end{pmatrix}-\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}3\\1\\-2\end{pmatrix}$