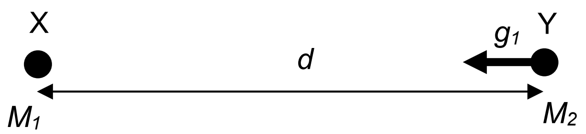
Dunman High School
(Senior High Physics)

Gravitational Field Tutorial Solution

LO	Sub-topic	Tut Qn
a	Show an understanding of the concept of a gravitational field as an example of field of force and define gravitational field strength at a point as the gravitational force exerted per unit mass placed at that point.	
b	Recognise the analogy between certain qualitative and quantitative aspects of gravitational and electric fields.	
c	Recall and use Newton's law of gravitation in the form $F = Gm_1m_2 / r^2$.	S1,S2
d	Derive, from Newton's law of gravitation and the definition of gravitational field strength, the equation $g = GM / r^2$ for the gravitational field strength of a point mass.	
e	Recall and apply the equation $g = GM / r^2$ for the gravitational field strength of a point mass to new situations or to solve related problems.	S3,S4,S7, S8,D1,D2
f	Show an understanding that near the surface of the Earth g is approximately constant and equal to the acceleration of free fall.	
g	Define potential at a point as the work done per unit mass in bringing a small test mass from infinity to the point.	
h	Solve problems using the equation $\phi = -GM / r$ for the gravitational potential in the field of a point mass.	S5,S6,S9, D3,D4,D5, D6,D10, D11, D12, D13, D14
i	Analyse circular orbits in inverse square law fields by relating the gravitational force to the centripetal acceleration it causes.	S10,D7,D9
j	Show an understanding of geostationary orbits and their application.	D8

No	Solution	Mark
S1	$F = \frac{G(3)(2)}{r^2}$ $F_1 = \frac{G(3)(2)}{(2r)^2} = \frac{1}{4}\left(\frac{G(3)(2)}{r^2}\right) = \frac{F}{4}$	1 1
S2	**Ans: C** On the Earth's surface, $F = \frac{GM_E m_s}{(5760 \times 10^3)^2}$ - - - - (1) At a height of 144 km, $F_1 = \frac{GM_E m_s}{[(5760+144)\times 10^3]^2} = \frac{GM_E m_s}{(5904 \times 10^3)^2}$ - - - - (2) $\frac{(2)}{(1)}$: $\frac{F_1}{F} = \frac{5760^2}{5904^2}$ = 0.952 ≈ 95% → i.e. less by 5%	
S3	**Ans: C** On the surface of the planet at distance R from its centre, gravitational field strength due to the planet $= \frac{GM}{R^2}$	
S4	**Ans: A** $g = \frac{GM}{r^2}$, is a vector. The gravitational field strength near Earth is directed towards Earth, which is to the left, represented by a negative value. The gravitational field strength near Moon is directed towards Moon, which is to the right, represented by a positive value.	
S5	**Ans: D** If gravitational potential energy is zero at infinity, it is always negative anywhere else, i.e. $U = -\frac{GMm}{r}$. Q is a point further away from the Earth's centre, hence U at point Q is greater than that at point P. i.e. U_Q is less negative than U_P.	
S6	**Ans: A** From $\phi = -\frac{GM}{r}$: at ϕ = –60 MJ kg^{-1}, r is smaller, i.e. closer to the Earth. $\Delta U = m(\phi_f - \phi_i)$ = (50)[(–60) – (–20)] = –2000 MJ (negative indicates a loss)	
S7(a) (i)		1

(ii)	$g_1 = \dfrac{GM_1}{d^2}$	1
(iii)	$F = (g_1)(M_2) = \dfrac{GM_1M_2}{d^2}$	1
(b)	force on M_3 by M_1 = force on M_3 by M_2 $\dfrac{GM_1M_3}{x^2} = \dfrac{GM_2M_3}{(d-x)^2} \rightarrow \dfrac{(d-x)^2}{x^2} = \dfrac{M_2}{M_1} \rightarrow \dfrac{d-x}{x} = \sqrt{\dfrac{M_2}{M_1}}$ Cross-multiply: $x\sqrt{M_2} = d\sqrt{M_1} - x\sqrt{M_1} \rightarrow x(\sqrt{M_1}+\sqrt{M_2}) = d\sqrt{M_1}$ $\rightarrow x = d\left(\dfrac{\sqrt{M_1}}{\sqrt{M_1}+\sqrt{M_2}}\right)$	1 1
	Note: • Since $d > x$, negative $\sqrt{\dfrac{M_2}{M_1}}$ is rejected.	
S8	Gravitational field strength at X due to the Earth: $g_E = -\dfrac{GM_E}{(2.0\times10^8)^2}$ (directed towards Earth, taken as negative) Gravitational field strength at X due to the Moon: $g_M = \dfrac{GM_M}{(1.8\times10^8)^2}$ (directed towards Moon, taken as positive) Resultant $g = g_M + g_E = 6.67 \times 10^{-11}\left[\dfrac{(7.4\times10^{22})}{(1.8\times10^8)^2} - \dfrac{(6.0\times10^{24})}{(2.0\times10^8)^2}\right]$ = **–9.85 × 10^{-3} N kg^{-1} (towards Earth)**	1 1
S9	**Ans: C** $F = -\dfrac{dE_p}{dr}$	
S10(a) (i)	$v = r\omega = r\left(\dfrac{2\pi}{T}\right) = (1.5 \times 10^{11})\left(\dfrac{2\pi}{365\times24\times60\times60}\right)$ = **2.99 × 10^4 m s^{-1}**	1 1
(ii)	$a = \dfrac{v^2}{r} = \dfrac{(2.99\times10^4)^2}{1.5\times10^{11}}$ = **5.96 × 10^{-3} m s^{-2}**	1 1
(iii)	$F = ma = (6.0 \times 10^{24})(5.96 \times 10^{-3})$ = **3.58 × 10^{22} N**	1 1
(iv)	$g = \dfrac{F}{m}$ = **5.96 × 10^{-3} N kg^{-1}**	1

(b)	Using $T^2 = \left(\frac{4\pi^2}{GM_s}\right)r^3$ → $T^2 = kr^3$ (Kepler's 3rd law, M_s = mass of Sun) For Earth, $1^2 = k(1.5 \times 10^{11})^3$ - - - - (1) For Mars, $T^2 = k(2.3 \times 10^{11})^3$ - - - - (2) $\frac{(2)}{(1)}$: $\frac{T^2}{1^2} = \frac{2.3^3}{1.5^3}$ → T = **1.9 years**	1 1 1
D1	**Ans: C** $M = \rho V = \rho\left(\frac{4}{3}\pi r^3\right)$ $g = \frac{GM}{r^2} = \frac{G\rho\left(\frac{4}{3}\pi r^3\right)}{r^2} = \frac{4\pi r\rho G}{3}$	
D2	**Ans: A** Using $g = \frac{GM}{r^2}$, when r = 1, g = 64 → $64 = \frac{GM}{(1)^2}$ → GM = 64. So, when r = 2, $g_2 = \frac{GM}{(2)^2} = \frac{64}{(2)^2} = 16$ When r = 4, $g_4 = \frac{64}{(4)^2} = 4$	
	Note: • Using Desmos and the three reference points in (**A**), if a graph of g against $\frac{1}{r^2}$ is plotted and extrapolated, a straight line graph passing the origin will be obtained.	
D3	**Ans: C** A: True. A mass will move from position Y of less negative gravitational potential to X of more negative potential when released, hence the work done by an external force is negative. B: True. Points that lie on the same equipotential line have the same potential. C: False. A mass will move from a position of less negative gravitational potential to one of more negative potential when released, hence the work done by the gravitational field to move a mass further away from Earth is negative. D: True. $\phi \propto -\frac{1}{r}$	
D4	From the graph, $g = -\frac{d\phi}{dr}$, i.e. resultant g is zero at ϕ = –1.3 MJ kg^{-1} Minimum energy required (to move the object to zero resultant g from Earth's surface) = $m(\phi_f - \phi_i)$ = (2.0)[(–1.3) – (–62.3)] = **122 MJ**	 1 1

	(Upon reaching the point of zero resultant *g*, if the object has a velocity towards the Moon, it will move towards the Moon pass the point of zero resultant *g*. Subsequently, the Moon's gravitational pull would predominate and a resultant force accelerates it to the Moon's surface.)	
D5	By conservation of energy, Loss in U = Gain in KE → $U_i - U_f = KE_f - KE_i$ $\left(-\frac{GMm}{3R}\right)-\left(-\frac{GMm}{R}\right) = \frac{1}{2}mv^2 - 0$ $\frac{2GMm}{3R} = \frac{1}{2}mv^2$ → $v = \left(\frac{4GM}{3R}\right)^{\frac{1}{2}}$	1 1
D6(a)	The gravitational potential at a point is defined as the work done per unit mass in bringing a small test mass from infinity to that point.	1
(b)	Potential at infinity is taken to be zero. Due to the **attractive** nature of the gravitational force, work done by an external force to bring any mass from infinity to that point is always negative, thus a decrease in potential (from zero). Hence the potential at any point must always be negative.	1 1
D6(c) (i)	$\Delta\phi = \phi_f - \phi_i$ $= GM\left[\left(-\frac{1}{r_f}\right)-\left(-\frac{1}{r_i}\right)\right]$ $= (6.67 \times 10^{-11})(6.0 \times 10^{24})\left[-\frac{1}{(6.4\times10^6+1.3\times10^7)}-\left(-\frac{1}{6.4\times10^6}\right)\right]$ = $\mathbf{4.19 \times 10^7\ J\ kg^{-1}}$	1 1
(ii)	By conservation of energy, KE loss = U gain $\frac{1}{2}mv^2 - 0 = m\Delta\phi$ $v = \sqrt{2\Delta\phi} = \sqrt{2(4.19\times10^7)}$ = $\mathbf{9.15 \times 10^3\ m\ s^{-1}}$	1 1
(iii)	The escape velocity $v = \sqrt{\frac{2GM}{R_{Earth}}} = \sqrt{\frac{2(6.67\times10^{-11})(6.0\times10^{24})}{6.4\times10^6}}$ = $\mathbf{1.12 \times 10^4\ m\ s^{-1}}$ **OR** $v = \sqrt{2gR_{Earth}} = \sqrt{2(9.81)(6.4\times10^6)} = 1.12 \times 10^4\ m\ s^{-1}$	1 1
(d)	The equation is not appropriate because the acceleration is **not uniform.**	1
D7(a)	Gravitational force provides the centripetal force for the satellite in orbit:	1

	$\frac{GMm}{r^2} = \frac{mv^2}{r}$ $\rightarrow \frac{GM}{r} = v^2 \rightarrow v = \sqrt{\frac{GM}{r}}$	1
(b)	$v = \sqrt{\frac{GM}{r}}$ $= \sqrt{\frac{(6.67\times10^{-11})(6.0\times10^{24})}{(6890\times10^3)}}$ = 7620 m s^{-1}	1 1
(c)(i)1.	$\Delta KE = \frac{1}{2}m(v^2 - u^2)$ $= \frac{1}{2}(120)(7620^2 - 7780^2)$ = **–1.48 × 10^8 J**	1 1
2.	$\Delta U = GMm\left[\left(-\frac{1}{r_f}\right)-\left(-\frac{1}{r_i}\right)\right]$ $= (6.67\times10^{-11})(6.0\times10^{24})(120)\left[-\frac{1}{6890\times10^3}-\left(-\frac{1}{6610\times10^3}\right)\right]$ = **2.95 × 10^8 J**	1 1
3.	$\Delta E = \Delta U + \Delta KE$ = 2.95 × 10^8 + (–1.48 × 10^8) = **1.47 × 10^8 J**	1
D7(c)(ii)	Total energy **increases** (since ΔE is positive)	1
D8(a)(i)	A point on the Earth surface will take 1 (Earth) day to travel an angular displacement of 2π. $\omega = \frac{2\pi}{T} = \frac{2\pi}{(24)(60)(60)}$ = 7.27 × 10^{-5} rad s^{-1}	1
(ii)	$a = r\omega^2$ = (6.38 × 10^6)(7.27 × 10^{-5})^2 = **3.38 × 10^{-2} m s^{-2}**	1 1
(b)(i)	A point on the **equator undergoes circular motion** about the Earth's axis, whereas a point on the **pole does not**. At the equator, **part of the gravitational acceleration has to provide for the centripetal acceleration**, hence the measured acceleration of free fall is lower.	1 1
(ii)	The Earth's angular velocity is low and **the measur** **fall at the equator is *just slightly lesser* (0.0338 m** **the measured acceleration of free fall at the pol** is small.	

Acceleration required to maintain the circular path of the mass at the equator:
$a = R\omega^2 = 6.38\times10^6 \times [2\pi/(24\times3600)]^2$
$= 0.034$ m s^{-2}

Equation of motion at the equator:
$mg - mg_{measured} = ma$
$\Rightarrow g - g_{measured} = a$
$\Rightarrow g_{measured} = g - a$

At the poles,
$mg - mg_{measured} = 0$
$\Rightarrow g_{measured} = g$

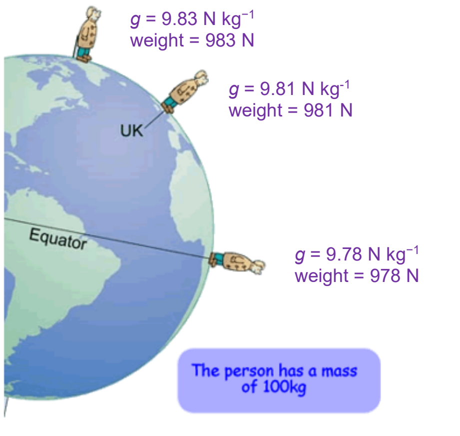
Extra Reading Notes 01:

the *gravitation field strength* near the Earth's surface (g_E) = 9.81 N kg^{-1}

and the *acceleration due to free-fall* near the Earth's surface (g_E) = 9.81 m s^{-2}

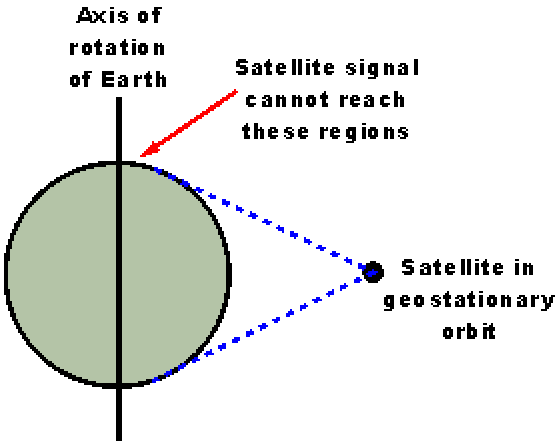
However, there might be small variation to the above values due to the following reasons.

1. The Earth is not a perfect sphere. The Earth's radius at the poles is smaller than that at the equator. The value of g_E is greater at the poles as it is nearer to the dense core of the Earth.
2. The Earth's surface is not uniform. Higher altitudes will give rise to slightly smaller values of g as r increases.
3. The density of the Earth varies from region to region beneath the Earth's surface.
4. The Earth is spinning about its own axis. This will reduce the value of the *apparent weight* especially at the equator where the apparent weight of an object there is the lowest.



The apparent weight varies with latitude

(c)(i)	**Geostationary orbits** are orbits of satellites orbiting around the Earth such that these satellites **would appear stationary when observed from a fixed location from the Earth.** The period of the satellites' orbits must be the same as that of the rotation of Earth, i.e. **24 hours**. These satellites would also have to orbit about the Earth's axis of rotation and must move from West to East as **the Earth rotates from West to East.** The gravitational force by the Earth provides the centripetal force for the satellite and is directed towards the centre of the Earth. Since the centripetal force is directly towards the centre of the orbit, the **centre of the orbit must be the centre of the Earth** and the **axis of the satellite's orbit is the same as that of the Earth's rotation**. From the reasons above, a geostationary satellite must be placed vertically above the equator. If its orbit is not over the equator it will sometimes be over the northern hemisphere and sometimes the southern hemisphere and so cannot be geostationary.	1 1 1
(ii)	Consider a mass m located on the surface of the Earth. From **Newton's law of gravitation**, the attractive gravitational force acting on mass m due to the gravitational field of the Earth can be expressed as $F = \frac{GMm}{R^2}$ - - - - - (1) Based on definition of gravitational field strength, ***g* is the gravitational force per unit mass acting at that point,**	

	$g = \frac{F}{m} \rightarrow \quad F = mg \qquad$ - - - - - (2) From equations (1) and (2), $\quad mg = \frac{GMm}{R^2} \rightarrow g = \frac{GM}{R^2}$	1 1
D8(c) (iii)	Mass of the Earth is not known. Radius of the Earth, $R = 6.38 \times 10^6$ m Consider the geostationary orbit of radius r, $\frac{GMm}{r^2} = mr\omega^2 \rightarrow r^3 = \frac{GM}{\omega^2} = \frac{gR^2}{\omega^2}$ (using the relation $GM = gR^2$ from (ii)) $\rightarrow r = \left(\frac{gR^2}{\omega^2}\right)^{1/3} = \left(\frac{(9.81)(6.38 \times 10^6)^2}{(7.27 \times 10^{-5})^2}\right)^{1/3}$ $= 4.23 \times 10^7$ m	1 1 1
(iv)	$v = r\omega = (4.23 \times 10^7)(7.27 \times 10^{-5})$ $= \mathbf{3.08 \times 10^3}$ **m s^{-1}**	1 1
(v)	$a = r\omega^2 = (4.23 \times 10^7)(7.27 \times 10^{-5})^2$ $=$ **0.226 m s^{-2}**	1 1
(d)	Polar advantage 1: **Satellites in low orbits are closer to the Earth hence their signals are received more strongly**. (*They also have better resolutions if used as photographic satellites: 50 m per pixel versus* 2.5 *km per pixel for geostationary satellites*). Polar advantage 2: **Satellites can fly over and serve every part of the world equally** (*instead of being limited by the Earth's curvature to between latitudes of about 70 N and 70* S, *with progressively weaker signal strength for higher latitudes*).  Geostationary advantage 1: **Continuous transmission between the ground station and the satellite**, since they are always at the same spot above the equator. (The ground station antenna need not be reorientated.) Geostationary advantage 2: **Able to video and track changes in a particular part of the Earth continuously** since it is fixed above a location on Earth, instead of piecing up bits and pieces as in the case of polar satellites which are above different places at different times.	1 1

Extra Reading Notes 02:

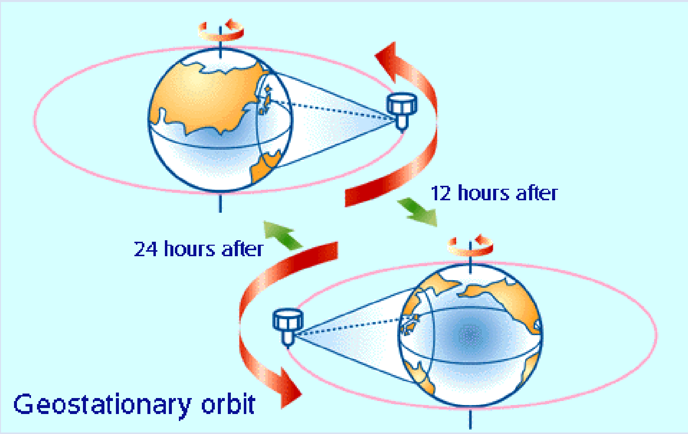
Advantages and disadvantages of geostationary satellites:

Advantages:

1. A geostationary satellite remains 'stationary' above the same spot on the Earth's surface at any time and so it is ideal for telecommunication purposes. A permanent line of 'vision' between the transmitter and the receiver via the satellite allows for uninterrupted telecommunication.
2. No need to constantly track the satellite to receive its downlink signal as it is always at the same relative position above the Earth's surface. Thus, there is no need to keep adjusting the direction of the satellite dish to receive telecommunication signal from a particular geostationary satellite.
3. As geostationary satellites are positioned at a high altitude (a distance of 3.6×10^7 m away from the surface of the Earth), it can view the whole Earth below them, rather than a small subsection and they can scan the same area frequently. Hence, they are ideal for meteorological applications and remote imaging.

Disadvantages:

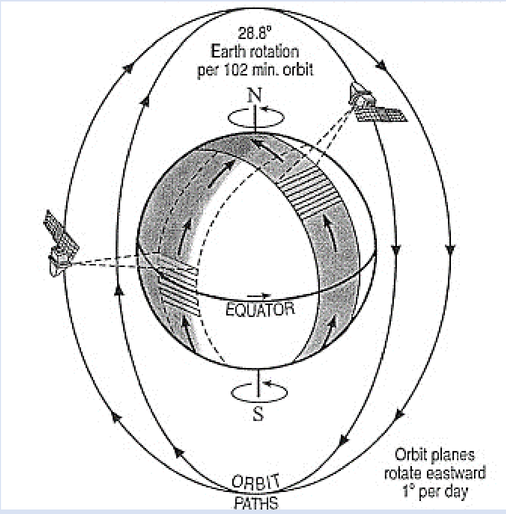
1. As geostationary satellites are positioned at such a high altitude, the spatial resolution of the images of the Earth's surface captured by them may not be as good as those captured by the low orbiting satellites.
2. Because of its high altitude also, there may be a delay in the reception of the telecommunication signals resulting in a lag time especially when conducting international video conferencing.



3. There is limitation to the geographical coverage, since ground stations at latitudes higher than 60 degrees would have difficulty reliably receiving signals at low elevations. Satellite dishes at such high latitudes would need to be pointed almost directly towards the horizon. The signals would have to pass through the largest amount of atmosphere, and could even be blocked by land topography, vegetation or buildings.

   The signals received by countries beyond the 60 degrees latitude 'belt', both on north and south sides would be rather weak.

Advantages and disadvantages of low altitude orbit satellites (e.g. polar orbit):



Satellites in polar orbits circle the Earth over the poles in a constant plane while the Earth rotates beneath them. Satellites in this type of orbit can view only a strip of Earth's surface on each orbit. Strips of images must be "stitched together," to produce a larger view. Polar satellites has a much lower altitude (about 850 km) providing more detailed information about the weather and cloud formation.

Advantages:

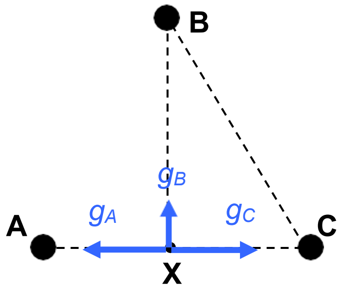
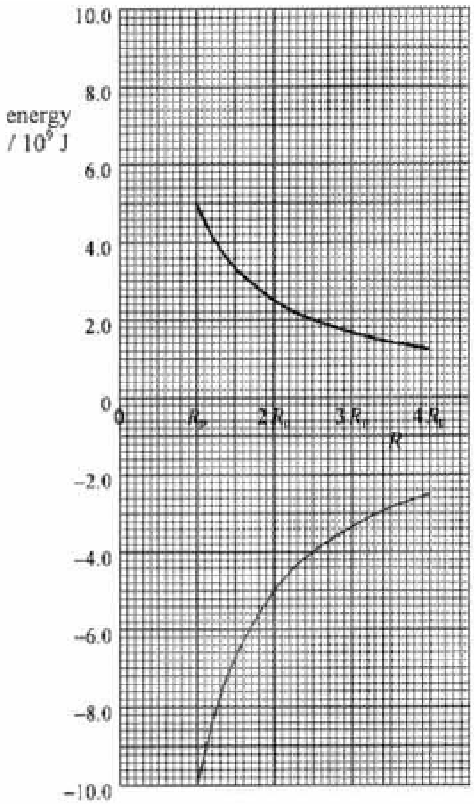
1. Due to their lower altitudes, these satellites can capture images of the Earth's surface with greater spatial resolution. Polar satellites have the advantage of photographing close up images of Earth. Geostationary satellite images of the polar regions are distorted because of the low angle that the satellite 'sees' the region with.

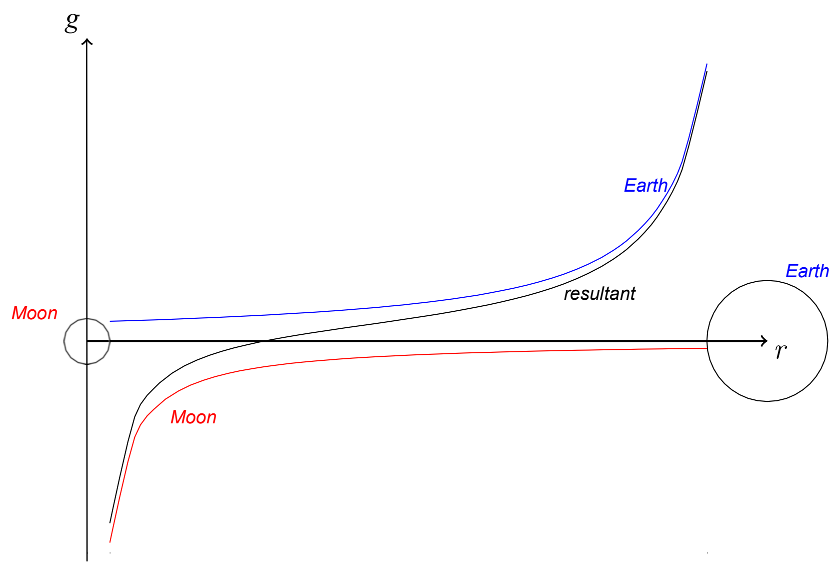
2. There is reduced lag time or delay between the transmission and reception of the signal.

Disadvantages:

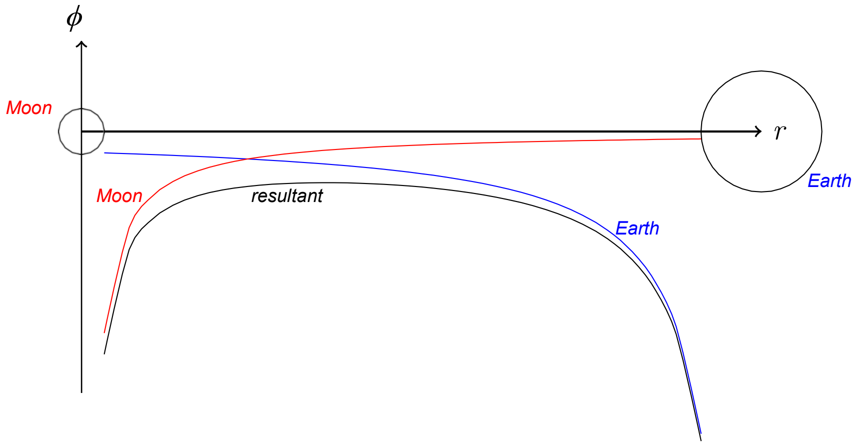
1. No one spot on the Earth's surface can be viewed continuously by any one satellite in a polar orbit. A typical low orbit satellite would take about 2 hr to make one revolution round the Earth. In order to have a continuous relay of data, there must be a series or chain of satellites in the same orbit so that one 'takes over' the predecessor's function as they move along in the orbit.

2. Because the satellite changes its location constantly with respect to the Earth's surface, the direction of the satellite dish would need to be adjusted constantly as well.

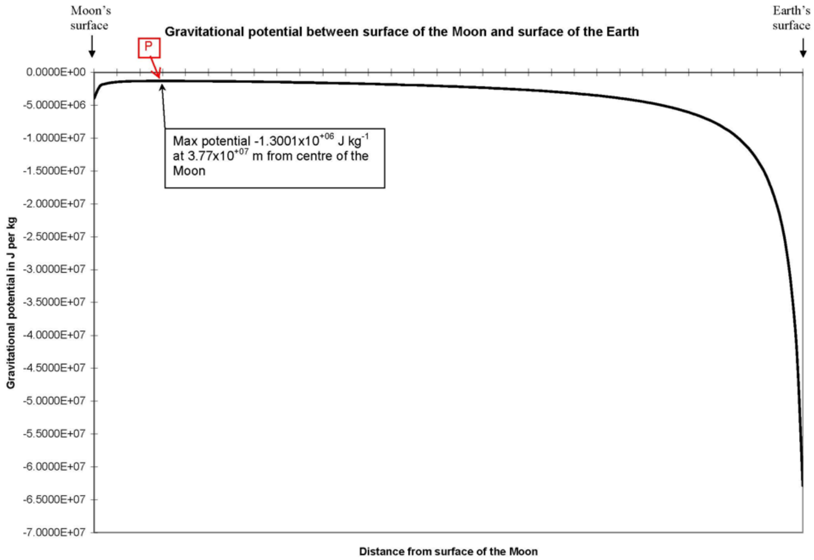
D9(a)	AX = CX = 1000 / 2 = 500 m BX = $\sqrt{1000^2 - 500^2} = 866\,\text{m}$ Since g_A will cancel out g_C, net g at X = $\dfrac{GM_B}{\text{BX}^2} = \dfrac{(6.67\times10^{-11})(2.0\times10^5)}{(866)^2} \approx 1.78\times10^{-11}\ \text{N kg}^{-1}$ 	
D9(b)	Net gravitational potential at X, ϕ_X $= \phi_A + \phi_B + \phi_C$ $= -\dfrac{GM}{\text{XA}} + -\dfrac{GM}{\text{XB}} + -\dfrac{GM}{\text{XC}}$ $= -(6.67\times10^{-11})(2.0\times10^5)\left(\dfrac{1}{500}+\dfrac{1}{866}+\dfrac{1}{500}\right)$ $\approx -6.9\times10^{-8}\ \text{J kg}^{-1}$	
D10(a) (i)	For a mass m in orbit, centripetal force is provided by gravitational force i.e. $\dfrac{mv^2}{R} = \dfrac{GMm}{R^2} \Rightarrow \text{KE} = \tfrac{1}{2}mv^2 = \dfrac{GMm}{2R}$	1 1
(ii)	For a satellite of mass m in orbit, $\dfrac{\text{GPE of satellite}}{\text{KE of satellite}} = \dfrac{-GMm/R}{GMm/2R} = -2$	1
(b)(i)	Sketch KE of satellite : (GPE reflected about R-axis, ½ magnitude): 	2

(ii)	From given graph, at $R = 4R_p$, KE = $\frac{1}{2}mv^2 = 1.25 \times 10^9$ J i.e. $\frac{1}{2}(1600)v_4^2 = 1.25 \times 10^9$ ⇒ speed $v_4 = 1250$ m s^{-1} at $R = 2R_p$, $\frac{1}{2}mv_2^2 = 2.5 \times 10^9$ J i.e. $\frac{1}{2}(1600)v_2^2 = 2.5 \times 10^9$ ⇒ speed $v_2 = 1767$ m s^{-1} Hence change in orbital speed, = $v_2 - v_4$ =1767 – 1250 = 517 m s^{-1}	1 1 1 1 1
D11(a)	**B** TE = GPE + KE if KE = 0, TE = GPE ⇒ For C, D & E, TE > GPE ⇒ GPE + KE > GPE ⇒ KE > 0 ⇒ For A, TE < GPE ⇒ GPE + KE < GPE ⇒ KE < 0 (Physically impossible)	
D11(b)	**C** 'falling' ⇒ KE > 0 Did not reach infinity ⇒ TE < 0	
D11(c)	**D, E** 'moving' ⇒ KE > 0 Can reach infinity ⇒ TE ⩾ 0	
D12	Sketch:  	

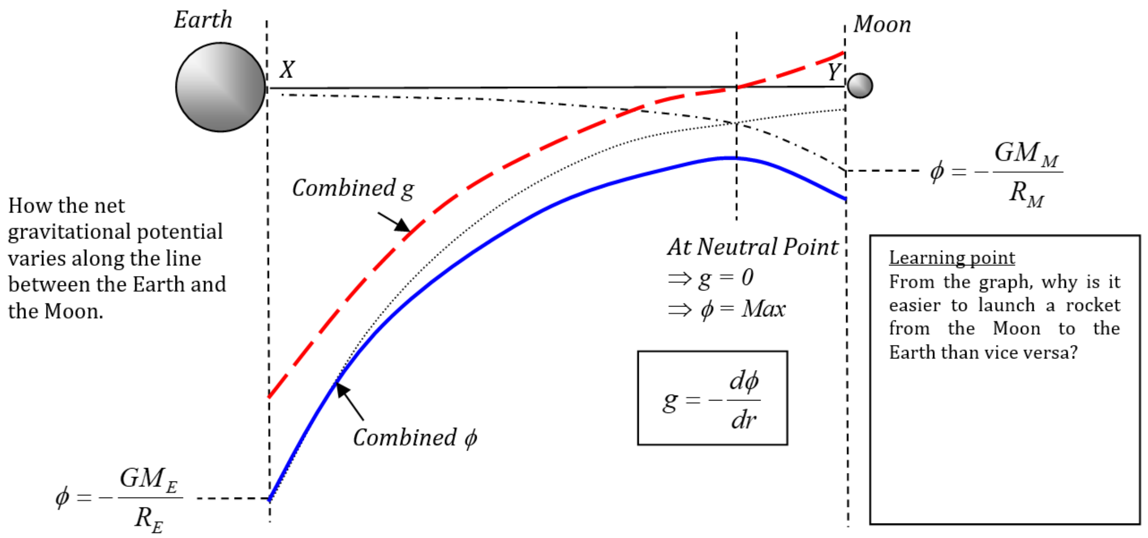
D13 Sketch:



Actual variation of gravitational potential between the Moon and the Earth



Extra Reading Notes 03:



D14(a)(i)	$g = -\dfrac{d\phi}{dr}$	1
(ii)	From the equation in (i), the **magnitude of the gravitational field strength is numerically equal to the gradient of the potential against distance graph**. The **acceleration of free fall at that point is numerically equal to the gravitational field strength**.	1 1
(iii)1.	From the figure, for zero acceleration, **gradient of the graph is equal to zero** i.e. at d = **13.6 × 10^6 m**	1 1
2.	A tangent is drawn on the point at the surface of Charon. Points (14.8, –29.52) and (18, –29.7) lie on the tangent. Acceleration of free fall on surface of Charon, a = potential gradient at surface of Charon $= \dfrac{[-29.7-(-29.52)]\times 10^6}{(18-14.8)\times 10^6}$ = **0.0562 m s^–2**	 1 1 1
(b)(i)	From (a)(iii)1., the point X where $g = 0$ is 13.6 × 10^6 m from Pluto surface. For the rock projected from Charon to reach Pluto, it must be projected with a minimum KE such that it would reach point X before its KE becomes zero. Once moving beyond point X, the rock will be accelerated towards Pluto because the resultant field strength (vector sum of field strengths due to Pluto and Charon) is directed towards Pluto. At X, ϕ_i = –29.56 MJ kg^–1, On the surface of Pluto, ϕ_f = –30.0 MJ kg^–1 By conservation of energy, GPE lost = KE gained upon reaching Pluto's surface $m(\phi_i - \phi_f) = \dfrac{1}{2}mv^2 - 0 \rightarrow v = \sqrt{2(\phi_i - \phi_f)}$ $= \sqrt{2[(-29.56)-(-30.0)]\times 10^6}$ = **938 m s^–1**	 1 1 1 1
(ii)	The minimum speed is different because the **loss in potential** from X, the point with zero acceleration of free fall, to the surface of Charon is **different.**	1

~ THE END ~