Complex Numbers Tutorial 9B:
Polar and Exponential Forms
Solutions

Additional Practice Questions

1	$\left\|z^2\right\| = 2 \Rightarrow \|z\| = \sqrt{2}$ $\arg(-iz) = \frac{\pi}{4} \Rightarrow \arg(-i) + \arg(z) = \frac{\pi}{4}$ $\arg(z) = \frac{\pi}{4} - \left(-\frac{\pi}{2}\right)$ $= \frac{3\pi}{4}$ $\|wz\| = 2\sqrt{2}$ $\|w\|\|z\| = 2\sqrt{2}$ $\therefore \|w\| = 2$ $\arg\left(\frac{z^2}{w}\right) = -\frac{5}{6}\pi$ $2\arg(z) - \arg(w) = -\frac{5}{6}\pi$ $\arg(w) = 2\left(\frac{3}{4}\pi\right) + \frac{5}{6}\pi$ $= \frac{7}{3}\pi$ $= \frac{\pi}{3} \ (pv)$ $w = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$ $= 1 + \sqrt{3}i$

2 (i)	$w = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^3$ Let $w_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$ $w_2 = \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}$ $\therefore w = 2(w_1)(w_2)^3$ $\lvert w\rvert = 2\lvert w_1\rvert\lvert w_2\rvert^3 = 2(1)(1) = 2$ $\arg(w) = \arg\left(2(w_1)(w_2)^3\right)$ $= \arg(2) + \arg(w_1) + 3\arg(w_2)$ $= 0 + \frac{3\pi}{4} + 3\left(-\frac{\pi}{6}\right)$ $= \frac{\pi}{4}$ $\therefore w = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$
(ii)	$\lvert w^n\rvert = \lvert w\rvert^n = 2^n$ $\arg w^n = n\arg(w) = \frac{n\pi}{4}$ $\therefore w^n = 2^n(\cos\frac{n\pi}{4} + \mathrm{i}\sin\frac{n\pi}{4})$ When n is a muliple of 4, let $n = 4k$, where k is an integer. $w^n = 2^{4k}(\cos k\pi + \mathrm{i}\sin k\pi)$ $= (-1)^k 2^{4k}$ (because $\cos k\pi$ =1 when k even, –1 when k odd and $\sin k\pi$ =0 for all k.

3 $a = 1 + \mathrm{i}\sqrt{3} = 2e^{\mathrm{i}\left(\frac{\pi}{3}\right)}$

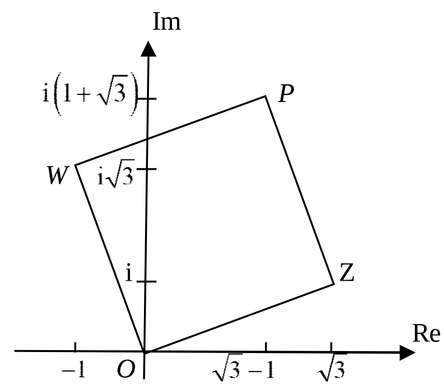
$$\therefore\ 1 + a + a^2 + a^3 + \ldots + a^9 = \frac{1-a^{10}}{1-a} = \frac{1-\left[2e^{\mathrm{i}\left(\frac{\pi}{3}\right)}\right]^{10}}{1-(1+\mathrm{i}\sqrt{3})}$$

$$= \frac{1-2^{10}e^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}}{-\mathrm{i}\sqrt{3}}$$

$$= \frac{1-2^{10}(-\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2})}{-\mathrm{i}\sqrt{3}}$$

$$= \frac{513+512\sqrt{3}\,\mathrm{i}}{-\mathrm{i}\sqrt{3}} \times \frac{\mathrm{i}\sqrt{3}}{\mathrm{i}\sqrt{3}}$$

$$= -512+171\sqrt{3}\,\mathrm{i} \quad //$$

4(a)	$z=\sqrt{3}+\mathrm{i}=2\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)$ $z^n=2^n\left(\cos\frac{n\pi}{6}+\mathrm{i}\sin\frac{n\pi}{6}\right)$ $z^n-\left(z^*\right)^n=2\mathrm{i}\,\mathrm{Im}(z^n)=2\mathrm{i}\left(2^n\sin\frac{n\pi}{6}\right)=0$ $\sin\frac{n\pi}{6}=0$ $\frac{n\pi}{6}=k\pi, k\in\mathbb{Z}^+$ The set of values of n is $\left\{n:n=6k,k\in\mathbb{Z}^+\right\}$
(b)	$z+w=\left(\sqrt{3}-1\right)+\mathrm{i}\left(1+\sqrt{3}\right)$  Note that $OWPZ$ is a parallelogram, $\arg(w)=\pi-\tan^{-1}\frac{\sqrt{3}}{1}=\frac{2\pi}{3}$ $\angle WOZ=\frac{2\pi}{3}-\frac{\pi}{6}=\frac{\pi}{2}$ $OZ=\lvert z\rvert=2$ $OW=\lvert w\rvert=\left\lvert-1+\mathrm{i}\sqrt{3}\right\rvert=2$ Since $\angle WOZ=\frac{\pi}{2}$ and $OZ=OW$, $OWPZ$ is a square. (shown) $\arg(z+w)=\arg(z)+\angle POZ=\frac{\pi}{6}+\frac{\pi}{4}=\frac{5\pi}{12}$ Also, $\arg(z+w)=\tan^{-1}\frac{\sqrt{3}+1}{\sqrt{3}-1}$ $\therefore\frac{5\pi}{12}=\tan^{-1}\frac{\sqrt{3}+1}{\sqrt{3}-1}$ $\therefore\tan\frac{5\pi}{12}=\frac{\sqrt{3}+1}{\sqrt{3}-1}=\frac{\left(\sqrt{3}+1\right)^2}{3-1}=\frac{4+2\sqrt{3}}{2}=2+\sqrt{3}$ (deduced)

5

Given $z = \cos\theta + i\sin\theta$

$$= e^{i\theta}$$

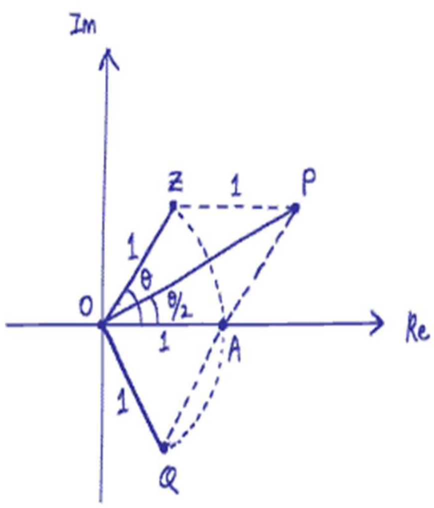
$$\frac{1+z}{1-z} = \frac{1+e^{i\theta}}{1-e^{i\theta}}$$

$$= \frac{e^{i\frac{\theta}{2}}\left[e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}\right]}{e^{i\frac{\theta}{2}}\left[e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}\right]}$$

$$= \frac{2\cos\frac{\theta}{2}}{-2i\sin\frac{\theta}{2}}$$

$$= \cot\frac{\theta}{2}\left(-\frac{1}{i}\cdot\frac{i}{i}\right)$$

$$= i\cot\frac{\theta}{2} \ \#$$



From the diagram,

OAPZ is a rhombus $\because$ OA = OZ

Also, ZA // OQ, ZA $\perp$ OP.

$\therefore \measuredangle POA = \frac{\theta}{2}$

$\measuredangle PZA = 90^\circ - \frac{\theta}{2}$

$\Rightarrow \measuredangle AOQ = 90^\circ - \frac{\theta}{2}$

$\therefore \measuredangle POQ = \frac{\theta}{2} + 90^\circ - \frac{\theta}{2}$

$= 90^\circ = \frac{\pi}{2}$ rad. (proven) #

$$\left|\frac{OP}{OQ}\right| = \frac{|1+z|}{|1-z|} = \left|\cot\frac{\theta}{2}\, i\right|$$

$$= \left|\cot\frac{\theta}{2}\right| \ \#$$

Alternate method:

$\angle POQ$

$= \arg P + (-\arg Q)$

$= \arg\left(\frac{P}{Q}\right)$

$= \arg\left(\frac{1+z}{1-z}\right)$

$= \arg\left(\left(\cot\frac{\theta}{2}\right)i\right)$

$= \frac{\pi}{2}$ #

6(i)	$z^4=\left(k+\mathrm{i}\sqrt{3}\right)^4$ $=k^4+4k^3\left(\mathrm{i}\sqrt{3}\right)+6k^2\left(\mathrm{i}\sqrt{3}\right)^2+4k\left(\mathrm{i}\sqrt{3}\right)^3+\left(\mathrm{i}\sqrt{3}\right)^4$ $=k^4-18k^2+9+\mathrm{i}\left(4\sqrt{3}k^3-12\sqrt{3}k\right)$
(ii)	z^4 real $\Rightarrow$ $4\sqrt{3}k^3-12\sqrt{3}k=0$ $4\sqrt{3}k\left(k^2-3\right)=0$ $k=0$ (rej) or $\pm\sqrt{3}$ $z=\sqrt{3}+\mathrm{i}\sqrt{3}$ or $z=-\sqrt{3}+\mathrm{i}\sqrt{3}$
(iii)	$\arg\left[\left(-\sqrt{3}+\mathrm{i}\sqrt{3}\right)^n\right]=-\frac{\pi}{4}$ $n\arg\left(-\sqrt{3}+\mathrm{i}\sqrt{3}\right)=-\frac{\pi}{4},\frac{7\pi}{4},\frac{15\pi}{4},...$ $n\left(\frac{3\pi}{4}\right)=-\frac{\pi}{4},\frac{7\pi}{4},\frac{15\pi}{4}...$ $n\left(\frac{3}{4}\right)=-\frac{1}{4},\frac{7}{4},\frac{15}{4}...$ Least $n=5$ $\left\|z^n\right\|=\left\|z\right\|^5=\left(\sqrt{6}\right)^5=36\sqrt{6}$

7(a)	Since $z=\mathrm{i}$ is a root, $\mathrm{i}^3+2\mathrm{i}+k=0 \Rightarrow k=-\mathrm{i}$ Hence the equation becomes $z^3+2z-\mathrm{i}=0$. $z^3+2z-\mathrm{i}=(z-\mathrm{i})(z^2+az+1)$ Comparing coefficient of z^2, $a=\mathrm{i}$. For $z^2+\mathrm{i}z+1=0$, $z=\frac{-\mathrm{i}\pm\sqrt{\mathrm{i}^2-4(1)(1)}}{2(1)}=\frac{-\mathrm{i}\pm\sqrt{-5}}{2}=\frac{-\mathrm{i}\pm\mathrm{i}\sqrt{5}}{2}$ Hence the other 2 roots are $\frac{-\mathrm{i}+\mathrm{i}\sqrt{5}}{2}$ and $\frac{-\mathrm{i}-\mathrm{i}\sqrt{5}}{2}$.

(b)(i)	$\frac{1}{w}=\frac{1}{r}[\cos(-\theta)+\mathrm{i}\sin(-\theta)]=\frac{1}{r}(\cos\theta-\mathrm{i}\sin\theta)$
(ii)	$\frac{500}{w}=3\lvert w\rvert+40\mathrm{i}$ $\frac{500}{r}(\cos\theta-\mathrm{i}\sin\theta)=3r+40\mathrm{i}$ Comparing real and imaginary parts, $\frac{500}{r}\cos\theta=3r,\quad -\frac{500}{r}\sin\theta=40$ $\cos\theta=\frac{3r^2}{500},\quad \sin\theta=-\frac{40r}{500}$ $\left(\frac{3r^2}{500}\right)^2+\left(\frac{40r}{500}\right)^2=1$ $9r^4+1600r^2-250000=0$ $(r^2-100)(9r^2+2500)=0$ $(r-10)(r+10)(9r^2+2500)=0$ Since $r\in\mathbb{R}^+$, $r=10$
(iii)	Subst $r=10$, $\cos\theta=\frac{3}{5},\quad \sin\theta=-\frac{4}{5}$ $\therefore\ w=10\left(\frac{3}{5}-\frac{4}{5}\mathrm{i}\right)=6-8\mathrm{i}$
	From $\frac{500}{w}=3\lvert w\rvert+40\mathrm{i}$, consider the modulus of both sides to get $\frac{500}{r}=\lvert 3r+40\mathrm{i}\rvert=\sqrt{(3r)^2+40^2}$. This leads directly to the same equation for r as above, and is solved similarly to get $r=10$. Then $w=\frac{500}{3(10)+40\mathrm{i}}=6-8\mathrm{i}$.
	From $\cos\theta=\frac{3r^2}{500},\quad \sin\theta=-\frac{40r}{500}$, eliminate r to get $\cos\theta=\frac{3}{500}\left(-\frac{500}{40}\sin\theta\right)^2=\frac{16}{15}\sin^2\theta=\frac{16}{15}\left(1-\cos^2\theta\right)$. Solve this quadratic to get $\cos\theta=\frac{3}{5}$, rejecting the other root, and proceed to get $r^2=\frac{500}{3}\cos\theta=100$. Thus $r=10$ and $\sin\theta=-\frac{40}{500}r=-\frac{4}{5}$. Finally $w=r\cos\theta+\mathrm{i}\sin\theta=6-8\mathrm{i}$.

8(i)	By conjugate root theorem, $z=1-\mathrm{i}\sqrt{3}$ is also a root. Hence $\left(z-\left(1+\mathrm{i}\sqrt{3}\right)\right)\left(z-\left(1-\mathrm{i}\sqrt{3}\right)\right)=z^2-2z+4$ is a factor of $\mathrm{f}(z)$. $(Az+B)\left(z^2-2z+4\right)=z^3-z^2+bz+4$ By comparison of coefficients, $A=1$, $B=1$. Hence $z=-1$ is also a root.
(ii)	$4w^3+bw^2-w+1=0$ $4+b\left(\frac{1}{w}\right)-\left(\frac{1}{w}\right)^2+\left(\frac{1}{w}\right)^3=0$ Hence $w=\frac{1}{z}$ $=\frac{1}{-1}, \frac{1}{1+\mathrm{i}\sqrt{3}}$, or $\frac{1}{1-\mathrm{i}\sqrt{3}}$ $=-1, \frac{1-\mathrm{i}\sqrt{3}}{4}$, or $\frac{1+\mathrm{i}\sqrt{3}}{4}$
(iii)	$z=1-\mathrm{i}\sqrt{3}=2e^{-\frac{\pi}{3}\mathrm{i}}$ $z^n\in\mathbb{R}\Rightarrow n\left(-\frac{\pi}{3}\right)=k\pi, k\in\mathbb{Z}$ Hence $n=-3k$ Thus least $n=3$ and $z^3=2^3e^{-\pi\mathrm{i}}=-8$