Q1	Solution
	Let $\mathrm{P}(N)$ be the statement $\sum_{n=1}^{N}\frac{n+3}{(n+1)(n+2)2^n}=\frac{1}{2}-\frac{1}{(N+2)2^N}$ for integers $N\geq 1$. When $N=1$: LHS = $\frac{1+3}{(1+1)(1+2)(2)}=\frac{1}{3}$ RHS = $\frac{1}{2}-\frac{1}{(1+2)(2)}=1-\frac{1}{6}=\frac{1}{3}$ LHS = RHS, hence $\mathrm{P}(1)$ is true. Assume $\mathrm{P}(k)$ is true for some $k\geq 1$, i.e. $\sum_{n=1}^{k}\frac{n+3}{(n+1)(n+2)2^n}=\frac{1}{2}-\frac{1}{(k+2)2^k}$. Claim $\mathrm{P}(k+1)$ is true, i.e. $\sum_{n=1}^{k+1}\frac{n+3}{(n+1)(n+2)2^n}=\frac{1}{2}-\frac{1}{(k+3)2^{k+1}}$. *Proof:* LHS $=\sum_{n=1}^{k+1}\frac{n+3}{(n+1)(n+2)2^n}$ $=\sum_{n=1}^{k}\frac{n+3}{(n+1)(n+2)2^n}+\frac{k+4}{(k+2)(k+3)2^{k+1}}$ $=\frac{1}{2}-\frac{1}{(k+2)2^k}+\frac{k+4}{(k+2)(k+3)2^{k+1}}$ $=\frac{1}{2}-\frac{(k+3)(2)-(k+4)}{(k+2)(k+3)2^{k+1}}$ $=\frac{1}{2}-\frac{2k+6-k-4}{(k+2)(k+3)2^{k+1}}$ $=\frac{1}{2}-\frac{k+2}{(k+2)(k+3)2^{k+1}}$ $=\frac{1}{2}-\frac{1}{(k+3)2^{k+1}}$ = RHS Hence $\mathrm{P}(k)$ is true $\Rightarrow$ $\mathrm{P}(k+1)$ is true. Since $\mathrm{P}(1)$ is true, and if $\mathrm{P}(k)$ is true then $\mathrm{P}(k+1)$ is also true, then by mathematical induction, $\mathrm{P}(N)$ is true for all positive integers $N\geq 1$. $\sum_{n=1}^{\infty}\frac{n+3}{(n+1)(n+2)2^n}=\lim_{N\to\infty}\frac{1}{2}-\frac{1}{(N+2)2^N}=\underline{\frac{1}{2}}$

Q2	Solution
(i)	The distributive axiom $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$ is violated. $c(\mathbf{u}+\mathbf{v})=c\begin{pmatrix}u_1v_1\\u_2v_2\end{pmatrix}=\begin{pmatrix}cu_1v_1\\cu_2v_2\end{pmatrix}$ $c\mathbf{u}+c\mathbf{v}=\begin{pmatrix}cu_1\\cu_2\end{pmatrix}+\begin{pmatrix}cv_1\\cv_2\end{pmatrix}=\begin{pmatrix}c^2u_1v_1\\c^2u_2v_2\end{pmatrix}$ So $c(\mathbf{u}+\mathbf{v})\neq c\mathbf{u}+c\mathbf{v}$ in general

	Alternative The axiom $(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ is violated. $(c+d)\mathbf{u} = \begin{pmatrix}(c+d)u_1 \\ (c+d)u_2\end{pmatrix}$ $c\mathbf{u} + d\mathbf{u} = \begin{pmatrix}cu_1 \\ cu_2\end{pmatrix} + \begin{pmatrix}du_1 \\ du_2\end{pmatrix} = \begin{pmatrix}cdu_1{}^2 \\ cdu_2{}^2\end{pmatrix}$ So $(c+d)\mathbf{u} \neq c\mathbf{u} + d\mathbf{u}$ in general
(ii)(a)	Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be matrices such that $\mathbf{A}_1\mathbf{B} = \mathbf{B}\mathbf{A}_1$ and $\mathbf{A}_2\mathbf{B} = \mathbf{B}\mathbf{A}_2$ $\left(\mathbf{A}_1 + \mathbf{A}_2\right)\mathbf{B} = \mathbf{A}_1\mathbf{B} + \mathbf{A}_2\mathbf{B} = \mathbf{B}\mathbf{A}_1 + \mathbf{B}\mathbf{A}_2 = \mathbf{B}(\mathbf{A}_1 + \mathbf{A}_2)$ $\left(k\mathbf{A}_1\right)\mathbf{B} = k\left(\mathbf{A}_1\mathbf{B}\right) = k\left(\mathbf{B}\mathbf{A}_1\right) = \mathbf{B}\left(k\mathbf{A}_1\right)$ The set is closed under addition and scalar multiplication. Also, the set is non-empty since $\mathbf{0B} = \mathbf{B0} = \mathbf{0}$. Hence it is a subspace.
(ii)(b)	$\mathbf{I}^{\mathrm{T}}\mathbf{I} = \mathbf{I}$ but $(2\mathbf{I})^{\mathrm{T}}(2\mathbf{I}) = 4(\mathbf{I}^{\mathrm{T}}\mathbf{I}) = 4\mathbf{I} \neq \mathbf{I}$ The set is not closed under scalar multiplication. Hence it is not a subspace.

Q3	**Solution**
(i)	$I = \int_{-2}^{2} 3^x \,\mathrm{d}x$ Let $\mathrm{f}(x) = 3^x$ and $h = \dfrac{2-(-2)}{4} = 1$

n	t_n	$y_n = \mathrm{f}(t_n)$
0	-2	$\frac{1}{9}$
1	-1	$\frac{1}{3}$
2	0	1
3	1	3
4	2	9

Let T denotes the approximation to $I = \int_{-2}^{2} 3^x \,\mathrm{d}x$, found using trapezium rule with 5 ordinates.

$T = \dfrac{h}{2}\left[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right]$ --- (1)

$T = 8\dfrac{8}{9}$

(ii)

$\mathrm{f}(x) = 3^x$

$\mathrm{f}'(x) = (\ln 3)(3^x)$

$\mathrm{f}''(x) = (\ln 3)^2 (3^x) > 0$ for $-2 \leq x \leq 2$

$\mathrm{f}(x) = 3^x$ is concave upwards over the interval $[-2, 2]$

Trapezium rule produces an overestimate T to $I = \int_{-2}^{2} 3^x \,\mathrm{d}x$.

(iii)

Let S denotes the approximation to $I = \int_{-2}^{2} 3^x \,\mathrm{d}x$, found using Simpson rule with 5 ordinates.

$S = \dfrac{1}{3}h\left[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right]$ --- (2)

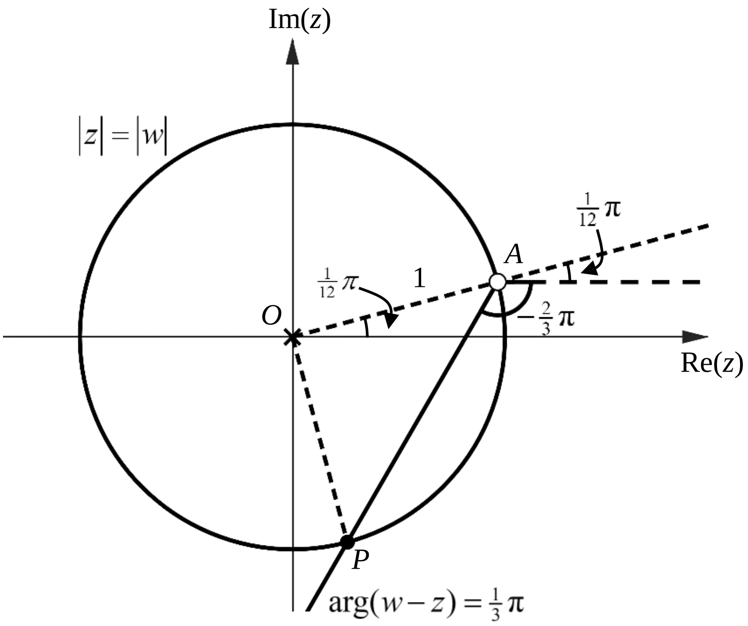
	$S=8\frac{4}{27}$
(iv)	$I=\int_{-2}^{2}3^x\,\mathrm{d}x$ $=\frac{1}{\ln 3}\left[3^x\right]_{-2}^{2}$ $=\frac{1}{\ln 3}\left[3^2-3^{-2}\right]$ $=\frac{80}{9}\left(\frac{1}{\ln 3}\right)$
(v)	Numerical integration using the Simpson rule produces a more accurate approximation compared to the Trapezium rule, with the same number of ordinates. The Simpson rule makes use of a quadratic approximation as opposes to the Trapezium rule which makes use of a linear approximation. Hence Simpson rule uses a better approximation to the curve $y=3^x$.
(vi)	Absolute percentage error $=\frac{\lvert I-S\rvert}{I}\times 100\% \approx 0.706\%$

Q4	**Solution**
(a)	Differentiate (1) with respect to x: $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}-2\frac{\mathrm{d}z}{\mathrm{d}x}=0$ From (2), $\frac{\mathrm{d}z}{\mathrm{d}x}=y-5z+16x$ $\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}-2(y-5z+16x)=0$ $\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}-2y+10z-32x=0$ From (1), $2z=\frac{\mathrm{d}y}{\mathrm{d}x}+4y-8$ $\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}-2y+5\left(\frac{\mathrm{d}y}{\mathrm{d}x}+4y-8\right)-32x=0$ $\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}-2y+5\frac{\mathrm{d}y}{\mathrm{d}x}+20y-40-32x=0$ $\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}+9\frac{\mathrm{d}y}{\mathrm{d}x}+18y=32x+40$ (shown)
(b)	Auxiliary equation: $m^2+9m+18=0$ $\Rightarrow (m+3)(m+6)=0$ $\Rightarrow m=-6$ or -3 Complementary function: $y=A\mathrm{e}^{-6x}+B\mathrm{e}^{-3x}$ for arbitrary constants A, B For particular integral, let $y=cx+d \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=c$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ Substitute into DE: $0+9c+18(cx+d)=32x+40$ Comparing coefficients:

Q4	Solution
	x: $18c = 32 \Rightarrow c = \dfrac{16}{9}$ x^0 : $9c + 18d = 40 \Rightarrow d = \dfrac{40-16}{18} = \dfrac{4}{3}$ $\therefore$ General solution for y is $y = Ae^{-6x} + Be^{-3x} + \dfrac{16}{9}x + \dfrac{4}{3}$ $\dfrac{dy}{dx} = -6Ae^{-6x} - 3Be^{-3x} + \dfrac{16}{9}$ Substitute into (1): $-6Ae^{-6x} - 3Be^{-3x} + \dfrac{16}{9} + 4\left(Ae^{-6x} + Be^{-3x} + \dfrac{16}{9}x + \dfrac{4}{3}\right) - 2z = 8$ $\Rightarrow -2Ae^{-6x} + Be^{-3x} + \dfrac{64}{9}x - \dfrac{8}{9} = 2z$ $\Rightarrow z = -Ae^{-6x} + \dfrac{B}{2}e^{-3x} + \dfrac{32}{9}x - \dfrac{4}{9}$ Sub. $x = 0$, $y = 0$: $A + B + \dfrac{4}{3} = 0 \Rightarrow A + B = -\dfrac{4}{3}$ ....(3) Sub. $x = 0$, $z = 0$: $-A + \dfrac{B}{2} - \dfrac{4}{9} = 0 \Rightarrow -2A + B = \dfrac{8}{9}$ ....(4) Using GC to solve (3) and (4), $A = -\dfrac{20}{27}$, $B = -\dfrac{16}{27}$ $\therefore$ Solutions for y and z are: $y = -\dfrac{20}{27}e^{-6x} - \dfrac{16}{27}e^{-3x} + \dfrac{16}{9}x + \dfrac{4}{3}$ $z = \dfrac{20}{27}e^{-6x} - \dfrac{8}{27}e^{-3x} + \dfrac{32}{9}x - \dfrac{4}{9}$

Q5	Solution
(a)	$2v^4 = 1 + \sqrt{3}i$ $2v^4 = 2e^{i\left(\frac{\pi}{3}\right)}$ $v^4 = e^{i\left(\frac{\pi}{3}+2k\pi\right)}$, where $k \in \mathbb{Z}$ $v = e^{i\left[\frac{1}{4}\left(\frac{\pi}{3}+2k\pi\right)\right]}$ $v = e^{i\left(\frac{\pi}{12}+\frac{k\pi}{2}\right)} = e^{i(6k+1)\frac{\pi}{12}}$ For arguments in the principal range, choose $k = 0, \pm 1, -2$ $\therefore v = \underline{e^{-i\left(\frac{11\pi}{12}\right)},\ e^{-i\left(\frac{5\pi}{12}\right)},\ e^{i\left(\frac{\pi}{12}\right)},\ e^{i\left(\frac{7\pi}{12}\right)}}$
(b)	Let $w = e^{ip} = \cos p + i\sin p$ where $p = \dfrac{\pi}{12}$. By De Moivre's Theorem, for any positive integer n,

Q5	Solution
	$w^n + \frac{1}{w^n} = w^n + w^{-n}$ $= \cos np + \mathrm{i}\sin np + \cos(-np) + \mathrm{i}\sin(-np)$ $= \cos np + \mathrm{i}\sin np + \cos np - \mathrm{i}\sin np$ $= \underline{2\cos np}$ (shown) $2\cos p = w + \frac{1}{w}$ $\Rightarrow (2\cos p)^4 = \left(w + \frac{1}{w}\right)^4$ $\Rightarrow 16\cos^4 p = \left(w^4 + \frac{1}{w^4}\right) + 4\left(w^2 + \frac{1}{w^2}\right) + 6$ $= 2\cos 4p + 4(2\cos 2p) + 6$ $= 2\cos\frac{\pi}{3} + 8\cos\frac{\pi}{6} + 6$ $= 2\left(\frac{1}{2}\right) + 8\left(\frac{\sqrt{3}}{2}\right) + 6$ $= 7 + 4\sqrt{3}$ $\Rightarrow \underline{\cos^4 p = \frac{7+4\sqrt{3}}{16}}$ (shown)
(c)	$\lvert z\rvert = \lvert w\rvert = 1$ Locus is a circle centred at the origin *O* with radius 1 unit. $\arg(w-z) = \frac{\pi}{3} \Rightarrow \arg(-(z-w)) = \frac{\pi}{3}$ $\Rightarrow \arg(-1) + \arg(z-w) = \frac{\pi}{3}$ $\Rightarrow \arg(z-w) = \frac{\pi}{3} - \pi$ $\Rightarrow \arg(z-w) = -\frac{2\pi}{3}$ Locus is a half-line starting from (and excluding the point *A* representing *w*), at an argument of $-\frac{2\pi}{3}$ rad.

Q5	Solution
	
(d)	Triangle *OAP* is isosceles triangle. $\angle OAP = \angle OPA$ $= \pi - \frac{2\pi}{3} - \frac{\pi}{12} = \frac{\pi}{4}$ $\angle AOP = \pi - 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$ The argument of the complex number represented by *P* is $\frac{\pi}{12} - \frac{\pi}{2} = -\frac{5\pi}{12}$ Since *P* lies on the circle centred at origin with radius 1, the complex number represented by *P* is $\underline{e^{-i\left(\frac{5\pi}{12}\right)}}$, which is one of the roots of the equation in **(a)**.

Q6	Solutions
(a)	$\hat{p} = \frac{7}{9}$, $z_{0.95} = 1.6449 \quad (\text{or } 1.645)$, $n = 900$ Since $\hat{P} \sim \mathrm{N}\left(p, \frac{p(1-p)}{n}\right)$ approx. by CLT, 90% confidence interval for p $= \left(\frac{7}{9} - 1.645\sqrt{\frac{\frac{7}{9}\left(1-\frac{7}{9}\right)}{900}},\ \frac{7}{9} + 1.645\sqrt{\frac{\frac{7}{9}\left(1-\frac{7}{9}\right)}{900}} \right)$ $= (0.7550, 0.8006)$ Answer is 4 dp as interval width is 0.0456 to 3 sf

(b)	
	$$2\times z_{0.95}\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}=0.04$$ $$1.6449\sqrt{\frac{\frac{7}{9}\left(1-\frac{7}{9}\right)}{n}}=0.02$$ Using GC, $n=1169.06\approx 1169$ = 1170 (3sf)

Q7	Solutions
	H_0: the number of heads obtained follows a binomial distribution with $p=0.6$. H_1: the number of heads obtained does not follow a binomial distribution with $p=0.6$. Level of significance: 5%

No of heads	0	1	2	3	4
Frequency	5	35	64	66	30
Expected frequency	5.12	30.72	69.12	69.12	25.92

Degree of freedom is 4.

$$\chi^2=\sum_{i=0}^{4}\frac{\left(O_i-E_i\right)^2}{E_i}\sim\chi^2\left(4\right)$$

By GC, the p-value is 0.780 > 0.05.
Hence, we do not reject H_0 and conclude at 5% significance level that there is insufficient evidence that a binomial distribution with $p=0.6$ is not a good fit.

If the experiment is repeated 1000 times, the new χ^2 value will be $1.7614\times\frac{1000}{200}=8.807<9.488$ and so there is no change to the result of the test.

Q8	Solutions
(a)	The depths of tread on the front tyres and those on the rear tyres are not known to be normally distributed.
(b)	

	A	B	C	D	E	F	G	H
	2.4	1.5	2.3	2.4	2.6	2.5	2.1	2.6
	2.3	1.9	2.1	1.8	1.8	2.8	1.4	2.1
Diff	0.1	–0.4	0.2	0.6	0.8	–0.3	0.7	0.5
Rank	1	–4	2	6	8	–3	7	5

H_0: $m_d=0$ H_1: $m_d\neq 0$
where m_d mm is the population median of depth of front tyre subtracts depth of rear tyre.

$P=29$, $Q=7$, so $T=Q=7$, the 10% two-tail critical region for $n=8$ is $T\le 5$.
Therefore, we do not reject H_0 at 10% significance level and conclude there is insufficient evidence that there is a difference between the average wear for the front and rear tyres.

(c)	
	After correcting the mistakes, we have only one positive difference. For the given conclusion, we need $T\le 5$.

Q8	Solutions
	This means that $-0.5 < b-1.9 < -0.3$. Hence $1.4 < b < 1.6$.

Q9	Solutions
(a)	$\mathrm{P}(Y \le y)$ $= \mathrm{P}(1-5X^2 \le y)$ $= \mathrm{P}\left(\frac{1-y}{5} \le X^2\right)$ $= \mathrm{P}\left(X \ge \sqrt{\frac{1-y}{5}}\right) + \mathrm{P}\left(-\sqrt{\frac{1-y}{5}} \ge X\right)$ $= \mathrm{P}\left(X \ge \sqrt{\frac{1-y}{5}}\right) + 0$ $= \int_{\sqrt{\frac{1-y}{5}}}^{1} 4x^3 \, \mathrm{d}x \quad \text{if} \quad 0 < \sqrt{\frac{1-y}{5}} < 1$ $= \left[x^4\right]_{\sqrt{\frac{1-y}{5}}}^{1} \quad \text{for} \quad -4 < y < 1$ $= 1 - \left(\frac{1-y}{5}\right)^2$ p.d.f of Y $= \frac{\mathrm{d}}{\mathrm{d}y}\left[1 - \left(\frac{1-y}{5}\right)^2\right]$ $= \frac{2}{25}(1-y)$ $\mathrm{g}(y) = \frac{2}{25}(1-y), -4 < y < 1$.
(b)	$\mathrm{E}(Y) = \int_{-4}^{1} \frac{2}{25}(1-y)y \, \mathrm{d}y = -\frac{7}{3}$

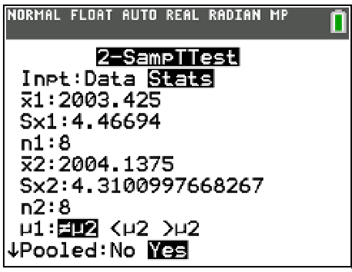
Q10	Solutions
(i)	1. The average rate of survey responses received remains constant. 2. The event of receiving a survey response is independent of the event of receiving another survey response.
(ii)	Let S be the number of surveys received in an hour, i.e. $S \sim \mathrm{Po}\left(\frac{5}{6}\right)$. $\mathrm{P}(S_1 = 0, S_2 = 0 \text{ and } S_3 \ge 1)$ $= \mathrm{P}(S_1 = 0)\mathrm{P}(S_2 = 0)\mathrm{P}(S_3 \ge 1)$ $= \mathrm{P}(S_1 = 0)\mathrm{P}(S_2 = 0)\left[1 - \mathrm{P}(S_3 = 0)\right]$ $= 0.10679$ $\approx 0.107 \text{ (3 s.f.)}$

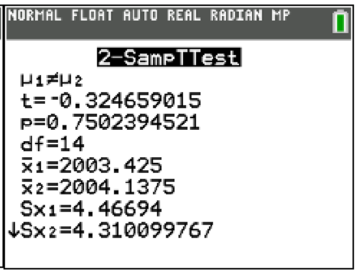
Q10	Solutions
	Alternative Let W h be the waiting time to receive the first survey. $W \sim \text{Exp}\left(\frac{5}{6}\right)$ $\text{P}(2 < W < 3) = 0.107$
(iii)	Let X be the number of surveys received in a day, i.e. $X \sim \text{Po}(20)$ Let W be the total number of surveys received in 2 days, i.e. $W \sim \text{Po}(40)$. Required probability $= \dfrac{2\left(\begin{array}{l}[\text{P}(X=16)][\text{P}(X=14)]+[\text{P}(X=17)][\text{P}(X=13)]\\ +[\text{P}(X=18)][\text{P}(X=12)]\end{array}\right)}{\text{P}(W=30)}$ $= \dfrac{2(0.0060479)}{0.018465}$ $= 0.65505$ $= 0.655$ (3s.f.)
(iv)	$\text{f}(t) = \dfrac{1}{72}\text{e}^{-\frac{1}{72}t}, t > 0$
(v)	$\text{P}(T > n) \geq 0.3$ $\text{e}^{-\frac{1}{72}n} \geq 0.3$ $n \leq \ln 0.3(-72) = 86.7$ $\therefore$ greatest $n = 86$

n	$\text{P}(T > n)$
86	0.3029
87	0.2987

Q11	Solutions
(i)	(1-sample) t test
(ii)	By GC, $\bar{x} = 2003.425$ and $s_X{}^2 = 4.46694^2$ Let μ_X g be the population mean mass of rice in a packet reported by Machine A. H_0: $\mu_X = 2000$ H_1: $\mu_X > 2000$ $T = \dfrac{\bar{X} - 2000}{s/\sqrt{8}} \sim t(7)$ Test statistic $t = \dfrac{2003.425 - 2000}{4.46694/\sqrt{8}} = 2.16868$ By GC, p-value = 0.0334. Since the null hypothesis is rejected, $\dfrac{\alpha}{100} \geq 0.0334 \Rightarrow \alpha \geq 3.34$.

<table>
<tr><td>Q11</td><td>Solutions</td></tr>
<tr><td></td><td>Thus the minimum value of α is 3.34.</td></tr>
<tr><td>(iii)</td><td>Appropriate hypothesis test is the 2-sample t test.
We need to assume that the variances of the masses of packets of rice of both batches are the same.

We have $\bar{y} = 2004.1375$ and
$s_Y{}^2 = \frac{1}{7}\left(266.99 - \frac{33.1^2}{8}\right) = 18.57696$.
Let μ_Y g be the mean mass of rice in a packet reported by Machine B.
H_0: $\mu_X - \mu_Y = 0$
H_1: $\mu_X - \mu_Y \neq 0$
$s_p{}^2 = \frac{7s_X{}^2 + 7s_Y{}^2}{14}$
$T = \frac{\overline{X} - \overline{Y}}{s_p\sqrt{\frac{1}{8}+\frac{1}{8}}} \sim t(14)$




By GC, p-value = 0.750 > 0.05.
Hence we do not reject H_0 and there is insufficient evidence at the 5% significance level to conclude that the mean masses of packets in the two batches are different.</td></tr>
<tr><td>(iv)</td><td>It is possible that the second batch of rice consisted of packets that are heavier.</td></tr>
</table>