2015 H2 Mathematics C2 Prelim Paper 1 Solutions

Qn	Solutions
1(a)	Let $u=2x-1$ $\int (x-1)\sqrt{2x-1}\ \mathrm{d}x=\frac{1}{2}\int\left(\frac{u+1}{2}-1\right)\sqrt{u}\ \mathrm{d}u$ $\frac{1}{2}\int\left(\frac{u}{2}+\frac{1}{2}-1\right)\sqrt{u}\ \mathrm{d}u$ $=\frac{1}{4}\int(u-1)\sqrt{u}\ \mathrm{d}u=\frac{1}{4}\int u^{\frac{3}{2}}-u^{\frac{1}{2}}\mathrm{d}u=\frac{1}{4}\left[\frac{2}{5}u^{\frac{5}{2}}-\frac{2}{3}u^{\frac{3}{2}}\right]+C$ $=\frac{1}{10}u^{\frac{5}{2}}-\frac{1}{6}u^{\frac{3}{2}}+C=\frac{1}{10}(2x-1)^{\frac{5}{2}}-\frac{1}{6}(2x-1)^{\frac{3}{2}}+C$
1(b)	Using MF15 $\frac{P+Q}{2}=3x \qquad \frac{P-Q}{2}=x$ $P+Q=6x \qquad P-Q=2x$ $2P=8x$ => $P=4x$ $Q=2x$ $\therefore \int \sin 3x \sin x\,\mathrm{d}x=\int -\frac{1}{2}(\cos 4x-\cos 2x)\,\mathrm{d}x$ $=-\frac{1}{2}\left[\frac{\sin 4x}{4}-\frac{\sin 2x}{2}\right]+C=-\frac{\sin 4x}{8}+\frac{\sin 2x}{4}+C$
2	$\mathrm{f}(x)=\ln(1-x)-\ln(1+\cos x), \quad \mathrm{f}(0)=-\ln 2$ $\mathrm{f}'(x)=\frac{-1}{1-x}+\frac{\sin x}{1+\cos x}, \quad \mathrm{f}'(0)=-1$ $\mathrm{f}''(x)=\frac{-1}{(1-x)^2}+\frac{(1+\cos x)\cos x-\sin x(-\sin x)}{(1+\cos x)^2}$ $\mathrm{f}''(x)=\frac{-1}{(1-x)^2}+\frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2}$ $\mathrm{f}''(x)=\frac{-1}{(1-x)^2}+\frac{1}{1+\cos x}, \quad \mathrm{f}''(0)=-\frac{1}{2}$ $\mathrm{f}'''(x)=\frac{-2}{(1-x)^3}+\frac{\sin x}{(1+\cos x)^2}, \quad \mathrm{f}'''(0)=-2$ Using series formula in MF15: $\mathrm{f}(x)=\mathrm{f}(0)+x\mathrm{f}'(0)+\frac{x^2}{2!}\mathrm{f}''(0)+\frac{x^3}{3!}\mathrm{f}'''(0)+\ldots$ $\mathrm{f}(x)=-\ln 2-x-\frac{1}{4}x^2-\frac{1}{3}x^3+\ldots$ $\ln\left(\frac{1+\cos x}{1+x}\right)=-\ln\left(\frac{1-(-x)}{1+\cos(-x)}\right)=\ln 2-x+\frac{1}{4}x^2-\frac{1}{3}x^3+\ldots$

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Let $FG = h_1$ and let $BC = h_2$.

Form equation of the ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{2000^2} = 1$ --- (1)

Since the areas of $ABCD$ and $EFGH$ are equal: $1000h_2 = 1435h_1$ --------- (2)

Substitute the point $G\left(\dfrac{1435}{2}, h_1\right)$ into (1):

$$\frac{717.5^2}{a^2} + \frac{h_1^{\,2}}{2000^2} = 1 \quad \text{--------- (3)}$$

Substitute the point $C\left(\dfrac{1000}{2}, h_2\right)$ into (1):

$$\frac{500^2}{a^2} + \frac{h_2^{\,2}}{2000^2} = 1 \quad \text{--------- (4)}$$

Substitute (2) into (4): $\dfrac{500^2}{a^2} + \dfrac{(1.435h_1)^2}{2000^2} = 1$ --- (5)

From (3) and (5): $a^2 = 764806.25 \Rightarrow a = 874.532 \Rightarrow MN = 2a = 1749.06 \approx 1749$ mm

4(i)

$$\text{sloped length} = \frac{3}{\sin\theta}$$

$$\text{total area} = \left(8 - \frac{3}{\sin\theta}\right)(3) + \frac{1}{2}\left(\frac{3}{\tan\theta}\right)(3) = 24 - 9\operatorname{cosec}\theta + \frac{9}{2}\cot\theta$$

4(ii)

$$A = 24 - 9\operatorname{cosec}\theta + \frac{9}{2}\cot\theta$$

$$\frac{dA}{d\theta} = 9\cot\theta\operatorname{cosec}\theta - \frac{9}{2}\operatorname{cosec}^2\theta$$

$$= \frac{9}{2}\cdot\frac{2\cos\theta - 1}{(\sin\theta)^2}$$

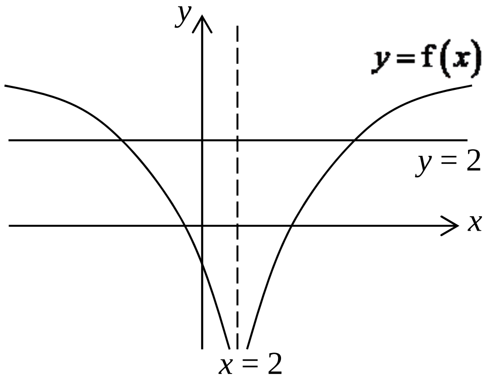
Let $\dfrac{dA}{d\theta} = 0$, $2\cos\theta = 1$.

So, $\theta = \dfrac{\pi}{3}$.

$$A = 24 - 9\operatorname{cosec}\frac{\pi}{3} + \frac{9}{2}\cot\frac{\pi}{3} = 24 - \frac{18}{\sqrt{3}} + \frac{9}{2\sqrt{3}} = 24 - \frac{9\sqrt{3}}{2} = 16.2 \ (3\text{ s.f.})$$

θ	$\left(\frac{\pi}{3}\right)^-$	$\left(\frac{\pi}{3}\right)$	$\left(\frac{\pi}{3}\right)^+$
$\frac{dA}{d\theta}$	+	0	–
	/	—	\

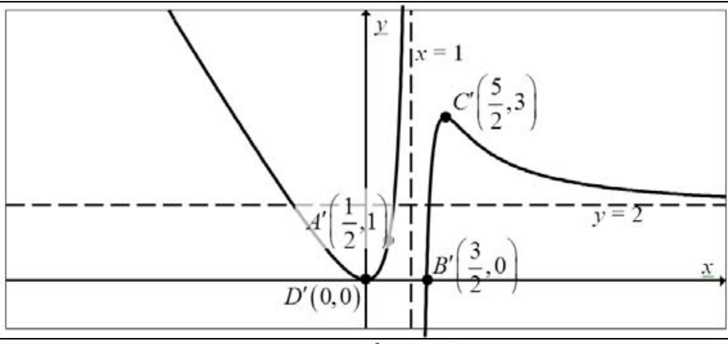
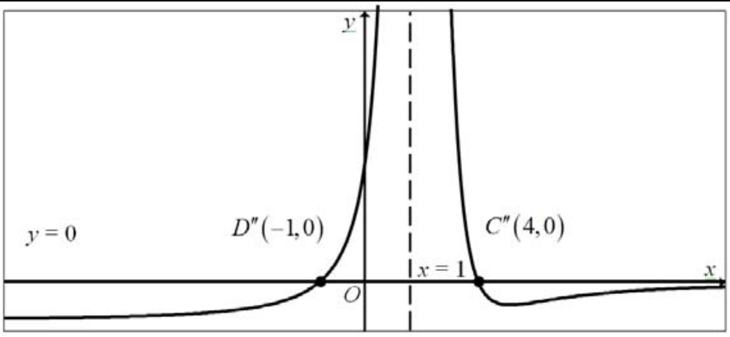
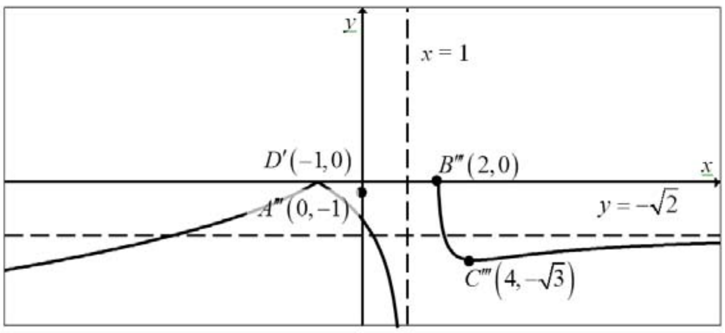
The maximum area is 16.2 m^2.

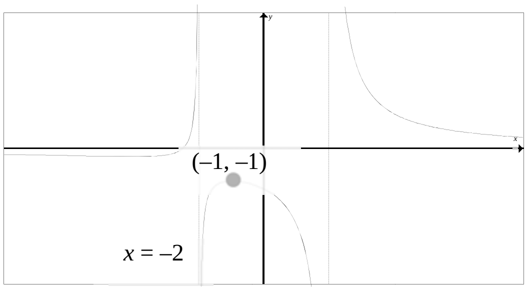
5(i)	 Since a line such as $y=2$ intersects the curve $y=\mathrm{f}(x)$ at 2 points, f is not one-to-one. $\therefore$ f^{-1} does not exist.
5(ii)	Largest possible domain for h is $[3,\infty)$.
5(iii)	$y=\ln(2-x)^2-2 \Rightarrow x=2\pm \mathrm{e}^{\frac{y+2}{2}}$ Since $x\ge 3$, $\mathrm{h}^{-1}: x \to 2+\mathrm{e}^{\frac{x+2}{2}}, x\in\mathbb{R}, x\ge -2$
5(iv)	$\mathrm{hh}^{-1}(x)=\mathrm{h}^{-1}\mathrm{h}(x)=x$ which is always valid since the inverse function exists. Since domain of hh^{-1} is domain of $\mathrm{D}_{\mathrm{h}^{-1}}=[-2,\infty)$ and domain of $\mathrm{h}^{-1}\mathrm{h}$ is domain of $\mathrm{D}_{\mathrm{h}}=[3,\infty)$. Thus the set of values is $\{x\in\mathbb{R} : x\ge 3\}$.
6(i)	As $t\to-\infty$, $x\to\sin t$ and $y\to-\cos t$. Using trigonometric identity $\sin^2 t+\cos^2 t=1$, the Cartesian equation of C is $x^2+y^2=1$. Thus the shape of C is a <u>circle</u> with <u>centre at the origin</u> and with <u>unit length radius</u>.
6(ii)	$\frac{\mathrm{d}x}{\mathrm{d}t}=\mathrm{e}^t+\cos t$, $\frac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{e}^t+\sin t$. $\therefore$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{e}^t+\sin t}{\mathrm{e}^t+\cos t}$ At P, gradient of normal is $-\frac{\mathrm{e}^\theta+\cos\theta}{\mathrm{e}^\theta+\sin\theta}$. $\therefore$ equation of normal is $y-\left(\mathrm{e}^\theta-\cos\theta\right)=-\frac{\mathrm{e}^\theta+\cos\theta}{\mathrm{e}^\theta+\sin\theta}\left(x-\left(\mathrm{e}^\theta+\sin\theta\right)\right)$ $\Rightarrow y=-\frac{\mathrm{e}^\theta+\cos\theta}{\mathrm{e}^\theta+\sin\theta}\left(x-\left(\mathrm{e}^\theta+\sin\theta\right)\right)+\left(\mathrm{e}^\theta-\cos\theta\right)\Rightarrow y=-\frac{\mathrm{e}^\theta+\cos\theta}{\mathrm{e}^\theta+\sin\theta}x+2\mathrm{e}^\theta$
6(iii)	Using equation of normal found in (ii), point D is $\left(0, 2\mathrm{e}^\theta\right)$. E is a point on C. From $y=\mathrm{e}^t-\cos t$, when $y=0$, $\Rightarrow$ $\mathrm{e}^t=\cos t$ By inspection, $t=0$. $\therefore$ $x=\mathrm{e}^0+\sin 0=1$. Hence point E is $(1,0)$.
6(iv)	The mid-point of DE is $\left(\frac{1}{2}, \mathrm{e}^\theta\right)$. Since $x=\frac{1}{2}$ is a fixed value and $\theta>0 \Rightarrow y=\mathrm{e}^\theta>1$, $\therefore$ required locus is a half-line $x=\frac{1}{2}$, with $y>1$.

7(i)	$\overrightarrow{OC}=\overrightarrow{OB}+\overrightarrow{BC}=\begin{pmatrix}0\\0\\h\end{pmatrix}+\begin{pmatrix}0\\3\\0\end{pmatrix}=\begin{pmatrix}0\\3\\h\end{pmatrix}$
7(ii)	By ratio theorem $\overrightarrow{OP}=\dfrac{2\overrightarrow{OC}+\overrightarrow{OB}}{3}$ $=\dfrac{1}{3}\left[\begin{pmatrix}0\\6\\2h\end{pmatrix}+\begin{pmatrix}0\\0\\h\end{pmatrix}\right]=\begin{pmatrix}0\\2\\h\end{pmatrix}$
7(iii)	Select a suitable direction vector parallel to the plane such as $\overrightarrow{BE}=\overrightarrow{OE}-\overrightarrow{OB}$ $=\begin{pmatrix}2h\\3\\0\end{pmatrix}-\begin{pmatrix}0\\0\\h\end{pmatrix}=\begin{pmatrix}2h\\3\\-h\end{pmatrix}$. Thus $\overrightarrow{BE}\cdot\left(a\,\mathbf{i}+b\,\mathbf{k}\right)=0$ $\Rightarrow\begin{pmatrix}2h\\3\\-h\end{pmatrix}\cdot\begin{pmatrix}a\\0\\b\end{pmatrix}=0\Rightarrow\dfrac{a}{b}=\dfrac{1}{2}$ Since C is on the plane, $\begin{pmatrix}0\\3\\h\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=2h$ $\Rightarrow\mathbf{r}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=2h\Rightarrow x+2z=2h$
7(iv)	Given that $h=3$, $\overrightarrow{OP}=\begin{pmatrix}0\\2\\3\end{pmatrix}$**,** $\overrightarrow{OF}=\begin{pmatrix}6\\0\\0\end{pmatrix}$ $\mathbf{n}=\begin{pmatrix}0\\2\\3\end{pmatrix}\times\begin{pmatrix}6\\0\\0\end{pmatrix}=\begin{pmatrix}0\\18\\-12\end{pmatrix}=6\begin{pmatrix}0\\3\\-2\end{pmatrix}$ The equation of plane *OPF* is $\mathbf{r}\cdot\begin{pmatrix}0\\3\\-2\end{pmatrix}=0$ Shortest distance = $\dfrac{\left\lvert\begin{pmatrix}1\\2\\2\end{pmatrix}\cdot\begin{pmatrix}0\\3\\-2\end{pmatrix}\right\rvert}{\sqrt{0^2+3^2+2^2}}=\dfrac{2}{\sqrt{13}}$ units.

8(i)	$\frac{dA}{dt} = kA - 4000 \Rightarrow \int \frac{1}{kA-4000}\,dA = \int 1\,dt$ $\Rightarrow \frac{1}{k}\int \frac{1}{A-\frac{4000}{k}}\,dA = \int 1\,dt$ $\Rightarrow \frac{1}{k}\ln\left\lvert A-\frac{4000}{k}\right\rvert = t + C \Rightarrow \left\lvert A-\frac{4000}{k}\right\rvert = e^{kt+kC}$ $\Rightarrow A-\frac{4000}{k} = \alpha e^{kt}$ $\left(\text{where } \alpha = \pm e^{kC}\right)$ $\Rightarrow A = \alpha e^{kt} + \frac{4000}{k}$ Sub $(t, A) = (0, 60000)$: $60000 = \alpha + \frac{4000}{k} \Rightarrow \alpha = 60000 - \frac{4000}{k}$ ---- (1) Sub $(t, A) = (3, 69500)$: $69500 = \alpha e^{3k} + \frac{4000}{k}$ ---- (2) Sub (1) into (2): $69500 = \left(60000 - \frac{4000}{k}\right)e^{3k} + \frac{4000}{k}$ Using G.C., $k \approx 0.1111343 = 0.111$ (3 s.f.) Thus $A = \left(60000 - \frac{4000}{k}\right)e^{kt} + \frac{4000}{k}$ $\Rightarrow A = 24000e^{0.111t} + 36000$ with $\alpha = 24000$ (3 s.f.) and $\lambda = 36000$ (3 s.f.)
8(ii)	Since $A = \pi r^2$, $\frac{dA}{dr} = 2\pi r$ Thus $\frac{dr}{dt} = \frac{dA}{dt} \div \frac{dA}{dr} = \frac{kA-4000}{2\pi r} = \frac{k\pi r^2 - 4000}{2\pi r}$ Sub $r = 200$, $\frac{dr}{dt} \approx \frac{(0.1111343)\pi(200)^2 - 4000}{2\pi(200)} = 7.93$ (3 s.f.)
8(iii)	Let the rate Mac needs to cut the weeds be n m^2 per month. $\frac{dA}{dt} \approx (0.1111343)A - n$ $0 = (0.1111343)(69500) - n \Rightarrow n = 7720$ (3 s.f.)
8(iv)	$\frac{dA}{dt} = 0$ means that the rate which Mac needs to cut the weeds is equal to the rate the weeds grow. Thus, the area covered in weeds is unchanged.
9(i)	Amount after 16 days = $1000 \times \left(\frac{1}{2}\right)^2 = 250$ mg

9(ii)	Amount of I-131 on Day 49 $=\left[\left[\left[1000\times\left(\frac{1}{2}\right)^2+1000\right]\times\left(\frac{1}{2}\right)^2+1000\right]\times\left(\frac{1}{2}\right)^2+1000\cdots\cdots(*)\right.$ $=1000\left[1+\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3\right]$ $=1000\left[\frac{1-\left(\frac{1}{4}\right)^4}{1-\frac{1}{4}}\right]=1328.125\text{ mg}=1328\text{ mg (nearest mg)}$
9(iii)	$S_\infty=\frac{1000}{1-\frac{1}{4}}=1333.33<1334\text{ mg}$ Amount of I-131 will never exceed 1334 mg.
9(iv)	Amount of I-125 on Day 121 $=1000\times\left(\frac{1}{2}\right)^2=250\text{ mg}$ I-131 is added on Day 17, …, 113, $\Rightarrow$ total 7 times Amount of I-131 on Day 121 $=1000\left[1+\frac{1}{4}+\left(\frac{1}{4}\right)^2+\cdots+\left(\frac{1}{4}\right)^6\right]\times\frac{1}{2}=500\left[\frac{1-\left(\frac{1}{4}\right)^7}{1-\frac{1}{4}}\right]=666.626\text{ mg}$ Total amount of radioisotopes $=250+666.626=917\text{ mg (nearest mg)}$
10(a)(i)	It is not necessarily true because to conclude that i^* is a root, the coefficients of the equation must be real.
10(a)(ii)	Sub $w=\mathrm{i}$ into $z^3-az^2+2az-4\mathrm{i}=0$ $\mathrm{i}^3-a\mathrm{i}^2+2a\mathrm{i}-4\mathrm{i}=0\Rightarrow-\mathrm{i}+a+2a\mathrm{i}-4\mathrm{i}=0\Rightarrow a(1+2\mathrm{i})=5\mathrm{i}\Rightarrow a=2+\mathrm{i}$
10(a)(iii)	Let $\left(bz^2+cz+d\right)(z-\mathrm{i})=0$ By inspection, $b=1$ $d=4$, $\left(z^2+cz+4\right)(z-\mathrm{i})=0$ Compare z terms: $-\mathrm{i}c+4=2a\Rightarrow c=\frac{4+2\mathrm{i}-4}{-\mathrm{i}}=-2$ Thus $z^2+cz+4=0$ $\Rightarrow z=\frac{2\pm\sqrt{4-4(1)(4)}}{2}=1\pm\frac{\sqrt{-12}}{2}=1\pm\sqrt{3}\mathrm{i}$ $\Rightarrow z=1+\sqrt{3}\mathrm{i}$ or $z=1-\sqrt{3}\mathrm{i}$

10(b)	$\arg(z-2\text{i})=\frac{\pi}{4}$ $\Rightarrow \tan\frac{\pi}{4}=\frac{y-2}{x} \Rightarrow y=x+2$ where $y>2$, $x>0$ --------- (1) From $\lvert z^{*}-1+\text{i}\rvert=2$ $\Rightarrow \lvert x-\text{i}y-1+\text{i}\rvert=2 \Rightarrow \lvert (x-1)-\text{i}(y-1)\rvert=2$ $\Rightarrow (x-1)^2+(y-1)^2=4$ --------- (2) Sub (1) into (2): $x^2-2x+1+x^2+2x+1=4 \Rightarrow 2x^2=2 \Rightarrow x=\pm 1$ Since $x>0$, therefore $x=1 \Rightarrow z=x+\text{i}y=1+3\text{i}$
11(a) (i)	
11(a) (ii)	
11(a) (iii)	
11(b) (i)	Since $x=-2$ is the asymptote, $(-2)^2-b=0 \Rightarrow b=4$ Substitute the point $\left(-4,-\frac{1}{4}\right)$ into G, $-\frac{1}{4}=\frac{-8+a}{16-4} \Rightarrow a=5$.

11(b) (ii)	 From G.C, we find the maximum point $\left(-1,-1\right)$. For increasing and concave downwards, the only range is $-2<x<-1$.
11(b) (iii)	Use long division to obtain $\frac{5x^3+2x^2-14x+7}{x^2-4}=5x+2+\frac{6x+15}{x^2-4}$ Thus $5x+2+\frac{6x+15}{x^2-4}=0 \Rightarrow \frac{2x+5}{x^2-4}=\frac{-1}{3}\left(5x+2\right)$ Sketch the equation of the line $y=\frac{-1}{3}\left(5x+2\right)$ onto the diagram to obtain number of intersections = 3. Thus there are 3 distinct real roots to the equation.