Check your Understanding (Applications of Differentiation)

a) Graphing and tangent

1) **(i)** Use a non-calculator method to find the coordinates of each stationary point on the graph of $y = 6x^3 - 27x^2$ and determine the nature of each stationary point. [4]

(ii) Sketch the graph of $y = 6x^3 - 27x^2$. [2]

1i) $y = 6x^3 - 27x^2$

$$\frac{dy}{dx} = 18x^2 - 54x$$

For stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$18x^2 - 54x = 0$

$18x(x-3) = 0$

$\therefore\ x = 0$ or $x = 3$

Coordinates of the turning points are

$(0,0), (3,-81)$

when $x = 0$, $\left.\frac{dy}{dx}\right|_{o^-} > 0$ and $\left.\frac{dy}{dx}\right|_{o^+} < 0$ $\quad\therefore$ max pt

when $x = 3$, $\left.\frac{dy}{dx}\right|_{3^-} < 0$ and $\left.\frac{dy}{dx}\right|_{3^+} > 0$ $\quad\therefore$ min pt

Hence $(0,0)$ is a maximum point and $(3,-81)$ is a minimum point.

Alternative method

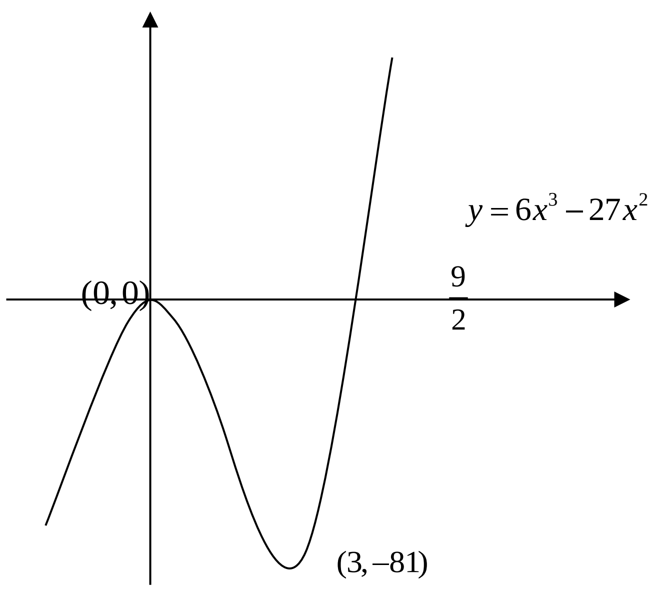
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 36x - 54$$

When $x = 0$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -54 < 0$ $\quad\therefore$ Max pt

When $x = 3$, $\frac{d^2y}{dx^2} = 54 > 0 \quad \therefore$ Min pt

Hence $(0,0)$ is a maximum point and $(3,-81)$ is a minimum point.

ii)



2) Use a non-calculator method to find the coordinates of each stationary point on the graph of $y = x(2x^2 - 9x + 12)$ and determine the nature of each stationary point. [5]

2) $y = x(2x^2 - 9x + 12)$

$$\frac{dy}{dx} = 6x^2 - 18x + 12$$

For stationary points, $\frac{dy}{dx} = 0$

$6x^2 - 18x + 12 = 0$

$6(x-1)(x-2) = 0$

$\therefore\ x=1$ or $x=2$

Coordinates of the turning points are

$(1,5),(2,4)$

when $x=1$, $\left.\frac{dy}{dx}\right|_{1^-} > 0$ and $\left.\frac{dy}{dx}\right|_{1^+} < 0 \quad \therefore$ max pt

when $x=2$, $\left.\frac{dy}{dx}\right|_{2^-} < 0$ and $\left.\frac{dy}{dx}\right|_{2^+} > 0 \quad \therefore$ min pt

Hence $(1,5)$ is a maximum point and $(2,4)$ is a minimum point.

Alternative method

$$\frac{d^2y}{dx^2}=12x-18$$

When $x = 1$, $\frac{d^2y}{dx^2}=-6 < 0 \quad \therefore$ Max pt

When $x = 2$, $\frac{d^2y}{dx^2}=6 > 0 \quad \therefore$ Min pt

Hence $(1,5)$ is a maximum point and $(2,4)$ is a minimum point.

3) The curve C has its equation defined as $y=\ln\frac{\sqrt{x}}{4x+1}$.

(i) Find the gradient of the tangent to the curve at P where $x = 1$. [3]

(ii) Find the equation of the tangent at P. [3]

(iii) If the tangent to the curve at P meets the x-axis at Q, calculate the exact coordinates of Q. [2]

3) $y=\ln\frac{\sqrt{x}}{4x+1}=\frac{1}{2}\ln x-\ln(4x+1)$

$$\frac{dy}{dx}=\frac{1}{2x}-\frac{4}{4x+1}$$

(i) when x = 1 $\frac{dy}{dx}=\frac{1}{2}-\frac{4}{5}=-\frac{3}{10}$

(ii) when x = 1, $y=-\ln 5$

$y=-\frac{3}{10}x+C \Rightarrow -\ln 5=-\frac{3}{10}+C$

$C=\frac{3}{10}-\ln 5$

$y=-\frac{3}{10}x+\frac{3}{10}-\ln 5$

(iii) y = 0 $\ln 5-\frac{3}{10}=-\frac{3}{10}x \Rightarrow x=1-\frac{10}{3}\ln 5$

$Q\left(1-\frac{10}{3}\ln 5,\quad 0\right)$

4) The equation of a curve is given by $y=e^{1+x^2}$.

(i) Find $\frac{\text{dy}}{\text{dx}}$ in terms of x. [2]

(ii) Find the equation of the tangent to the curve at the point *x* = *a*. [3]

If the tangent passes through the origin, find the value of *a*. [3]

4i) $y=e^{1+x^2}$

$\frac{\text{dy}}{\text{dx}}=2x.e^{1+x^2}$

ii) when $x=a,\ y=e^{1+a^2}$, $\frac{\text{dy}}{\text{dx}}=2ae^{1+a^2}$

Equation of tangent at $x = a$ is

$$y - e^{1+a^2} = 2ae^{1+a^2}(x-a)$$

$$y = 2axe^{1+a^2} + (1-2a^2)e^{1+a^2}$$

iii) Since tangent passes through origin

$$\Rightarrow (1-2a^2)e^{1+a^2} = 0$$

Since $e^{1+a^2} \neq 0 \ \Rightarrow 1-2a^2 = 0 \Rightarrow \ a = \pm\dfrac{1}{\sqrt{2}}$

5) The curve C has its equation defined as $y = \ln\sqrt{\dfrac{x^2+2x+1}{2x-1}}$, for $x > \dfrac{1}{2}$.

(i) Find the equation of the tangent to the curve at P, where $x = 1$. [5]

(ii) If the tangent to the curve at P meets the x-axis at Q, calculate the exact coordinates of Q. [2]

5)

At $x=1$, $\dfrac{dy}{dx} = -\dfrac{1}{2}$.

$$y - \ln 2 = -\frac{1}{2}(x-1)$$

$$y = -\frac{1}{2}(x-1-2\ln 2)$$

At $y = 0$, $x = 1 + 2\ln 2$ $\quad\therefore\ (1+2\ln 2\,,\,0)$

$$\frac{dy}{dx} = \frac{x-2}{(x+1)(2x-1)}$$

At $x = k$, $\dfrac{dy}{dx} = 0$ $\quad\Rightarrow x = 2$

$\Rightarrow k = 2$

$$\frac{d}{dx}\left(\ln\sqrt{\frac{x^2+2x+1}{2x-1}}\right)$$

$$= \frac{d}{dx}\left(\ln\sqrt{x^2+2x+1} - \ln\sqrt{2x-1}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}\ln(x^2+2x+1) - \frac{1}{2}\ln(2x-1)\right)$$

$$= \frac{2x+2}{2(x+1)^2} - \frac{(1)(2)}{2(2x-1)}$$

$$= \frac{1}{(x+1)} - \frac{1}{(2x-1)}$$

$$= \frac{x-2}{(x+1)(2x-1)}$$

6) **(i)** Find the numerical value of the derivative of x^x when $x=2$, correct to 1 decimal place. [1]

(ii) Hence find the equation of the tangent to the graph of $y=x^x$ at the point where $x=2$, giving your answer in the form $y=mx+c$. [3]

6)

i) From G.C, $\left.\frac{\mathrm{d}}{\mathrm{d}x}\left(x^x\right)\right|_{x=2}=6.8$

ii) When $x=2$, $y=4$.

Equation of tangent at $x=2$:

$$y-4=6.8(x-2)$$

$$y=6.8x-9.6$$

7) Given that $\mathrm{h}(x)=x+\ln x^2$, $x\in□$, $x\neq 0$.

(i) Find $\mathrm{h}'(x)$ and hence show that $\mathrm{h}(x)$ is an increasing function for $x>0$. [2]

(ii) Find the equation of the tangent to the curve $y=\mathrm{h}(x)$ at the point where $x=1$. [2]

(iii) The same tangent to the curve $y=\mathrm{h}(x)$ in 8(ii), is also the tangent of another curve $y=px^2+qx$ at the point where $x=-1$. Determine the constants p and q. [4]

7i)

$\mathrm{h}(x)=x+\ln x^2$

$\mathrm{h}'(x)=1+\dfrac{2}{x}$

For $x>0$, $\mathrm{h}'(x)>0$

$\Rightarrow$ h (x) is an increasing function for $x>0$.

ii) When x = 1, y = h(1) = 1,

$$h'(1) = 1 + \frac{2}{1} = 3$$

Eqn of tangent : $\begin{aligned} y - 1 &= 3(x-1) \\ y &= 3x - 2 \end{aligned}$

iii)

$$y = px^2 + qx$$

$$\frac{dy}{dx} = 2px + q$$

At $x = -1,\ y = 3(-1) - 2 = -5,$

$$\frac{dy}{dx} = -2p + q = 3$$

$\Rightarrow q = 2p + 3$ ....(1)

Sub $(-1, -5)$ into $y = px^2 + qx$:

$\Rightarrow -5 = p - q$

$\Rightarrow q = p + 5$ ....(2)

Solve (1) & (2) : $p = 2,\ q = 7$

8(i) Find, in terms of k, the coordinates of the stationary points of the curve $y = 2x^3 - 6x + k$, where k is a constant. [5]

(ii) Determine the nature of each of the stationary points found in (i). [4]

(iii) Sketch the curve for the case k = 0, labelling clearly all stationary points and intercepts. [3]

State the range of values of x for which y is

(a) strictly increasing, [2]

(b) strictly decreasing.

8) $y = 2x^3 - 6x + k$

$\frac{dy}{dx} = 6x^2 - 6$

Let $\frac{dy}{dx} = 0$

$6x^2 - 6 = 0$

$x = 1$ or -1

$y = 2(1)^3 - 6(1) + k$ or $2(-1)^3 - 6(-1) + k$

$= k - 4$ $\quad k + 4$

$\therefore$ Coordiates of stationary points are $(1, k - 4)$ and $(-1, k + 4)$

(ii)

x	1^-	1	1^+
dy/dx	-ve	0	+ve
	\	–	/

x	-1^-	-1	-1^+
dy/dx	+ve	0	-ve
	/	–	\

Therefore, (1, k - 4) is a minimum point and (-1, k+4) is a maximum point.

(iii) $k = 0$, so

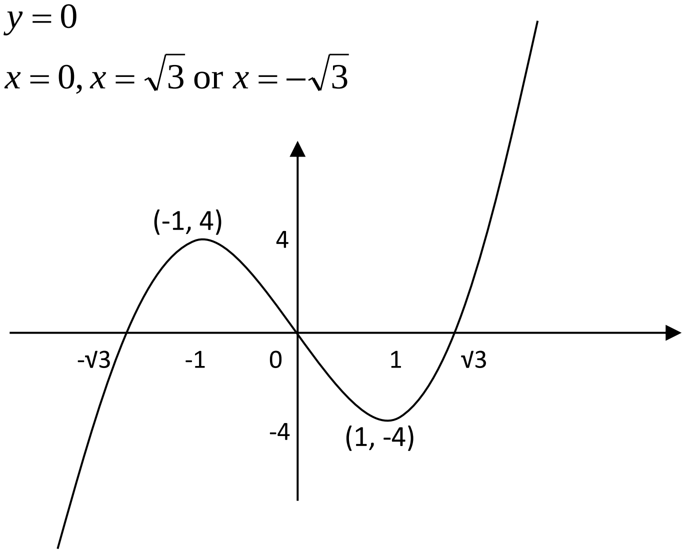
$y = 2x^3 - 6x$

$= 2x(x - \sqrt{3})(x + \sqrt{3})$

x - intercepts :

$y = 0$

$x = 0, x = \sqrt{3}$ or $x = -\sqrt{3}$



(a) $x < -1$ or $x > 1$

(b) $-1 < x < 1$

b) Rate of change

1) The height of a rain tree can be expressed by $h = 10 - \frac{8}{t+4}$ m, where t is the number of years after the tree is planted from an established juvenile tree.

(i) How high was the tree when it was planted? [1]

(ii) Show that $\frac{\mathrm{d}h}{\mathrm{d}t} > 0$ for all $t \geq 0$. What is the significance of this result? [3]

(iii) State the maximum height of the rain tree. [1]

1a) When $t = 0$,

$$h = 10 - \frac{8}{0+4} = 8\text{m}$$

b) $\frac{dh}{dt} = \frac{8}{(t+4)^2}$ Since $(t+4)^2 \geq 4$ or $(t+4)^2 > 0$, thus $\frac{dh}{dt} > 0$

The tree is **always growing** or its height increases with time or

Height = $10 - 0 = 10\text{m}$

$\frac{dh}{dt} \to 0$ as ***t* increases**

c) Maximum height = 10 m

When $t \to \infty$, $\frac{8}{t+4} \to 0$.

2) Variables x and y are related by the equation $y = e^{2x}$. Given that the rate of change of y is 0.3 units per second, find the corresponding rate of change of x when $y = 1$. [4]

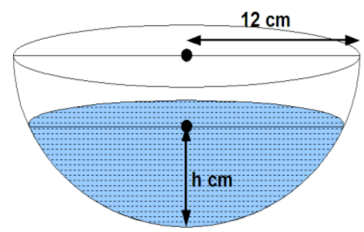
2) $given\ y = e^{2x}\ and\ \frac{dy}{dt} = 0.3$

$$\frac{dy}{dx} = 2e^{2x}$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \Rightarrow 0.3 = 2e^{2x} \times \frac{dx}{dt}$$

when $y = 1$, $x = 0$ $\Rightarrow \frac{0.3}{2} = \frac{dx}{dt} = 0.15$ units/s

3) The diagram shows a hemispherical bowl of radius 12 cm. Water is poured into the bowl and at any time t , the height of the water level from the lowest point of the hemisphere is h cm. The rate of change of the height of the water level is 0.4 cm/s.

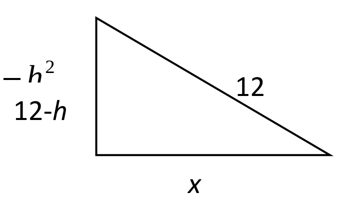


(i) Show that the area of the water surface, A, is given by $A = \pi(24h - h^2)$. [2]

(ii) Find the rate of change of A at instant when $h = 5$ cm.
(Leave your answer in terms of π) [4]

3) Let the radius of the circle be x cm

(i) $x^2 = 12^2 - (12 - h)^2$ $= 24h - h^2$



$\therefore A = \pi(24h - h^2)$

(ii) $\frac{dA}{dh} = \pi(24 - 2h)$

$$\frac{dA}{dt} = \frac{dA}{dh} \bullet \frac{dh}{dt} = \pi(24 - 2h)(0.4)$$

At $h = 5$, $\frac{dA}{dt} = \pi(14)(0.4) = 5.6\pi$ cm^2/s

4) Given that $y = e^{1-3x}$, find the exact value of $\frac{dy}{dx}$ at the point when $x = -1$. If $\frac{dx}{dt} = 2p$ units per second, hence, find in terms of e and p, the rate of change of y at the instant when $x = -1$. [4]

4) $y = e^{1-3x}$

$$\frac{dy}{dx} = -3e^{1-3x}$$

At $x = -1$, $\frac{dy}{dx} = -3e^{1-3(-1)} = -3e^4$

$$\frac{dx}{dt} = 2p$$

$$\frac{dy}{dt} = \frac{dy}{dx} . \frac{dx}{dt} = -3e^4 .2p$$

$$= -6pe^4 \text{ units/s}$$

5) A metal cube is being expanded by heat.

(i) At the instant when the length of an edge is 2 cm, the length of an edge is increasing at the rate of 0.001 cm s^{-1}. At what rate is the volume of the cube increasing at this instant?

(ii) At the instant when the surface area of the cube is increasing at the rate of 0.036 cm^2 s^{-1}, an edge is increasing at the rate of 0.001 cm s^{-1}. Find the length of the edge at this instant.

5i) Let the edge be x cm

Then, volume of cube is $V = x^3$

$$\frac{dV}{dx} = 3x^2$$

Now, $\dfrac{dV}{dt} = \dfrac{dV}{dx}\dfrac{dx}{dt} = 3x^2\dfrac{dx}{dt}$

When $x = 2$ cm, $\dfrac{dx}{dt} = 0.001$ cm s^{-1},

So, $\dfrac{dV}{dt} = 3x^2 (0.001)$

$= 0.012$ cm^3 s^{-1}

ii) Surface area, $A = 6x^2$

$$\frac{dA}{dx} = 12x$$

$$\frac{dA}{dt} = \frac{dA}{dx}\frac{dx}{dt}$$

$$\frac{dA}{dt} = 12x\frac{dx}{dt}$$

$0.036 = (12x)(0.001)$

$$x = \frac{0.036}{(12)(0.001)} = 3 \text{ cm}$$

c) Maxima and Minima

1) The volume, V cm^3, of a ball bearing with radius x cm, $x<1$, can be expressed as
$V=\pi x^2\sqrt{4x^2-8x+4}$.

Show that $V=(2\pi x^2-2\pi x^3)$ cm^3. [2]

Find the maximum volume of the ball bearing. [3]

1) $V=\pi x^2\sqrt{4x^2-8x+4}$

$$=\pi x^2\sqrt{4(x^2-2x+1)}$$
$$=\pi x^2\sqrt{4(1-x)^2}$$
$$=\pi x^2(2)(1-x)$$
$$=2\pi x^2-2\pi x^3$$

or

$$=\pi x^2\sqrt{4(x-1)^2}$$
$$=\pi x^2(-2(x-1)) \text{ since } x<1$$

$$\frac{dV}{dx}=4\pi x-6\pi x^2$$

$$\frac{dV}{dx}=0$$
$$4\pi x-6\pi x^2=0$$
$$2\pi x(2-3x)=0$$

$x=0$ (reject) or $2-3x=0$

$$x=\frac{2}{3}cm$$

Maximum volume $=2\pi x^2-2\pi x^3$

$$=2\pi\left(\frac{2}{3}\right)^2-2\pi\left(\frac{2}{3}\right)^3$$
$$=\frac{8}{27}\pi \text{ cm}^3$$

2) A closed container takes the form of a hollow cylinder of radius r and height h surmounted by a hemispherical cap at the top and a circular disc at the bottom, as shown in figure 1.

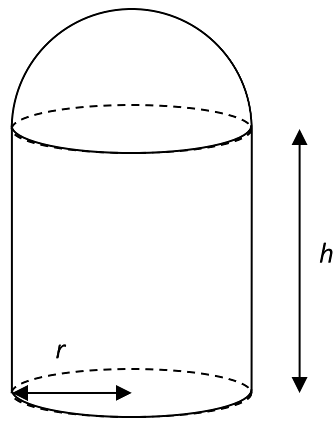


Figure 1

Given that the volume of this container is 45π, express h in terms of r and show that the total surface area of the container is given by

$$A = \frac{5}{3}\pi r^2 + \frac{90\pi}{r}.$$

Find the radius of the cylinder if its total surface area is to be a minimum. [8]

(The formulae for the volume and surface area of a sphere are $V = \frac{4}{3}\pi r^3$, $A = 4\pi r^2$ respectively)

2) $45\pi = \pi r^2 h + \frac{1}{2} \times \frac{4}{3}\pi r^3$

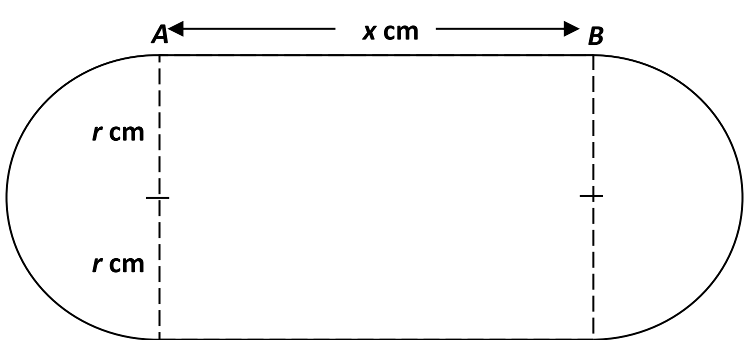
$$h = \frac{45 - \frac{2}{3}r^3}{r^2}$$

$$A = \frac{1}{2}(4\pi r^2) + 2\pi rh + \pi r^2$$
$$= 3\pi r^2 + 2\pi rh$$
$$= 3\pi r^2 + 2\pi r\left(\frac{45 - \frac{2}{3}r^3}{r^2}\right)$$
$$= \frac{5\pi r^2}{3} + \frac{90\pi}{r}$$

$$\frac{dA}{dr} = \frac{5\pi}{3} \times 2r + 90\pi\left(-\frac{1}{r^2}\right) = 0$$
$$\Rightarrow r = 3$$

r	3^-	3	3^+
$\frac{dA}{dr}$	-ve	o	+ve

3) The figure shows the surface of a table top consisting of a rectangular region *ABCD* and a semicircular region at each end with *AD* and *BC* as diameters. *AD* = *BC* = 2*r* cm. The rectangular region *ABCD* has an area of 200 cm^2.



(a) Given that $AB = x$ cm, show that the perimeter P cm of the table top is given by

$P = 2x + \dfrac{200\pi}{x}$. **[3]**

(b) Find the minimum value of P as x varies, in terms of π. **[4]**

3) **(a)** Area of rect $ABCD$ = 200 cm^2

$$2rx = 200$$

$$\Rightarrow r = \frac{100}{x} \quad \dots (1)$$

Perimeter, $P = 2x + 2\pi r \quad \dots (2)$

Sub. (1) in (2): $P = 2x + 2\pi(\dfrac{100}{x})$

$$P = 2x + \frac{200\pi}{x} \text{ (Shown)}$$

(b) $\dfrac{dP}{dx} = 2 - \dfrac{200\pi}{x^2} = 0$

$x = 10\sqrt{\pi}$, taking the positive root.

Using GC, P is min when $x = 10\sqrt{\pi}$

Min. value of P = $2\left(10\sqrt{\pi}\right) + \dfrac{200\pi}{10\sqrt{\pi}}$

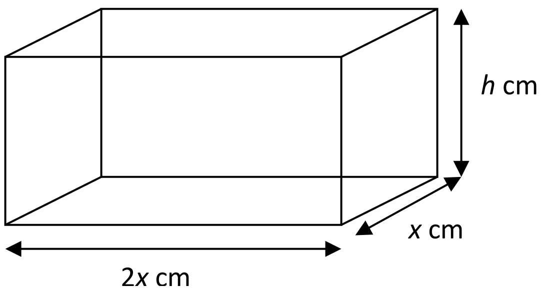
$$= 40\sqrt{\pi} \text{ cm}$$

4) An open rectangular box of height h cm has a horizontal rectangular base of sides x cm and $2x$ cm.

(i) Given that the volume of the box is 36 cm^3, express h in terms of x.

(ii) Show that the total surface area, A cm^2, of the box is given by $A = 2x^2 + \frac{108}{x}$.

(iii) Calculate the value of x and of h which make the total surface area, A, a minimum.



4i) Volume, $V = (2x)(x)(h)$

$$36 = 2x^2 h$$

$$h = \frac{36}{2x^2} = \frac{18}{x^2}$$

ii) $A = 2x(x) + 2(x)(h) + 2(2x)(h)$

$$= 2x^2 + 6xh$$

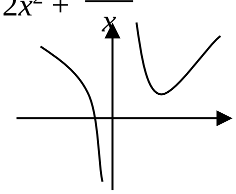
$$= 2x^2 + 6x\left(\frac{18}{x^2}\right)$$

$$= 2x^2 + \frac{108}{x} \text{ (shown)}$$

iii) <u>Method 1:</u>

Using GC, graph $y = 2x^2 + \frac{108}{x}$

The min pt is (3, 54)



Hence, V is min when

$x = 3$

$h = \frac{18}{3^2} = 2$

Method 2:

$\frac{dA}{dx} = 4x - \frac{108}{x^2}$

For stationary points, $\frac{dA}{dx} = 0$

So, $4x - \frac{108}{x^2} = 0$

$\frac{4x^3 - 108}{x^2} = 0$

$\frac{x^3 - 27}{x^2} = 0$

$x = 3$

x	3^-	3	3^+
$\frac{dA}{dx}$	– ve	0	+ ve

So $x = 3$ gives a minimum A.

$h = \frac{18}{3^2} = 2$

5 (a) A cylindrical metal can, open at the top, has height h cm and radius r cm. It is constructed to hold $125\pi\ \text{cm}^3$ of liquid. The thickness of the metal may be neglected.

(i) Show that the external surface area, $S\ \text{cm}^2$, of metal used for the tank is given by $S = \pi\left(r^2 + \frac{250}{r}\right)$. [3]

(ii) Given that r can vary, show, by differentiation, that the amount of material needed to construct the can is a minimum when $r = h$. [4]

(b) Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of $6\ \text{m}^2\text{s}^{-1}$. Find the rate of increase of the radius of the spill when the area is 15 m^2. [4]

5) (a)(i) $V=\pi r^2 h = 125\pi \Rightarrow h = \dfrac{125}{r^2}$

$S = \pi r^2 + 2\pi r h$

$= \pi r^2 + 2\pi r\left(\dfrac{125}{r^2}\right)$

$= \pi\left(r^2 + \dfrac{250}{r}\right)$

(ii) $\dfrac{dS}{dr} = \pi\left(2r - \dfrac{250}{r^2}\right)$

At stationary point,

$\dfrac{dS}{dr} = 0 \Rightarrow 2r - \dfrac{250}{r^2} = 0$

$\Rightarrow r^3 = 125$

$\Rightarrow r = 5$

Now, $\dfrac{dS}{dr} = \dfrac{2\pi}{r^2}\left(r^3 - 125\right)$.

r	5^-	5	5^+
$\dfrac{dS}{dr}$	– ve	0	+ ve
tangent	\	_	/

Therefore, S is minimum when $r = 5$.

When $r = 5$, $h = \dfrac{125}{5^2} = 5$.

Hence, S is minimum when $r = h$.

(b) $A = \pi r^2 \Rightarrow \dfrac{dA}{dr} = 2\pi r$

When $A = 15\ \text{m}^2$,

$15 = \pi r^2 \Rightarrow r = \sqrt{\dfrac{15}{\pi}}$.

$\dfrac{dA}{dt} = \dfrac{dA}{dr} \times \dfrac{dr}{dt} \Rightarrow 6 = 2\pi\sqrt{\dfrac{15}{\pi}} \times \dfrac{dr}{dt}$

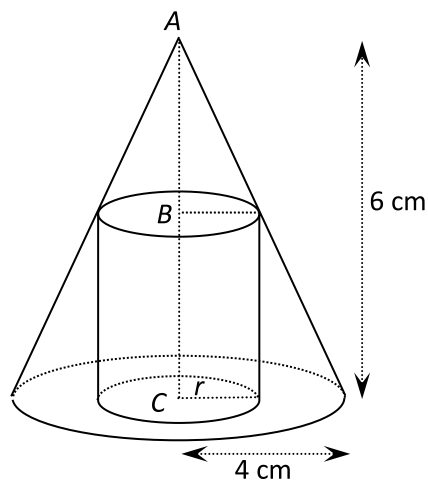
$\Rightarrow \dfrac{dr}{dt} = \sqrt{\dfrac{3}{5\pi}} = 0.437\ \text{ms}^{-1}$

6) The diagram shows a cylinder which is placed inside a right circular cone so that the circumference of the top of the cylinder is in contact with the inner surface of the cone. The base of the cylinder is level with the base of the cone. The base radius of the cylinder is r cm and the base radius of the cone is 4 cm. The height of the cylinder, BC, is h cm and the height of the cone, AC, is 6 cm.

(i) By considering AB in terms in terms of h, show that

$$h = 6 - \frac{3}{2}r\,. \qquad [1]$$

(ii) Using differentiation, find the maximum volume of the cylinder in terms of π. [5]



6i) AB = 6 – h

$$\frac{AB}{r} = \frac{6}{4} \qquad \Rightarrow \qquad \frac{6-h}{r} = \frac{3}{2}$$

$$\therefore \qquad 6 - h = \frac{3}{2}r \qquad \Rightarrow \qquad h = 6 - \frac{3}{2}r$$

ii) $V = \pi r^2 h = \pi r^2 \left(6 - \frac{3}{2}r\right) = \pi \left(6r^2 - \frac{3}{2}r^3\right)$

$$\frac{dV}{dr} = \pi\left(12r - \frac{9r^2}{2}\right) = 0$$

$\Rightarrow$ $r = 0$ (NA) or $r = \frac{8}{3}$

r	$\left(\frac{8}{3}\right)^-$	$\frac{8}{3}$	$\left(\frac{8}{3}\right)^+$
$\frac{dV}{dr}$	+ve	0	−ve
Slope	/	—	\

$\therefore$ max. V at $r = \frac{8}{3}$

Max. V = $128\pi/9$ cm^3 or 14.2π cm^3

Alternative test:

$$\frac{d^2V}{dr^2} = \pi(12 - 9r)$$

When $r = \frac{8}{3}$, $\frac{d^2V}{dr^2} = -24 < 0$

$\therefore$ max. V at $r = \frac{8}{3}$

Alternatively, may write V in terms of h:

$$V = \pi\left(4 - \frac{2}{3}h\right)^2 h = \pi\left(16h - \frac{16}{3}h^2 + \frac{4}{9}h^3\right)$$

$$\frac{dV}{dh} = \pi\left(16 - \frac{32}{3}h + \frac{4h^2}{3}\right)$$

$$= \frac{4}{3}\pi(h-6)(h-2) = 0$$

$h = 2$ cm (as $h \neq 6$ cm)