River Valley High School Integrated Programme
2023 JC2 H2 Mathematics (9758)
Lecture Test 1 (Term 1)

Name : ____________________ **Index Number** : ____________

Class : ____________________ **Date** : **3 Feb 2023**

Duration : **50 mins**

List of Formulae

Derivatives

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$

Integrals

(Arbitrary constants are omitted; *a* denotes a positive constant.)

$\mathrm{f}(x)$	$\int \mathrm{f}(x)\mathrm{d}x$	
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	$(\lvert x\rvert < a)$
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left(\dfrac{x-a}{x+a}\right)$	$(x > a)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left(\dfrac{a+x}{a-x}\right)$	$(\lvert x\rvert < a)$
$\tan x$	$\ln(\sec x)$	$(\lvert x\rvert < \frac{1}{2}\pi)$
$\cot x$	$\ln(\sin x)$	$(0 < x < \pi)$
$\operatorname{cosec} x$	$-\ln(\operatorname{cosec} x + \cot x)$	$(0 < x < \pi)$
$\sec x$	$\ln(\sec x + \tan x)$	$(\lvert x\rvert < \frac{1}{2}\pi)$

[Answer all the questions on writing papers. **Up to 1 mark** will be deducted for poor presentation.]

1 **(i)** Find $\displaystyle\int x\mathrm{e}^{x}\,\mathrm{d}x$. [2]

(ii) Find $\displaystyle\int \frac{1}{1-4x^2}\,\mathrm{d}x$. [2]

(iii) Hence find the general solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\mathrm{e}^{x} - 4xy^2\mathrm{e}^{x}.$$ [2]

2 The curve C has equation $y = \dfrac{x^{\frac{3}{2}}}{\left(4x^2+9\right)^{\frac{3}{4}}}$.

(i) The finite region bounded by the curve C, the line $x = \dfrac{3\sqrt{3}}{2}$ and the x-axis is rotated through 2π radians about the x-axis. Find the volume of the solid obtained, giving your answer correct to 3 significant figures [2]

(ii) The finite region bounded by the curve C, the lines $x = \dfrac{3\sqrt{3}}{2}$, $y = 1$ and the y-axis is rotated through 2π radians about the x-axis. Find the volume of the solid obtained, giving your answer correct to 3 significant figures. [3]

3 A 1500-litre tank contains 600 litres of water with 5 kg of sugar dissolved in it initially. Sugar solution of concentration $\dfrac{1}{5}$ kg per litre enters the tank at a constant rate of 9 litres per hour. A well-mixed solution leaves the tank at a constant rate of 6 litres per hour.

(i) Denoting the amount of sugar in the tank as Q kg, show that $\dfrac{\mathrm{d}Q}{\mathrm{d}t} + \dfrac{2Q}{200+t} = \dfrac{9}{5}$. [2]

(ii) Using the substitution $u = \left(200+t\right)^2 Q$, show that the differential equation reduces to

$\dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{9}{5}\left(200+t\right)^2$. [3]

(iii) Hence, find the amount of sugar in the tank at the time when it is full. [6]

4 Alice has 3 red, 3 blue and 3 yellow dominoes.

(a) Find the number of ways to arrange the dominoes in a line if

(i) there are no restrictions, [2]

(ii) the red dominoes cannot be all together. [3]

[redacted]

~ The End ~

What's wrong with the following?

Let $u = e^x$ $\frac{dv}{dx} = x$

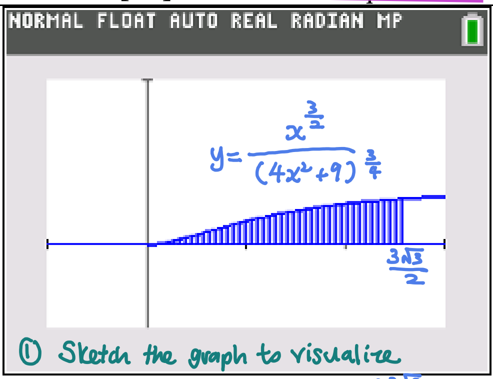
$\frac{du}{dx} = e^x$ $v = \frac{1}{2}x^2$

$\int xe^x\,dx = (e^x)(\tfrac{1}{2}x^2) - \int (\tfrac{1}{2}x^2)(e^x)\,dx$

1	Solution [10] Integration Techniques	
(a)	$\int xe^x\,dx = xe^x - \int e^x\,dx$ $= xe^x - e^x + c$ $u = x$ $\frac{dv}{dx} = e^x$ $\frac{du}{dx} = 1$ $v = e^x$	• You got to realize the integral is getting more complicated!!! • How do we determine the role of u and $\frac{dv}{dx}$? L - $\ln x$ I - $\tan^{-1}$, $\sin^{-1}$, $\cos^{-1}$ A - $1, x, x^2$ T - $\tan x$, $\sin x$ E - e^x Increasing order to be $\boxed{}$ Fill in this box with u or $\frac{dv}{dx}$
(b)	$\int \frac{1}{1-4x^2}\,dx$ $= \frac{1}{2}\int \frac{1}{1-(2x)^2}\,2\,dx$ $= \frac{1}{2}\left[\frac{1}{2(1)}\ln\left\|\frac{1+2x}{1-2x}\right\|\right] + c$ $= \frac{1}{4}\ln\left\|\frac{1+2x}{1-2x}\right\| + c$ • Note that from MF26 $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left\|\frac{a+x}{a-x}\right\| + c$ — (*) • Using (*), we deduce that $\int \frac{1}{a^2-f^2}\cdot[f']\,dx = \frac{1}{2a}\ln\left\|\frac{a+f}{a-f}\right\| + c$ where f is a function in x • In this case, suppose $f = 2x$, then $f' = 2$ • The actual representation of the formula in MF26. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + c$ where $\|x\| < a$ The domain $\|x\| < a$ is given and ensures $\left(\frac{a+x}{a-x}\right) > 0$, making $\ln\left(\frac{a+x}{a-x}\right)$ well defined. • In general when the limits in integral is not provided, use the absolute sign for the expression in logarithm i.e. $\ln\left\|\frac{a+f}{a-f}\right\|$ to ensure the log expression is well defined. Alternatively $\int \frac{1}{1-4x^2}\,dx$ $= \int \frac{1}{(1-2x)(1+2x)}\,dx$ $= \int \frac{1/2}{1+2x} + \frac{1/2}{1-2x}\,dx$ ← Partial Fractions $= \frac{1}{2}\left[\frac{1}{2}\ln\|1+2x\| - \frac{1}{2}\ln\|1-2x\|\right] + c$ Absolute value on logarithm $= \frac{1}{4}\ln\left\|\frac{1+2x}{1-2x}\right\| + c$	
(iii)	$\frac{dy}{dx} = xe^x - 4xy^2e^x = xe^x(1-4y^2)$ Factorize out all the x terms, so that (x terms) and (y terms) are separated. $\int \frac{1}{1-4y^2}\frac{dy}{dx}\,dx = \int xe^x\,dx$ • Move all the (y terms) to the same side as $\frac{dy}{dx}$ • Integrate both side w.r.t x. $\int \frac{1}{1-4y^2}\,dy = \int xe^x\,dx$ $\frac{1}{4}\ln\left\|\frac{1+2y}{1-2y}\right\| = xe^x - e^x + c$	

2 Solution [12] I[redacted]

(i)



Volume of solid formed $= \pi \int_0^{\frac{3\sqrt{3}}{2}} y^2 \, dx$ ← ② : Apply appropriate formula.

$$= \pi \int_0^{\frac{3\sqrt{3}}{2}} \left(\frac{x^{\frac{3}{2}}}{\left(4x^2+9\right)^{\frac{3}{4}}} \right)^2 dx$$

← ③ : Decide whether exact value or ans is to be rounded off to certain degree of accuracy ie. 3 dec place / 3 significant figures.
• Then decide whether actual integration OR GC is to be used to get the ans.

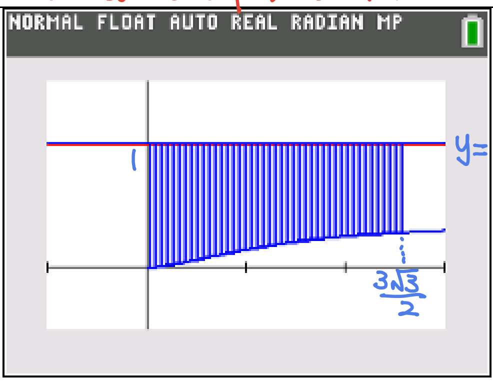
$= 0.09375\pi$

GC Step: $\boxed{\text{ALPHA}}$ $\boxed{\text{fnint}}$

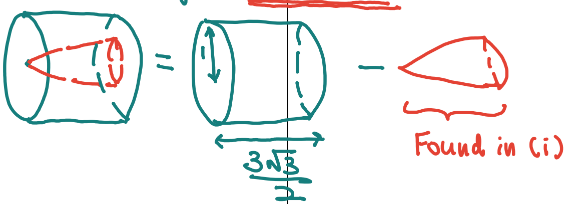
$= 0.295$ units^3 (to 3sf)

Qn asked for ans to be in 3.s.f ⇒ use GC.

(ii)



✻ Sketch appropriate diagrams to help you visualize



Volume of solid formed

=Volume of cylinder – volume of the solid in (i)

$$= \pi(1)^2\left(\frac{3\sqrt{3}}{2}\right) - 0.09375\pi$$

$= 7.86757$

$= 7.87$ units^3 (to 3sf)

3	
(i)	Show: This is a proving Q so this statement must be clearly written. $\dfrac{dQ}{dt} = \text{rate in} - \text{rate out}$ $= (9)\left(\dfrac{1}{5}\right) - (6)\left(\dfrac{Q}{600+3t}\right)$ The multiplication of (9) with $(\frac{1}{5})$ for the rate in quantity must be clearly shown here. Same for multiplication of (6) with $\frac{Q}{600+3t}$ for "rate out". Note: Volume of solution in tank after time t = Initial Amt + Net flow in after t hours. $= 600 + (9-6)t$ (9: flow in amt in an hour; 6: flow out amt in an hour) $= \dfrac{9}{5} - \dfrac{2Q}{200+t}$ $\Rightarrow \dfrac{dQ}{dt} + \dfrac{2Q}{200+t} = \dfrac{9}{5}$ (shown)
(ii)	$u = (200+t)^2 Q$ ← This is the given expression to use. The expression to prove, involves $\frac{du}{dt}$ ∴ Differentiate with respect to t $\dfrac{du}{dt} = (200+t)^2 \dfrac{dQ}{dt} + 2Q(200+t)$ ← Application of product rule, Q is a variable in terms of t, NOT a CONSTANT Common Mistake: $u = (200+t)^2 Q$, $\frac{du}{dt} = 2(200+t)Q$ } Treating Q as a constant ✗ $\dfrac{1}{(200+t)^2}\dfrac{du}{dt} = \boxed{\dfrac{dQ}{dt} + \dfrac{2Q}{200+t}}$ ← Observe that this is the RHS expression of $\frac{dQ}{dt} + \frac{2Q}{200+t} = \frac{9}{5}$ in (i) Sub into differential equation $\dfrac{dQ}{dt} + \dfrac{2Q}{200+t} = \dfrac{9}{5}$ $\dfrac{1}{(200+t)^2}\dfrac{du}{dt} = \dfrac{9}{5}$ $\dfrac{du}{dt} = \dfrac{9}{5}(200+t)^2$ (shown) Alternatively, $\frac{dQ}{dt} + \frac{2Q}{200+t} = \frac{9}{5}$ — (*) $u = (200+t)^2 Q$ ← The idea is to replace $\frac{dQ}{dt}$ and Q in (*) with $\frac{du}{dt}$ and u. $Q = \dfrac{u}{(200+t)^2}$ (1) $\dfrac{dQ}{dt} = \dfrac{1}{(200+t)^2}\dfrac{du}{dt} - \dfrac{2}{(200+t)^3}u$ (2) Sub (1) and (2) into $\dfrac{dQ}{dt} + \dfrac{2Q}{200+t} = \dfrac{9}{5}$ $\dfrac{1}{(200+t)^2}\dfrac{du}{dt} - \dfrac{2}{(200+t)^3}u + \dfrac{2}{(200+t)^3}u = \dfrac{9}{5}$ $\dfrac{1}{(200+t)^2}\dfrac{du}{dt} = \dfrac{9}{5}$ $\dfrac{du}{dt} = \dfrac{9}{5}(200+t)^2$ (shown)

(iii)	$\dfrac{\mathrm{d}u}{\mathrm{d}t}=\dfrac{9}{5}(200+t)^2$ $u=\dfrac{9}{5}\dfrac{(200+t)^3}{3}+c$ $=\dfrac{3}{5}(200+t)^3+c$ $(200+t)^2Q=\dfrac{3}{5}(200+t)^3+c$ $Q=\dfrac{3}{5}(200+t)+\dfrac{c}{(200+t)^2}$ When $t = 0$, $Q = 5$. $5=\dfrac{3}{5}(200)+\dfrac{c}{(200)^2}$ $c=-4600000$ $Q=\dfrac{3}{5}(200+t)-\dfrac{4600000}{(200+t)^2}$ Tank is full when $t = 300$ $\dfrac{\text{Capacity} - \text{initial Amt}}{\text{Amt that flow in per hour}}=\dfrac{1500-600}{3}$ $Q=\dfrac{3}{5}(500)-\dfrac{4600000}{(500)^2}=281.6$ kg	

Question 4

Alice has 3 red, 3 blue and 3 yellow dominoes.

(a) Find the number of ways to arrange the dominoes in a line if

(i) there are no restrictions,

(ii) the red dominoes cannot be all together.

4	Solution [8] P & C	
(a) (i)	Number of ways $=\dfrac{9!}{3!3!3!}=1680$	
(ii)	Number of ways $=1680-\dfrac{7!}{3!3!}=1540$, Alternatively, Case1: red dominoes are all separated Number of ways $=\dfrac{6!}{3!3!}\times\dbinom{7}{3}=700$, Case2: 2 red dominoes are together with the other one separated from the group of 2. Number of ways $=\dfrac{6!}{3!3!}\times\dbinom{7}{2}\times 2!=840$, Total number of ways $=700+840=1540$	Common Mistake: Cannot all together, is generally not just all separated. Reference alternative solution.