CATHOLIC JUNIOR COLLEGE
General Certificate of Education Advanced Level
Higher 2
JC2 Preliminary Examination

MATHEMATICS — 9740/01

Paper 1

26 Aug 2015
3 hours

Additional Materials: List of Formulae (MF15)
Graph Paper

Name: ____________________________ **Class:** ________________

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, arrange your answers in NUMERICAL ORDER.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of 5 printed pages, including the cover page.

1 A calculator is **not** to be used in answering this question.

By considering $w = z^2 - z$ or otherwise, solve

$$z^4 - 2z^3 - 2z^2 + 3z - 10 = 0 \text{ where } z \in \mathbb{C},$$

leaving the roots in exact form. **[4]**

2 With reference to the origin O, two points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The points O, A and B are not collinear. The point P divides AB in the ratio $AP:PB = 3:2$. It is given that $\mathbf{a}$ is a unit vector, $OB = 4$, angle $AOB = \frac{\pi}{3}$ and the foot of the perpendicular from P to the line passing through points O and A is F. Show that $\overrightarrow{OF} = \lambda\mathbf{a}$, where λ is a constant to be determined. **[5]**

3 John decided to embark on a 100-day skipping exercise challenge. The duration of his skipping exercise each day, in seconds, follows an arithmetic progression for the odd days, and another arithmetic progression for the even days.

Day	**Duration (s)**
Day 1	20
Day 2	20
Day 3	29
Day 4	30
Day 5	38
Day 6	40
Day 7	47
Day 8	50
Day 9	56
Day 10	60
⋮	⋮

(i) Find the duration of the skipping exercise that he does on the 75th day. **[2]**

(ii) Find the total duration of the skipping exercise that he does for the first 75 days. **[3]**

(iii) After the 75th day, John thinks that the workout is too strenuous so he decides to modify the workout such that he does 80% of the previous day's duration. Find the duration of the skipping exercise that he does on the 100th day. **[2]**

(iv) Comment on the practicality of this modification. **[1]**

4 **(i)** Show that $r(r+1)=\frac{1}{3}\left[r(r+1)(r+2)-(r-1)r(r+1)\right]$. **[1]**

(ii) Hence, using the method of difference, find the sum $1\times2+2\times3+3\times4+\cdots+n(n+1)$. **[3]**

(iii) Prove by mathematical induction that

$$1\times2\times3+2\times3\times4+3\times4\times5+\cdots+n(n+1)(n+2)=\frac{1}{4}n(n+1)(n+2)(n+3).$$ **[4]**

(iv) Based on the results in parts **(ii)** and **(iii)**, write a reasonable conjecture for the sum of the series $\sum_{r=1}^{n} r(r+1)(r+2)(r+3)$. **[1]**

5 Sequence U is defined by the following recurrence relation,

$$u_1=1,\quad u_{n+1}=\frac{1}{2}u_n+1 \text{ for all } n\in\mathbf{Z}^+.$$

Sequence V is defined by $v_n=u_n-2$ for all $n\in\mathbf{Z}^+$.

(i) Find the recurrence relation between v_{n+1} and v_n. Hence show that the sequence V is a geometric progression with common ratio $\frac{1}{2}$. **[3]**

(ii) Find the limit of the sequence V and that of the sequence U when $n\to\infty$. **[3]**

(iii) Find the sum to infinity of the sequence V. **[1]**

(iv) Find the sum of the first n terms of the sequence U.

Hence show that its sum to infinity does not exist. **[3]**

6 The equations of three planes p_1, p_2 and p_3 are

$$x+2y+4z=7$$
$$2x+5y+7z=8$$
$$x+ay+3z=b$$

respectively, where a and b are constants.

(i) Find the acute angle between p_1 and p_2. **[2]**

The planes p_1 and p_2 meet in the line l.

(ii) Find a vector equation for l. **[2]**

The plane p_3 contains the point $(1,1,1)$. The three planes, p_1, p_2 and p_3, have no point in common.

(iii) Find the values of a and b. **[3]**

(iv) Find the distance between l and p_3. **[2]**

7 It is given that $\mathrm{f}\left(x\right)=\dfrac{1}{1+x+x^2}$.

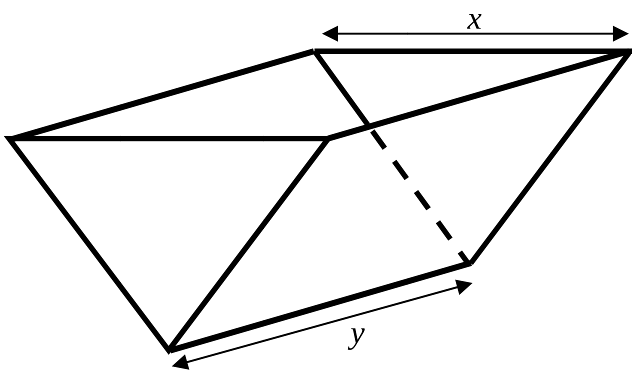
(i) Write down $\mathrm{f}'\left(x\right)$. **[1]**

(ii) Find the binomial expansion of $\mathrm{f}\left(x\right)$, up to and including the term in x^3. **[3]**

(iii) Hence, or otherwise, find the Maclaurin series for $\dfrac{-(1+2x)}{\left(1+x+x^2\right)^2}+\sin(3x)$, up to and including the term in x^2. [You may refer to the List of Formulae (MF15)]. **[2]**

(iv) Give a reason why the use of your answer in part **(ii)** to give an approximate value for $\int_0^1 \mathrm{f}\left(x\right)\,\mathrm{d}x$ is not valid. **[2]**

8



(i) An open water tank, y metres long, is in the form of an inverted prism so that its cross-section is an equilateral triangle, x metres on each side, as shown in the figure above. The tank is made of material of negligible thickness and its volume is $10\,\mathrm{m}^3$. Show that the total external surface area of the water tank, A, is

$$A=\frac{\sqrt{3}}{2}x^2+\frac{80\sqrt{3}}{3x}.$$

Use differentiation to find the value of x that will require the least amount of material.

[4]

(ii) It is given that $y=5$. The water tank is initially empty and water is being pumped in at a constant rate of $\dfrac{1}{3}$ m^3/s. At time t seconds, the depth of the water in the tank is h metres. Find the rate at which the depth is increasing at the instant when the depth is 1 m. **[4]**

9 The curve C has parametric equations

$$x = t^2 + t, \qquad y = t^2 - t, \qquad \text{where} \quad -2 \leqslant t \leqslant 2.$$

(i) Find $\dfrac{dy}{dx}$ in terms of t. What can be said about the tangents to C as $t \to -\dfrac{1}{2}$ and $t \to \dfrac{1}{2}$? **[3]**

(ii) Sketch C, showing clearly the features of the curve at the points where $t = -2, -\dfrac{1}{2}, \dfrac{1}{2}$ and 2. **[3]**

(iii) The normal to the curve at the point P where $t = 1$ meets C again at the point Q. Use a non-calculator method to find the coordinates of Q. **[4]**

(iv) Find a cartesian equation of C. **[2]**

10 The curve C has equation $y = \dfrac{8x}{x^2+1}$.

(i) Find the exact area of the region bounded by the curve C, the x–axis and the line $x = 1$. **[3]**

The region bounded by the curve C, the y–axis and the line $y = 4$ is rotated through 2π radians about the x–axis.

(ii) Using the substitution $x = \tan\theta$, show that the volume, V, of the solid generated is obtained by $V = p\pi - q\pi\int_a^b \sin^2\theta \, d\theta$ where a, b, p and q are constants to be determined exactly. **[5]**

(iii) Hence, evaluate V exactly. **[3]**

11 The functions f and g are defined as follows

$$\mathrm{f} : x \mapsto \frac{x^2 + a}{x} \quad \text{for } x \in \mathbb{R},\ x > 0,$$

$$\mathrm{g} : x \mapsto \mathrm{e}^x + 1 \quad \text{for } x \in \mathbb{R},$$

where $0 < a < 1$.

(i) Sketch the graph of $y = \mathrm{f}(x)$, indicating the equations of asymptotes and the coordinates of turning points, if any. **[4]**

(ii) Show that the composite function fg exists, and define fg in a similar form. **[4]**

(iii) Find the range of fg. **[3]**

(iv) If the domain of f is further restricted to $0 < x \leqslant k$, state the greatest value of k, for which the function f^{-1} exists. Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. **[5]**

— THE END —