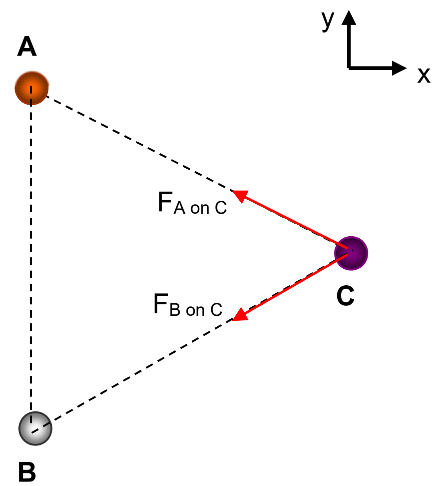
Example 1: Finding Resultant Gravitational Force

Three identical masses, each of mass *m*, are located on a table at the corners of an equilateral triangle of side *d*. Determine the resultant gravitational force on mass C due to masses A and B.



Solution:

> **Concept:** If several particles are present, each pair will experience a mutual gravitational attraction. The resultant gravitational force on a given particle is **the vector sum of the separate attractive forces acting on the particle due to individual particles interacting with it**.

Magnitude of force on C due to B $F_{B \text{ on } C} = \dfrac{Gm^2}{d^2}$

Also, |force on C due to A |= |force on C due to B|

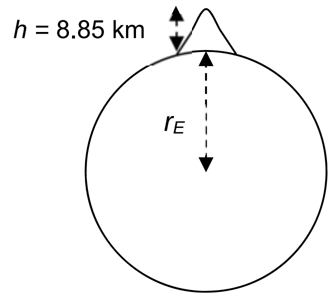
Hence by vector addition, the resultant force = 2 $|F_{B \text{ on } C}|$ cos 30º

$= \dfrac{\sqrt{3}Gm^2}{d^2}$ directed horizontally towards the left.

Example 2: Gravitational Field Strength near the Surface of the Earth

Assume that the Earth is not rotating and is a perfect sphere of uniform density.

(i) Calculate the gravitational field strength of the Earth at its surface.
(ii) Given that Mount Everest is 8.85 km high, compute the gravitational field strength of Earth at the peak of Mount Everest.
(iii) Comment on your values computed above.



You are given the following information:

- Mass of the Earth, m_E = 5.98 x 10^{24} kg
- Average radius of Earth, r_E = 6.37 x 10^6 m

Solution:

(i) The gravitational field strength at the surface of the Earth,

$$g_{surface} = G\frac{m_E}{{r_E}^2} = (6.67 \times 10^{-11}\text{ N m}^2\text{ kg}^{-2})\left[\frac{5.98\times10^{24}\text{ kg}}{(6.37\times10^6\text{ m})^2}\right] = 9.83\text{ m s}^{-2}.$$

(ii) The gravitational field strength at the peak of Mount Everest,

$$g_{everest} = G\frac{m_E}{(r_E+h)^2} = (6.67 \times 10^{-11}\text{ N m}^2\text{ kg}^{-2})\left[\frac{5.98\times10^{24}\text{ kg}}{(6.37\times10^6\text{ m} + 8.85\times10^3\text{ m})^2}\right] = 9.80\text{ m s}^{-2}$$

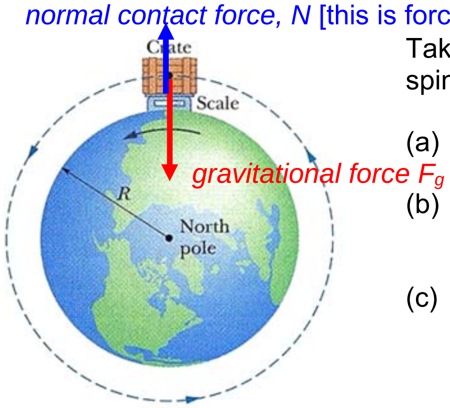
(iii) Comparing the values computed in (i) and (ii), we see that

- $g_{surface} > g_{everest}$ by only approx. 0.03 m s^{-2} (about 0.3%).
- Hence, for the purpose of calculations, g may be taken to be constant near the Earth surface. Furthermore, for heights h near the surface of the Earth such that $h \ll r_E$, the value of g may be taken to be constant.

Example 3: How Earth's rotation affects the weight measured

Consider a crate of mass m resting on a weighing machine at the equator.

normal contact force, N [this is force by scale on crate]



Take the Earth to be a uniform sphere of mass M and radius R, spinning on its axis.

(a) On the diagram, draw all the forces acting on the crate.

(b) Write down an expression for the reading of the weighing machine supporting this mass.

(c) Write down an expression for the 'apparent' gravitational acceleration indicated by the weighing machine.

Solution:

(b) By Newton's second law (considering the crate):

$$\sum F = ma_c$$
$$F_g - N = ma_c$$
$$\Rightarrow N = F_g - ma_c$$

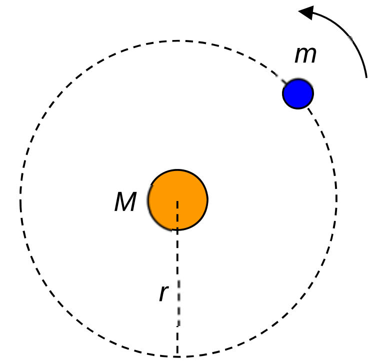
By Newton's 3rd law, the force by crate on scale = N.

(c) Apparent gravitational field strength $g' = \dfrac{N}{m} = \dfrac{F_g - ma_c}{m} = \dfrac{mg - ma_c}{m} = g - a_c$

What is the value of the centripetal acceleration? $a_c = r\omega^2 = 6.4 \times 10^6 \left(\dfrac{2\pi}{24 \times 3600}\right)^2 = 0.034 \text{ m s}^{-2}$

Hence the apparent gravitational field strength is 9.81 – 0.034 = 9.78 m s^{-2}

Example 4 Investigating satellite motion



Consider a satellite of mass m revolving around a larger mass M in a circle of radius r.

Determine the relationship between the radius of the orbit and

(a) the velocity of the satellite;

(b) the period of rotation.

Solution:

As the satellite revolves, its velocity keeps changing as the direction of motion keeps changing. Hence, the satellite is accelerating. From the topic of Circular Motion, we know that a resultant force must be providing this centripetal acceleration.

The gravitational attraction of the large mass M on the satellite keeps the satellite in orbit, i.e., **the gravitational force acting on the satellite (by the mass *M*) provides for the centripetal force required for the satellite to maintain circular orbit about the mass *M*.**

(a) By Newton's 2nd Law,

$$\sum F = ma_c$$
$$\frac{GMm}{r^2} = m\frac{v^2}{r}$$
$$v = \sqrt{\frac{GM}{r}}$$
$$\Rightarrow v \propto \frac{1}{\sqrt{r}}$$

(b) By Newton's 2nd Law,

$$\sum F = ma_c$$
$$\frac{GMm}{r^2} = mr\omega^2$$
$$\frac{GM}{r^3} = \frac{4\pi^2}{T^2}$$
$$\Rightarrow T^2 \propto r^3$$

Conclusion: For an orbit of radius r, there is only one allowed value of T and v.

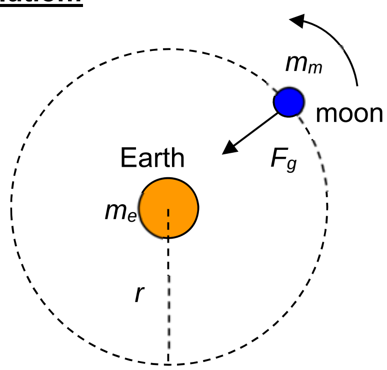
Example 5: How do you 'weigh' the Earth?

When Newton first discovered his Law of Gravitation, he had neither the value of *G* nor the mass of the Sun.

In 1798, a hundred and twenty-one years after Newton proposed the Universal Law of Gravitation, Henry Cavendish conducted the first experiment to measure the force of gravity between masses in the laboratory, and obtained accurate values for the gravitational constant, *G*, and the mass density of the Earth.
With the value of *G*, it was possible to obtain a value for the mass of the Earth.

By considering the motion of the moon about the Earth, show that the mass of the Earth m_e is approximately 6×10^{24} kg. You may assume that distance between Earth and the moon $r = 4.0 \times 10^8$ m.

Solution:



The gravitational force acting on the moon by the Earth's gravitational field provides for the centripetal force required by the moon to move in orbit about the Earth.

By Newton's 2nd Law,

$$F_g = m_m a_c$$

$$\Rightarrow G\frac{m_e m_m}{r^2} = m_m r \omega_m^2 = m_m r\left(\frac{2\pi}{T_m}\right)^2$$

$$\Rightarrow m_e = \frac{4\pi^2}{G}\frac{r^3}{T_m^{\ 2}}$$

Substituting for *G*, r_m and since we know that the moon takes approximately about 1 month to make one complete revolution around the Earth,

$$\Rightarrow \quad m_e = \frac{4\pi^2}{G}\frac{r^3}{T_m^{\ 2}} = \frac{4\pi^2}{G}\frac{(4\times10^8\text{ m})^3}{(30\times24\times60\times60\text{ s})^2} = 5.64 \times 10^{24}\text{ kg} \sim 6 \times 10^{24}\text{ kg}$$

Hence, we have found the mass of the Earth!

The Cavendish Experiment - Sixty Symbols

Example 6: Geostationary Satellite

A communications satellite of mass *m* is placed in a circular geostationary orbit.
Find its height above the Earth's surface.
(Take radius of the Earth to be 6.38 x 10^6 m and the mass of the Earth to be 5.98 x 10^{24} kg).

Solution:

The **gravitational force** on the satellite **provides** the required **centripetal force.**
(NOTE: the statement above is required to explain the following steps below. Please write this in exams!)

By Newton's second law:

$$\sum F = ma_c$$

$$\frac{GMm}{r^2} = mr\omega^2$$

$$\frac{GM}{r^3} = \frac{4\pi^2}{T^2}$$

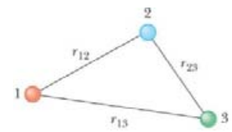
$$\Rightarrow r^3 = \frac{GMT^2}{4\pi^2} = \frac{6.67\times10^{-11}\times5.98\times10^{24}\times(24\times3600)^2}{4\pi^2}$$

$$r = 4.23\times10^7 \text{ m}$$

$$\Rightarrow h = 4.23\times10^7 - 6.38\times10^6 = 3.59\times10^7 \text{ m}$$

Example 7: Gravitational Potential Energy of a System of 3 Masses

A system consists of three particles, each of mass *m* located at the corners of an equilateral triangle of side *x*. Determine the gravitational potential energy of the system.



Solution:

In order to assemble the three particles at the corners of the equilateral triangle, we bring each particle (from infinity) to its assigned position one after another. The work done by us while bringing each particle to its position is equal to the change in the gravitational potential energy of the system.

Hence the total change in the gravitational potential energy of the system ΔU is the net work done W_{net} to position all three particles.

$$\Delta U = \Delta W_{net}$$

Since the gravitational potential energy in the system is zero initially when all the particles are infinitely far away,

$$U_f - 0 = W_1 + W_2 + W_3$$

Note that when positioning particle 3, both particle 1 and 2 are already positioned and they exert gravitational forces on the third particle.

$$U_f = 0 + (-\frac{Gm_1m_2}{r_{12}}) + \left[(-\frac{Gm_1m_3}{r_{13}}) + (-\frac{Gm_2m_3}{r_{23}})\right]$$

Since all particles have the same mass and they are at the vertices of an equilateral triangle of side x,

$$U_f = -\frac{3Gm^2}{x}$$

$\frac{3Gm^2}{x}$ would be the work needed to separate the particles by an infinite distance.

Example 8: Gravitational Potential Energy Changes near the Earth's Surface

Show that the expression $U = -G\frac{Mm}{r}$ can be reduced to the expression $mg\Delta h$ for calculations of changes in gravitational potential energy near the Earth's surface, i.e. for Δh << radius of the Earth R_E.

Solution

$$\Delta U = U_f - U_i = -\frac{GMm}{(R_E + \Delta h)} - \left(-\frac{GMm}{R_E}\right) = GMm\left(\frac{1}{R_E} - \frac{1}{R_E + \Delta h}\right)$$

$$= GMm\left(\frac{R_E + \Delta h - R_E}{R_E(R_E + \Delta h)}\right)$$

$$= GMm\left(\frac{\Delta h}{R_E(R_E + \Delta h)}\right)$$

Since Δh << R_E, $R_E(R_E + \Delta h) \approx R_E^{\,2}$

Hence, $\Delta U = GMm\left(\frac{\Delta h}{R_E^{\,2}}\right) = m\frac{GM}{R_E^{\,2}}\Delta h = mg\Delta h$

Example 9: Escape Speed

The minimum speed with which a mass should be projected from the Earth's surface in order to escape Earth's gravitational field is known as the **escape speed** of the body from Earth.

Show that the escape speed of the mass *m* placed at the surface of the Earth is $\sqrt{2gR_E}$, given that the radius of the Earth is R_E and gravitational field strength at the Earth's surface is given by *g*.

Solution:

The mass is taken to have escaped from the Earth's gravitational field when it is at an infinite distance away from Earth where the gravitational pull is zero. The escape speed is the speed that affords the mass just enough KE to reach "infinity".

Since the total energy of the Earth – mass system is conserved ($(KE + GPE)_{surface} = (KE + GPE)_{infinity}$)

Loss in KE of system = Gain in GPE of system

$KE_{surface} - KE_{infinity} = GPE_{infinity} - GPE_{surface}$

$$\frac{1}{2}mv_{esc}^2 - 0 = 0 - \left(-\frac{GMm}{R_E}\right)$$

$$v_{esc} = \sqrt{\frac{2GM}{R_E}} = \sqrt{\frac{2GMR_E}{R_E^2}} = \sqrt{2gR_E}$$

NOTE:

1. The escape speed of a mass is independent of the mass of the object.
2. Students are expected to show workings to derive the escape speed formula and not simply quote the formula by memory work for structured questions.

Example 10: Mechanical Energies of a Satellite in Orbit

A satellite of mass *m* orbits Earth of mass *M* in a circular path. The satellite has speed *v* and the radius of its orbit about Earth is *R.*

(a) Derive an expression for the KE and the total energy of the satellite in terms of G, *M, m* and *R*.

(b) Plot a graph showing how each of the following physical quantities associated with the satellite varies with distance from the centre of Earth:

(i) kinetic energy;

(ii) gravitational potential energy;

(iii) total energy.

Solution:

(a) The **gravitational force** on the satellite **provides** for the required **centripetal force.**

(NOTE: the statement above is required to explain the following steps below. Please write this in exams!)

$$\frac{GMm}{R^2} = \frac{mv^2}{R}$$

$$\frac{1}{2} \times \frac{GMm}{R^2} = \frac{mv^2}{R} \times \frac{1}{2}$$

$$\text{KE} = \frac{GMm}{2R}$$

The total energy of the satellite

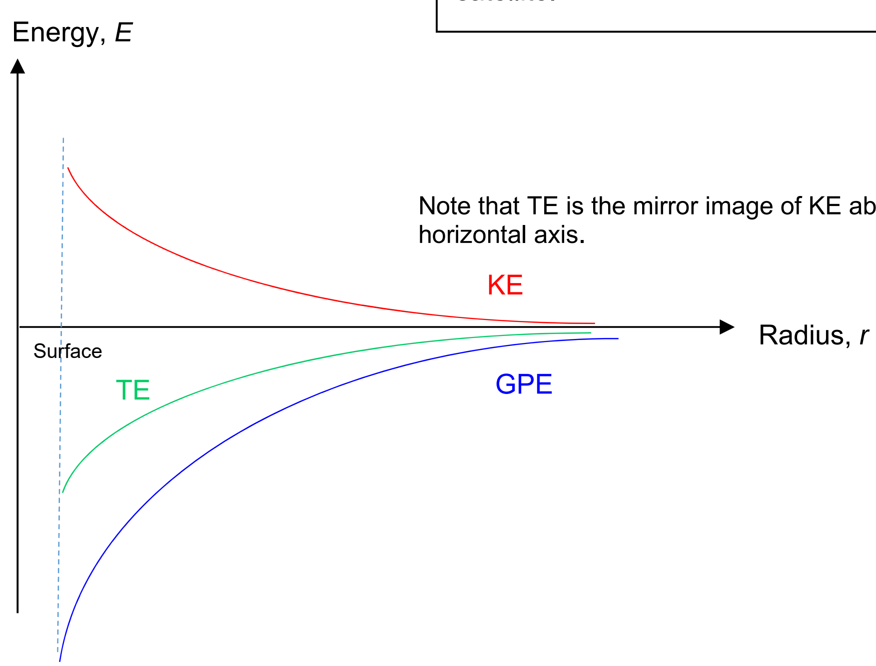
$= \text{KE} + \text{GPE}$

$$= \frac{GMm}{2R} + \left(-\frac{GMm}{R}\right)$$

$$= -\frac{GMm}{2R}$$

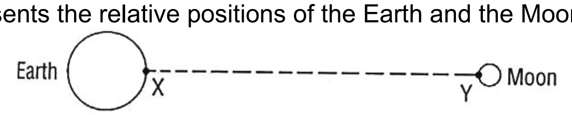
NOTE:
The phrase "total energy of satellite" actually refers to total energy of the Earth-Satellite system. As Earth is much more massive, it is assumed to be at rest. Hence the total energy of the system is just the kinetic energy of the satellite and gravitational potential energy of the Earth-Satellite system. This is commonly referred to as the total energy of the satellite.

(b)



Example 11: Gravitational Potential between 2 Masses (Mod TYS Pg 91 Qn 38 N93/I/5; N2000/I/7)

The diagram (not to scale) represents the relative positions of the Earth and the Moon.



The line XY joins the surface of the Earth to the surface of the Moon.

(a) Sketch on the same axes graphs representing

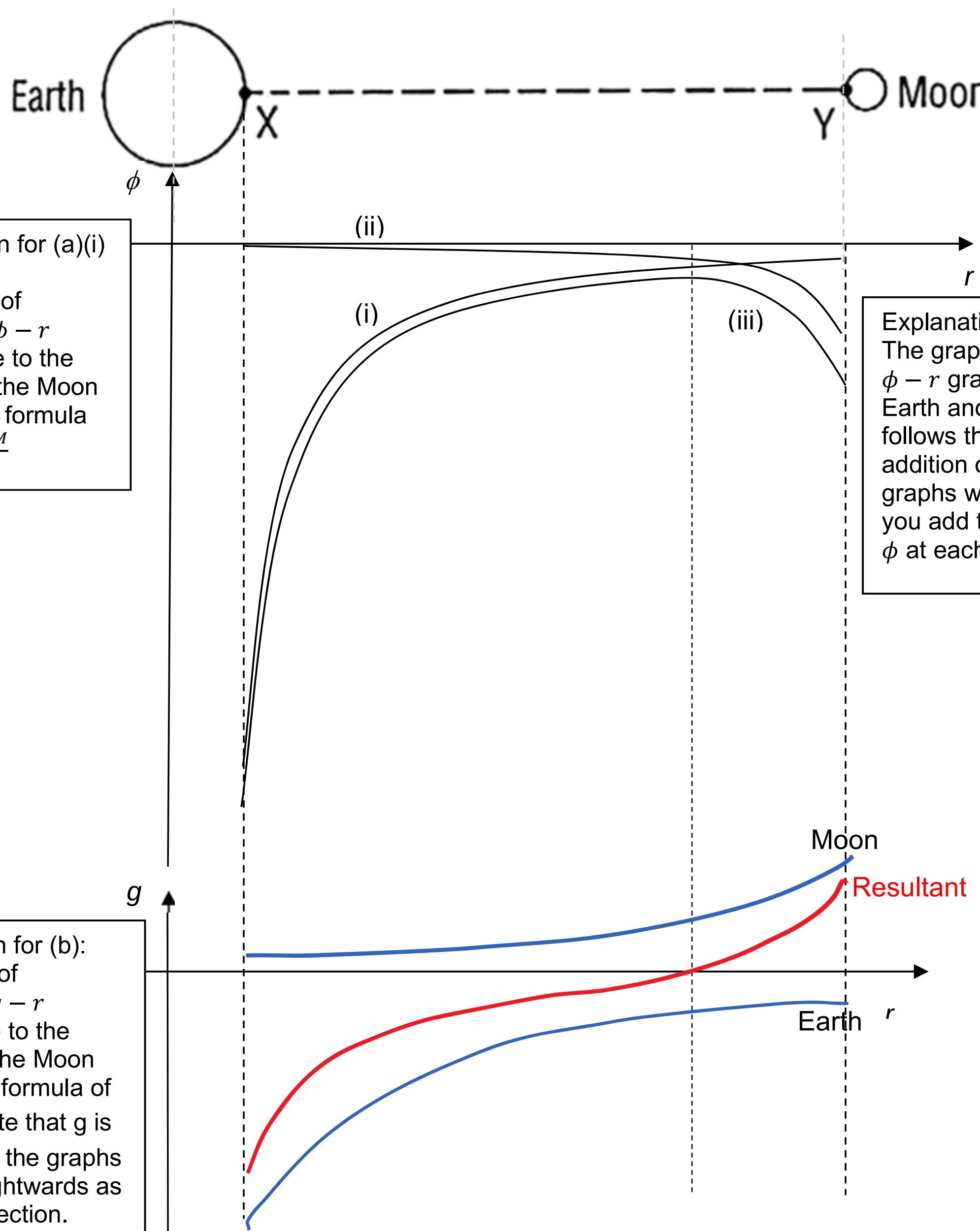
(i) the gravitational potential due to the Earth between X and Y;

(ii) the gravitational potential due to the Moon between X and Y;

(iii) the resultant gravitational potential between X and Y.

(b) Sketch the corresponding variation of gravitational field strength with distance on the axis below.

Solution:



Explanation for (a)(i) and (ii): The graph of individual $\phi - r$ graphs due to the Earth and the Moon follows the formula of $\phi = -\frac{GM}{r}$

Explanation for (a)(iii): The graph of resultant $\phi - r$ graph due to the Earth and the moon follows the scalar addition of the two graphs which means you add the values of ϕ at each r value.

Explanation for (b): The graph of individual $g - r$ graphs due to the Earth and the Moon follows the formula of $g = \frac{GM}{r}$. Note that g is a vector so the graphs is taking rightwards as positive direction.