2024 NJC SH1 H2 Maths Promotional Examinations Suggested Solutions

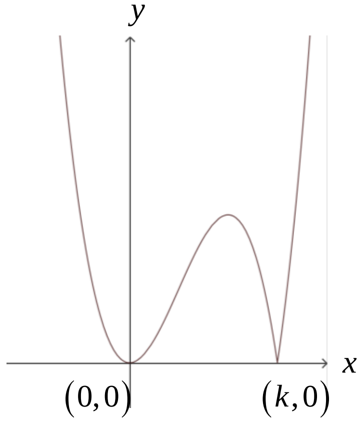
Question 1 (Inequality)

Suggested Solution
$\dfrac{x+3}{x^2+x-2} \geq -1$ $\dfrac{x+3}{x^2+x-2} + \dfrac{x^2+x-2}{x^2+x-2} \geq 0$ $\dfrac{x^2+x-2+x+3}{x^2+x-2} \geq 0$ $\dfrac{x^2+2x+1}{x^2+x-2} \geq 0$ $\dfrac{(x+1)^2}{(x-1)(x+2)} \geq 0$ $x<-2$ or $x>1$ or $x=-1$

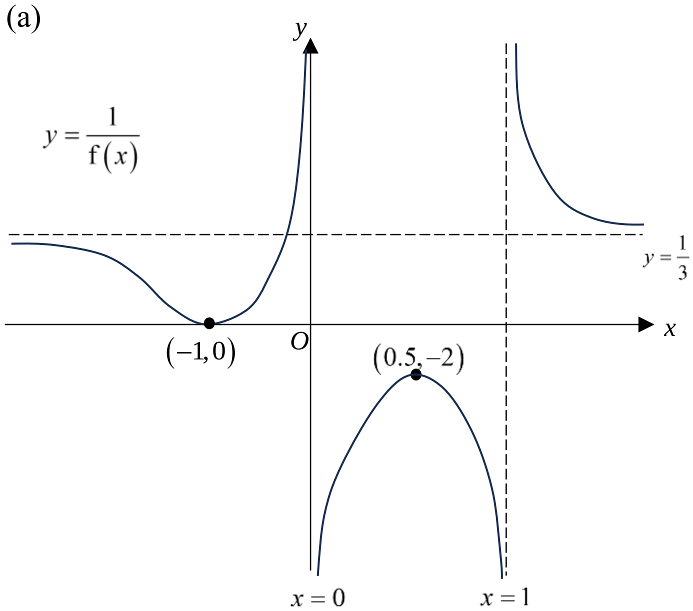
Question 2 (Applications of Integration)

Suggested Solution
$\int y^2 \mathrm{d}x$ $=\int x^2 \sin x \,\mathrm{d}x$ $=-x^2\cos x-\int 2x(-\cos x)\,\mathrm{d}x$ $=-x^2\cos x+2\int x(\cos x)\,\mathrm{d}x$ $=-x^2\cos x+2\left[x\sin x-\int \sin x\,\mathrm{d}x\right]$ $=-x^2\cos x+2\left[x\sin x+\cos x\right]$ $=-x^2\cos x+2x\sin x+2\cos x$ $=\left(2-x^2\right)\cos x+2x\sin x+c$ Volume required $=\pi\int_0^{\pi} y^2\mathrm{d}x$ $=\pi\left[\left(2-x^2\right)\cos x+2x\sin x\right]_0^{\pi}$ $=\pi\left[\left(2-\pi^2\right)(-1)-(2)1\right]$ $=\pi^3-4\pi$

Question 3 (Applications of Integration, Transformation)

Suggested Solution
(i) 
(ii) $\int_0^{2k} \left\| x^2(x-k) \right\| \mathrm{d}x$ $= \int_0^{2k} \left\| x^3 - kx^2 \right\| \mathrm{d}x$ $= -\int_0^{k} x^3 - kx^2 \ \mathrm{d}x + \int_k^{2k} x^3 - kx^2 \ \mathrm{d}x$ $= -\left[\frac{x^4}{4} - k\frac{x^3}{3} \right]_0^k + \left[\frac{x^4}{4} - k\frac{x^3}{3} \right]_k^{2k}$ $= \frac{3}{2}k^4$

Question 4 (Transformation)

Suggested Solution
(a) 

(b) $y=-\dfrac{x^2}{6x+45}$

$\rightarrow y=-\dfrac{(-x)^2}{6(-x)+45}=-\dfrac{x^2}{-6x+45}$

$\rightarrow y=-\dfrac{(3x)^2}{-6(3x)+45}=-\dfrac{9x^2}{-18x+45}=-\dfrac{x^2}{-2x+5}$

$\rightarrow y=-\dfrac{(x+2)^2}{-2(x+2)+5}=-\dfrac{(x+2)^2}{-2x-4+5}=\dfrac{(x+2)^2}{2x-1}$

$\mathrm{h}(x)=\dfrac{(x+2)^2}{2x-1}$

Question 5 (Vectors II)

Suggested Solution

(i) Equation of l_1:

$$\begin{pmatrix}7\\6\\5\end{pmatrix}-\begin{pmatrix}-5\\-2\\-5\end{pmatrix}=\begin{pmatrix}12\\8\\10\end{pmatrix}=2\begin{pmatrix}6\\4\\5\end{pmatrix}$$

$$\mathbf{r}=\begin{pmatrix}7\\6\\5\end{pmatrix}+\lambda\begin{pmatrix}6\\4\\5\end{pmatrix},\lambda\in\mathbf{R}.$$

From line l_2:

$$x-5=\frac{y-10}{k}=\frac{z-8}{2}(=\mu)$$

$$\left.\begin{aligned}x&=5+\mu\\y&=k\mu+10\\z&=2\mu+8\end{aligned}\right\}\mathbf{r}=\begin{pmatrix}5\\10\\8\end{pmatrix}+\mu\begin{pmatrix}1\\k\\2\end{pmatrix},\mu\in\mathbf{R}$$

Solving simultaneously:

$$\begin{pmatrix}7+6\lambda\\6+4\lambda\\5+5\lambda\end{pmatrix}=\begin{pmatrix}5+\mu\\10+k\mu\\8+2\mu\end{pmatrix}$$

$\Rightarrow \lambda=-1,\mu=-4,k=2.$

Sub $\lambda=-1$ into $x=7+6\lambda, y=6+4\lambda, z=5+5\lambda$:

The coordinates of P are $(1,2,0)$.

(ii) Method 1 (Projection)

Let the point on l_2 closest to A be F.

$$\overrightarrow{PA} = \overrightarrow{OA} - \overrightarrow{OP} = \begin{pmatrix} 7 \\ 6 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix}$$

$\overrightarrow{PF}$

$$= \overrightarrow{PA} \cdot \frac{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}}{\sqrt{1^2+2^2+2^2}} \left(\frac{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}}{\sqrt{1^2+2^2+2^2}} \right)$$

$$= \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix} \cdot \frac{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}}{\sqrt{1^2+2^2+2^2}} \left(\frac{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}}{\sqrt{1^2+2^2+2^2}} \right)$$

$$= \frac{8}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 8/3 \\ 16/3 \\ 16/3 \end{pmatrix}$$

$$\overrightarrow{OF} = \overrightarrow{OP} + \overrightarrow{PF} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 8/3 \\ 16/3 \\ 16/3 \end{pmatrix} = \begin{pmatrix} 11/3 \\ 22/3 \\ 16/3 \end{pmatrix}$$

Method 2 (Using perpendicular directions)

Let the point on l_2 closest to A be F.

Since F lies on l_2, $\overrightarrow{OF} = \begin{pmatrix} 5+\mu \\ 10+2\mu \\ 8+2\mu \end{pmatrix}$ for some $\mu \in \mathbf{R}$.

$$\overrightarrow{AF} = \overrightarrow{OF} - \overrightarrow{OA} = \begin{pmatrix} 5+\mu \\ 10+2\mu \\ 8+2\mu \end{pmatrix} - \begin{pmatrix} 7 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} -2+\mu \\ 4+2\mu \\ 3+2\mu \end{pmatrix}$$

Since $\overrightarrow{AF} \perp l_2$, we have $\overrightarrow{AF} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 0.$

$\begin{pmatrix}-2+\mu\\4+2\mu\\3+2\mu\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=0$ $-2+\mu+8+4\mu+6+4\mu=0$ $9\mu=-12$ $\mu=-\frac{4}{3}$ $\overrightarrow{OF}=\begin{pmatrix}5-\frac{4}{3}\\10+2\left(-\frac{4}{3}\right)\\8+2\left(-\frac{4}{3}\right)\end{pmatrix}=\begin{pmatrix}\frac{11}{3}\\\frac{22}{3}\\\frac{16}{3}\end{pmatrix}$
(iii) Let point *A'* be the point which is the reflection of *A* in l_2. $\overrightarrow{PF}=\frac{\overrightarrow{PA}+\overrightarrow{PA'}}{2}$ $\begin{pmatrix}8/3\\16/3\\16/3\end{pmatrix}=\left[\begin{pmatrix}6\\4\\5\end{pmatrix}+\overrightarrow{PA'}\right]\div 2$ $\overrightarrow{PA'}=2\times\begin{pmatrix}8/3\\16/3\\16/3\end{pmatrix}-\begin{pmatrix}6\\4\\5\end{pmatrix}=\begin{pmatrix}-2/3\\20/3\\17/3\end{pmatrix}=\frac{1}{3}\begin{pmatrix}-2\\20\\17\end{pmatrix}$ Equation of reflected line: $\mathbf{r}=\begin{pmatrix}1\\2\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\20\\17\end{pmatrix},\mu\in\mathbf{R}$.

Question 6 (Curve Sketching, Applications of Differentiation)

Suggested Solution
(i) $x^2y^2+3xy+2y^2-x-2=0$ $2xy^2+2y\frac{\mathrm{d}y}{\mathrm{d}x}x^2+3y+3x\frac{\mathrm{d}y}{\mathrm{d}x}+4y\frac{\mathrm{d}y}{\mathrm{d}x}-1=0$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2xy^2+1-3y}{2x^2y+3x+4y}$
(ii) When $x=0, y=1$ or $y=-1$. At $(0,1)$: $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2}$ At $(0,-1)$: $\frac{\mathrm{d}y}{\mathrm{d}x}=-1$
(iii) Let $x=k$. $(k^2+2)y^2+3ky-(k+2)=0$ $9k^2-4(k^2+2)[-(k+2)]<0$ $9k^2+4k^3+8k^2+8k+16<0$ $4k^3+17k^2+8k+16<0$ $(k+4)(4k^2+k+4)<0$ $4k^2+k+4=4\left[k^2+\frac{1}{4}k+\left(\frac{1}{8}\right)^2-\left(\frac{1}{8}\right)^2\right]+4$ $=4\left(k+\frac{1}{8}\right)^2-\frac{1}{16}+4$ $=4\left(k+\frac{1}{8}\right)^2+\frac{63}{16}>0$ for all real x. or Discriminant of $4k^2+k+4=0$ is $1^2-4(4)4=-63<0$ Since coefficient of $k^2=4>0$, $4k^2+k+4>0$ for all real x. Since $4k^2+k+4>0$ for all real x, we need $k+4<0$. We have $k<-4$. The curve does not intersect the vertical line $x=k$ when $k<-4$, thus has no parts where $x<-4$.

Question 7 (Vectors I and II)

Suggested Solution
(a) (i) p <u>is perpendicular to</u> **q** <u>or</u> **p** <u>is a zero vector</u> or **q** <u>is a zero vector.</u>
(a) (ii) The locus of *R* is a circle with diameter *OG*.
(b) $\begin{pmatrix}1\\-2\\2\end{pmatrix}\times\begin{pmatrix}2\\3\\-2\end{pmatrix}=\begin{pmatrix}-2\\6\\7\end{pmatrix}$ $\overrightarrow{QR}=\overrightarrow{OR}-\overrightarrow{OQ}=\begin{pmatrix}1\\2\\2\end{pmatrix}$ Projection of $\overrightarrow{QR}$ onto p $=\left\|\begin{pmatrix}1\\2\\2\end{pmatrix}\times\frac{\begin{pmatrix}-2\\6\\7\end{pmatrix}}{\sqrt{(-2)^2+6^2+7^2}}\right\|$ $=\left\|\frac{\begin{pmatrix}2\\-11\\10\end{pmatrix}}{\sqrt{89}}\right\|$ $=\frac{15}{\sqrt{89}}$

Question 8 (Applications of Differentiation, SLE)

Suggested Solution

(i) $x = 1$

The stationary point on $y = \mathrm{g}(x)$ with x-coordinate 1 is a minimum point since $\mathrm{g}''(1) > 0$ from the graph

or

x	1^-	1	1^+
$\mathrm{g}'(x)$	< 0	0	> 0
Shape of tangent	\	—	/

(ii) $\mathrm{g}'(-1) = 0$

$\Rightarrow a(-1)^4 + b(-1)^3 + c(-1)^2 + d(-1) + e = 0$

$\Rightarrow a - b + c - d + e = 0$

$\mathrm{g}'(1) = 0 \Rightarrow a + b + c + d + e = 0 \text{ -----}(1)$

We have $\mathrm{g}''(x) = 4ax^3 + 3bx^2 + 2cx + d$.

$\mathrm{g}''\left(\frac{1}{2}\right) = 0$

$\Rightarrow 4a\left(\frac{1}{2}\right)^3 + 3b\left(\frac{1}{2}\right)^2 + 2c\left(\frac{1}{2}\right) + d = 0$

$\Rightarrow \frac{1}{2}a + \frac{3}{4}b + c + d = 0 \text{ -----}(2)$

$\mathrm{g}''(-1) = 0$

$\Rightarrow -4a + 3b - 2c + d = 0 \text{ -----}(3)$

We have $\mathrm{g}'(x) = ax^4 + bx^3 + cx^2 + dx + e$.

$\mathrm{g}'\left(\frac{1}{2}\right) = -\frac{27}{8}$

$\Rightarrow a\left(\frac{1}{2}\right)^4 + b\left(\frac{1}{2}\right)^3 + c\left(\frac{1}{2}\right)^2 + d\left(\frac{1}{2}\right) + e = -\frac{27}{8}$

$\Rightarrow a\left(\frac{1}{16}\right) + b\left(\frac{1}{8}\right) + c\left(\frac{1}{4}\right) + d\left(\frac{1}{2}\right) + e = -\frac{27}{8}$

$\Rightarrow a + 2b + 4c + 8d + 16e = -54 \text{ -----}(4)$

From **(ii)**,

$a - b + c - d + e = 0 \text{ -----}(5)$

By G.C., $a = 2, b = 4, c = 0, d = -4, e = -2$.

Question 9 (Vectors I)

Suggested Solution
(i) Method 1

Area of triangle $ABO = \frac{1}{2}\left|\mathbf{a}\times\mathbf{b}\right|$

$\mathbf{a}+2\mathbf{b}+\lambda\mathbf{c}=\mathbf{0}$

$$\mathbf{c}=-\frac{\mathbf{a}+2\mathbf{b}}{\lambda}$$

$$\overrightarrow{CA}=\mathbf{a}-\mathbf{c}=\mathbf{a}+\frac{\mathbf{a}+2\mathbf{b}}{\lambda}=\frac{(\lambda+1)\mathbf{a}+2\mathbf{b}}{\lambda}$$

and

$$\overrightarrow{CB}=\mathbf{b}-\mathbf{c}=\mathbf{b}+\frac{\mathbf{a}+2\mathbf{b}}{\lambda}=\frac{\mathbf{a}+(\lambda+2)\mathbf{b}}{\lambda}$$

$$\begin{aligned}
\text{Area of }\Delta ABC &= \frac{1}{2}\left|\frac{(\lambda+1)\mathbf{a}+2\mathbf{b}}{\lambda}\times\frac{\mathbf{a}+(\lambda+2)\mathbf{b}}{\lambda}\right|\\
&=\frac{1}{2}\left|\frac{(\lambda+1)\mathbf{a}\times\mathbf{a}+2\mathbf{b}\times\mathbf{a}+(\lambda+1)\mathbf{a}\times(\lambda+2)\mathbf{b}+2\mathbf{b}\times(\lambda+2)\mathbf{b}}{\lambda^2}\right|\\
&=\frac{1}{2}\left|\frac{\mathbf{0}-2(\mathbf{a}\times\mathbf{b})+(\lambda^2+3\lambda+2)(\mathbf{a}\times\mathbf{b})+\mathbf{0}}{\lambda^2}\right|\\
&=\frac{1}{2}\left|\frac{(\lambda^2+3\lambda)(\mathbf{a}\times\mathbf{b})}{\lambda^2}\right|\\
&=\frac{1}{2}\left|\frac{\lambda+3}{\lambda}\right|\left|\mathbf{a}\times\mathbf{b}\right|
\end{aligned}$$

$$\frac{\text{Area of }\Delta ABC}{\text{Area of }\Delta ABO}=\frac{\frac{1}{2}\left|\frac{\lambda+3}{\lambda}\right|\left|\mathbf{a}\times\mathbf{b}\right|}{\frac{1}{2}\left|\mathbf{a}\times\mathbf{b}\right|}=\left|\frac{\lambda+3}{\lambda}\right| \text{ (shown)}$$

Method 2

$\mathbf{a}+2\mathbf{b}+\lambda\mathbf{c}=\mathbf{0}$

$$\mathbf{a}=-2\mathbf{b}-\lambda\mathbf{c}$$

$$\begin{aligned}
\text{Area of triangle }\Delta ABO &= \frac{1}{2}\left|\mathbf{a}\times\mathbf{b}\right|\\
&=\frac{1}{2}\left|(-2\mathbf{b}-\lambda\mathbf{c})\times\mathbf{b}\right|\\
&=\frac{1}{2}\left|\mathbf{0}-\lambda\mathbf{c}\times\mathbf{b}\right|\\
&=\frac{1}{2}\left|\lambda(\mathbf{b}\times\mathbf{c})\right|\\
&=\frac{1}{2}\left|\lambda\right|\left|\mathbf{b}\times\mathbf{c}\right|
\end{aligned}$$

$$\text{Area of } \Delta ABC = \frac{1}{2}\left|\overrightarrow{BA} \times \overrightarrow{BC}\right|$$
$$= \frac{1}{2}\left|(\mathbf{a}-\mathbf{b}) \times (\mathbf{c}-\mathbf{b})\right|$$
$$= \frac{1}{2}\left|(-2\mathbf{b}-\lambda\mathbf{c}-\mathbf{b}) \times (\mathbf{c}-\mathbf{b})\right|$$
$$= \frac{1}{2}\left|(-3\mathbf{b}-\lambda\mathbf{c}) \times (\mathbf{c}-\mathbf{b})\right|$$
$$= \frac{1}{2}\left|-3\mathbf{b}\times\mathbf{c}-\lambda\mathbf{c}\times\mathbf{c}-3\mathbf{b}\times(-\mathbf{b})-\lambda\mathbf{c}\times(-\mathbf{b})\right|$$
$$= \frac{1}{2}\left|-3\mathbf{b}\times\mathbf{c}-\mathbf{0}-\mathbf{0}-\lambda\mathbf{b}\times\mathbf{c}\right|$$
$$= \frac{1}{2}\left|-3-\lambda\right|\left|\mathbf{b}\times\mathbf{c}\right|$$
$$= \frac{1}{2}\left|\lambda+3\right|\left|\mathbf{b}\times\mathbf{c}\right|$$
$$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta ABO} = \frac{\frac{1}{2}\left|\lambda+3\right|\left|\mathbf{b}\times\mathbf{c}\right|}{\frac{1}{2}\left|\lambda\right|\left|\mathbf{b}\times\mathbf{c}\right|} = \left|\frac{\lambda+3}{\lambda}\right| \text{ (shown)}$$

(ii) For A, B and C to be collinear, the area of ΔABC must be 0, so $\lambda = -3$.

(iii)

Method 1

Since D is on AB, by Ratio Theorem, $\overrightarrow{OD} = (1-k)\mathbf{a} + k\mathbf{b}$ when $AD:DB = k:(1-k)$.

Since D lies on the line OC, $\overrightarrow{OD} = m\overrightarrow{OC} = -\frac{m}{\lambda}(\mathbf{a}+2\mathbf{b})$, in which the coefficient of **b** is twice the coefficient of **a**.

$k = 2(1-k)$

$k = \frac{2}{3}$

So $AD:DB = \frac{2}{3}:\left(1-\frac{2}{3}\right) = 2:1$

Method 2

$\mathbf{c} = -\frac{\mathbf{a}+2\mathbf{b}}{\lambda}$

As λ varies, C moves along the same line passing through the origin with direction vector $(\mathbf{a}+2\mathbf{b})$. The line cuts AB at the point D when $\lambda = -3$ from **(ii)** answer.

$\overrightarrow{OD} = -\frac{\mathbf{a}+2\mathbf{b}}{-3} = \frac{\mathbf{a}+2\mathbf{b}}{3}$

By the Ratio Theorem, D divides AB such that $AD:DB = 2:1$

(iv)

$$\angle OAC = 90^\circ$$

$$\overrightarrow{OA} \cdot \overrightarrow{CA} = 0$$

$$\mathbf{a} \cdot \frac{(\lambda+1)\mathbf{a} + 2\mathbf{b}}{\lambda} = 0$$

$$(\lambda+1)\mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot (2\mathbf{b}) = 0$$

$$(\lambda+1)|\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} = 0 \qquad \text{...(1)}$$

$$\angle OBC = 90^\circ$$

$$\overrightarrow{OB} \cdot \overrightarrow{CB} = 0$$

$$\mathbf{b} \cdot \frac{\mathbf{a} + (\lambda+2)\mathbf{b}}{\lambda} = 0$$

$$\mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \left[(\lambda+2)\mathbf{b}\right] = 0$$

$$\mathbf{a} \cdot \mathbf{b} + (\lambda+2)|\mathbf{b}|^2 = 0 \qquad \text{...(2)}$$

To eliminate $\mathbf{a} \cdot \mathbf{b}$, $(1) - (2) \times 2$,

$$(\lambda+1)|\mathbf{a}|^2 - 2(\lambda+2)|\mathbf{b}|^2 = 0$$

$$\lambda|\mathbf{a}|^2 + |\mathbf{a}|^2 - 2\lambda|\mathbf{b}|^2 - 4|\mathbf{b}|^2 = 0$$

$$\left(|\mathbf{a}|^2 - 2|\mathbf{b}|^2\right)\lambda = 4|\mathbf{b}|^2 - |\mathbf{a}|^2$$

$$\lambda = \frac{4|\mathbf{b}|^2 - |\mathbf{a}|^2}{|\mathbf{a}|^2 - 2|\mathbf{b}|^2}$$

Question 10 (Functions)

Suggested Solution

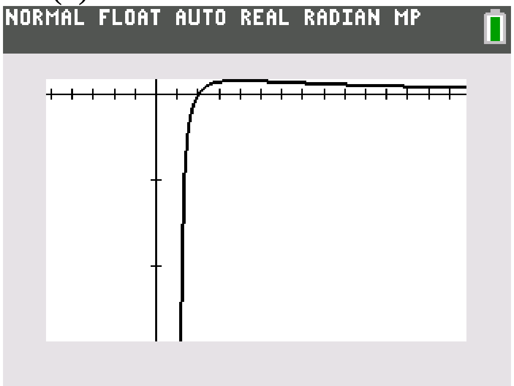
(a) (i) $R_g = (-1, \infty) \not\subseteq (1, \infty) = D_f$

fg does not exist.

$$f(x) = \frac{\sqrt{2}(x-2)}{(x+2)(x-1)}, \ x > 1$$

(a) (ii)

(b)



$$f'(x) = \frac{\sqrt{2}(x^2+x-2) - \sqrt{2}(2x+1)(x-2)}{(x^2+x-2)^2} = 0$$

$$\sqrt{2}(x^2+x-2) - \sqrt{2}(2x+1)(x-2) = 0$$

$$x^2+x-2-(2x^2-3x-2) = 0$$

$$x^2 - 4x = 0$$

$$\Rightarrow x = 0 \text{ or } x = 4$$

$x = 1$

$\left(4, \frac{\sqrt{2}}{9}\right)$ is a maximum turning point. Thus $R_f = \left(-\infty, \frac{\sqrt{2}}{9}\right]$.

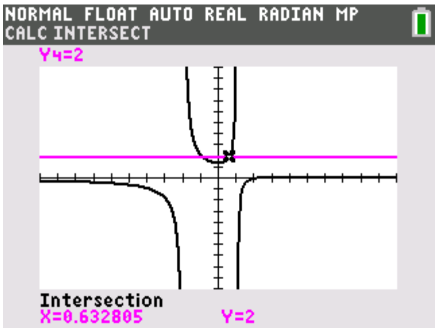
$$R_f = \left(-\infty, \frac{\sqrt{2}}{9}\right] \xrightarrow{g} R_{gf} = \left[e^{-\frac{\sqrt{2}}{9}} - 1, \infty\right)$$

(a) (iii)

Method 1 (Solving $f^{-1}(2)=g(k)$**)**

Let $g^{-1}f^{-1}(2)=k$.

$f^{-1}(2)=g(k)$



Solving for $f(\alpha)=2$, we get $\alpha=0.632805$ (since $0<\alpha<1$).

We note that $f(\alpha)=2 \Rightarrow f^{-1}(2)=\alpha$

Hence $f(0.632805)=2 \Rightarrow f^{-1}(2)=0.632805$

From $f^{-1}(2)=g(k)$, we have $0.632805=g(k)$.

$0.632805=e^{-k}-1$

$\therefore k=-0.490$ (3 s.f.).

Method 2 (Solving $fg(k)=2$**)**

Let $g^{-1}f^{-1}(2)=k$.

$f^{-1}(2)=g(k)$
$fg(k)=2$
$f\left(e^{-k}-1\right)=2$

$$\frac{\sqrt{2}\left(e^{-k}-1\right)-2\sqrt{2}}{\left(e^{-k}-1\right)^2+\left(e^{-k}-1\right)-2}=2$$

From GC, $k=-0.490$ (3 s.f.) or 2.60 (3 s.f.)

Justification based on $f\left[g(k)\right]=2$**:**

$g(-0.490)=e^{-(-0.490)}-1\approx 0.632\in D_f=(0,1)$

$g(2.60)=e^{-(2.60)}-1\approx -0.926\notin D_f=(0,1)$

Or justification based on $f^{-1}(2)=g(k)$**:**

$R_{f^{-1}}=D_f=(0,1)$

$g(2.60) \approx -0.926 \notin R_{f^{-1}}$.

$g(-0.490) \approx 0.632 \in R_{f^{-1}}$

So we have $k = -0.490$ (3 s.f.).

Method 3 (Solving by finding $f^{-1}(x)$) – refer to Remarks

Let $g^{-1}f^{-1}(2) = k$.

$f^{-1}(2) = g(k)$

To find f^{-1}:

$y(x^2 + x - 2) = \sqrt{2}x - 2\sqrt{2}$

$yx^2 + (y - \sqrt{2})x + (2\sqrt{2} - 2y) = 0$

$$x = \frac{-(y-\sqrt{2}) \pm \sqrt{(y-\sqrt{2})^2 - 4y(2\sqrt{2} - 2y)}}{2y}$$

$$= \frac{-y+\sqrt{2} \pm \sqrt{(y-\sqrt{2})^2 + 8y(y-\sqrt{2})}}{2y}$$

$$= \frac{-y+\sqrt{2} \pm \sqrt{(y-\sqrt{2})\left[(y-\sqrt{2}) + 8y\right]}}{2y}$$

$$= \frac{-y+\sqrt{2} \pm \sqrt{(y-\sqrt{2})(9y-\sqrt{2})}}{2y}$$

$$= \frac{-y+\sqrt{2}}{2y} \pm \frac{\sqrt{(y-\sqrt{2})(9y-\sqrt{2})}}{2y}$$

From $R_f = (\sqrt{2}, \infty)$, we have $y > \sqrt{2}$.

$$-y < -\sqrt{2} \Rightarrow -y + \sqrt{2} < 0 \Rightarrow \frac{-y+\sqrt{2}}{2y} < 0$$

Since $0 < x < 1$, we need $x = \dfrac{-y+\sqrt{2}}{2y} + \dfrac{\sqrt{(y-\sqrt{2})(9y-\sqrt{2})}}{2y}$.

$$\therefore f^{-1}(x) = \frac{-x+\sqrt{2} + \sqrt{(x-\sqrt{2})(9x-\sqrt{2})}}{2x}$$

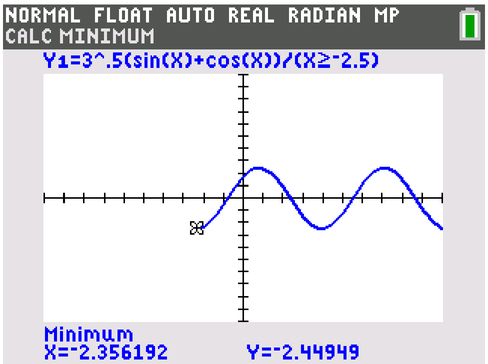
$$\mathrm{f}^{-1}(2)=\mathrm{g}(k)$$

$$\frac{-2+\sqrt{2}+\sqrt{\left(2-\sqrt{2}\right)\left(9(2)-\sqrt{2}\right)}}{2(2)}=\mathrm{e}^{-k}-1$$

$$0.632804984774=\mathrm{e}^{-k}-1$$

$k=-0.490$ (3 s.f.)

(b) (i) $\mathrm{h}: x \mapsto \sqrt{3}\sin x+\sqrt{3}\cos x, \qquad x \in \mathbb{R},\ -\frac{5}{2} \le x \le p$



To find the exact x-coordinate of the minimum point:

$$\mathrm{h}'(x)=0$$

$$\sqrt{3}\cos x-\sqrt{3}\sin x=0$$

$$\tan x=1$$

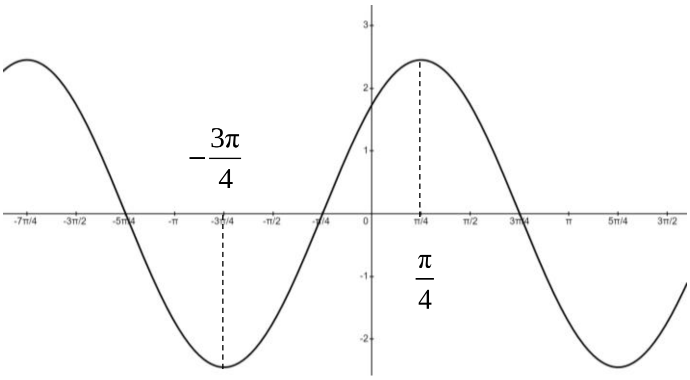
$$\text{basic angle}=\frac{\pi}{4}$$

Based on the graph, the minimum point has x-coordinate $=-\pi+\frac{\pi}{4}=-\frac{3\pi}{4}$ (≈ -2.36).

Since $-2.5<-2.36$, the largest value of p for h^{-1} to exist is $-\frac{3\pi}{4}$.

Alternative:

By R-formula, $y=\sqrt{3}\sin x+\sqrt{3}\cos x=\sqrt{6}\sin\left(x+\frac{\pi}{4}\right)$.



For h^{-1} to exist, we need h to be 1-1.

Since $-2.5<-\frac{3\pi}{4}\approx -2.36$, the largest p for h^{-1} to exist is $-\frac{3\pi}{4}$.

(b) (ii)

$$h : x \mapsto \sqrt{3}\sin x + \sqrt{3}\cos x, x \in \mathbb{R}, -\frac{5}{2} \le x \le -\frac{3\pi}{4}$$

$D_{hh^{-1}} = D_{h^{-1}}$

$= R_h$

$= \left[-\sqrt{6}, \sqrt{3}\sin(-2.5) + \sqrt{3}\cos(-2.5)\right]$

$\approx [-2.44949, -2.42421]$

$D_{h^{-1}h} = D_h \approx [-2.5, -2.35619]$

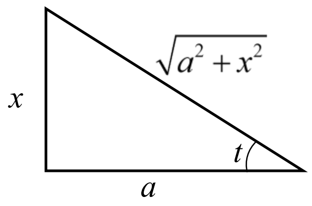
Number line:

For $hh^{-1}(x) = h^{-1}h(x)$, $x \in [-2.44949, -2.42421]$

To 3 decimal places: $x \in [-2.449, -2.424]$

Question 11 (Integration Techniques)

Suggested Solution
(a) (i) Let $\frac{13x-2}{(3-x)(1+4x^2)}=\frac{A}{3-x}+\frac{Bx+C}{1+4x^2}$. Then $$13x-2=A(1+4x^2)+(Bx+C)(3-x)$$ Substitute $x=3$, $$13(3)-2=A\left[1+4(3)^2\right]$$ $$37A=37$$ $$A=1$$ Substitute $x=0$ and $A=1$, $$-2=1(1)+3C$$ $$3C=-3$$ $$C=-1$$ Substitute $x=1$, $A=1$ and $C=-1$, $$13(1)-2=(1)\left[1+4(1)^2\right]+2(B-1)$$ $$2(B-1)=6$$ $$B=4$$ Therefore, $\frac{13x-2}{(3-x)(1+4x^2)}=\frac{1}{3-x}+\frac{4x-1}{1+4x^2}$.
(a) (ii) $\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{13x-2}{(3-x)(1+4x^2)}\ \mathrm{d}x$ $=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{3-x}+\frac{4x-1}{1+4x^2}\ \mathrm{d}x$ $=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{3-x}\ \mathrm{d}x+\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{4x}{1+4x^2}\ \mathrm{d}x-\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{1+4x^2}\ \mathrm{d}x$ $=-\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{-1}{3-x}\ \mathrm{d}x+\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{8x}{1+4x^2}\ \mathrm{d}x-\frac{1}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{\left(\frac{1}{2}\right)^2+x^2}\ \mathrm{d}x$ $=\left(-\ln\lvert 3-x\rvert\right)_{-\frac{1}{2}}^{\frac{1}{2}}+\frac{1}{2}\left(\ln\lvert 1+4x^2\rvert\right)_{-\frac{1}{2}}^{\frac{1}{2}}-\frac{1}{4}\left(2\tan^{-1}2x\right)_{-\frac{1}{2}}^{\frac{1}{2}}$ $=-\ln\frac{5}{2}+\ln\frac{7}{2}+\frac{1}{2}\ln 2-\frac{1}{2}\ln 2-\frac{1}{2}\left[\tan^{-1}1-\tan^{-1}(-1)\right]$

$= \ln\left(\frac{\frac{7}{2}}{\frac{5}{2}}\right) - \frac{1}{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$ $= \ln\frac{7}{5} - \frac{\pi}{4}$
(b) $x = a\tan t \Rightarrow \frac{dx}{dt} = a\sec^2 t$ $\int \frac{a^2 - x^2}{\left(a^2 + x^2\right)^2}\ dx$ $= \int \frac{a\sec^2 t\left(a^2 - a^2\tan^2 t\right)}{\left(a^2 + a^2\tan^2 t\right)^2}\ dt$ $= \int \frac{a^3\sec^2 t\left(1 - \tan^2 t\right)}{a^4\left(1 + \tan^2 t\right)^2}\ dt$ $= \int \frac{\sec^2 t\left(1 - \tan^2 t\right)}{a\sec^4 t}\ dt$ $= \frac{1}{a}\int \cos^2 t - \sin^2 t\ \ dt$  $= \frac{1}{a}\int \cos 2t\ \ dt$ $= \frac{1}{2a}\sin 2t + c$ $= \frac{1}{a}\sin t\cos t + c$ $= \frac{1}{a}\left(\frac{x}{\sqrt{a^2 + x^2}}\right)\left(\frac{a}{\sqrt{a^2 + x^2}}\right) + c$ $= \frac{x}{a^2 + x^2} + c$, where c is an arbitrary constant

Question 12 (Applications of Differentiation)

Suggested Solution

(i) Translation in the positive *y*-direction by $3\sqrt{6}$ units.

$y^2 - x^2 = 1$ has stationary points at $(0,1)$ and $(0,-1)$ and equation of asymptotes $y = x$ and $y = -x$.

For *D*, equations of asymptotes are $y = x + 3\sqrt{6}$ and $y = -x + 3\sqrt{6}$ and $S\left(0, -1 + 3\sqrt{6}\right)$.

(ii) Sub $x = \tan p, y = \sec p + 3\sqrt{6}$ into LHS of the equation of *D*:

$$\left(y - 3\sqrt{6}\right)^2 - x^2$$
$$= \left(\sec p + 3\sqrt{6} - 3\sqrt{6}\right)^2 - (\tan p)^2$$
$$= \sec^2 p - \tan^2 p$$
$$= 1$$

(iii) **Method 1**

$$\left(y - 3\sqrt{6}\right)^2 - x^2 = 1$$

Differentiate w.r.t *x*

$$2\left(y - 3\sqrt{6}\right)\frac{\mathrm{d}y}{\mathrm{d}x} - 2x = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y - 3\sqrt{6}}$$

Gradient of normal $= -\dfrac{1}{\dfrac{\mathrm{d}y}{\mathrm{d}x}} = -\dfrac{y - 3\sqrt{6}}{x}$

At *P*, gradient of normal

$$= -\frac{\left(\sec p + 3\sqrt{6}\right) - 3\sqrt{6}}{\tan p}$$
$$= -\frac{\sec p}{\tan p}$$
$$= -\frac{1}{\sin p}$$

Method 2

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}p}}{\frac{\mathrm{d}x}{\mathrm{d}p}} = \frac{\tan p \sec p}{\sec^2 p} = \sin p$$

Gradient of normal $= -\dfrac{1}{\sin p} = -\operatorname{cosec} p$

Equation of normal at *P*:

$$\frac{y - \sec p - 3\sqrt{6}}{x - \tan p} = -\frac{1}{\sin p}$$

$\Rightarrow y-\sec p-3\sqrt{6}=\left(-\frac{1}{\sin p}\right)x+\frac{\tan p}{\sin p}$

$\Rightarrow y=\left(-\frac{1}{\sin p}\right)x+2\sec p+3\sqrt{6}$

$\Rightarrow (\sin p)\,y+x=2\tan p+3\sqrt{6}\sin p$

(iv) Method 1: Cartesian Form

Gradient of $r=-\frac{y-3\sqrt{6}}{x}$

Gradient of $r=-1\div-\frac{1}{\sqrt{3}}=\sqrt{3}$

Equating the above:

$\sqrt{3}=-\frac{y-3\sqrt{6}}{x}$

$y=3\sqrt{6}-\sqrt{3}x$

Sub $y=3\sqrt{6}-\sqrt{3}x$ into $\left(y-3\sqrt{6}\right)^2-x^2=1$:

$\left(3\sqrt{6}-\sqrt{3}x-3\sqrt{6}\right)^2-x^2=1$

$\left(-\sqrt{3}x\right)^2-x^2=1$

$2x^2=1$

$x^2=\frac{1}{2}$

$x=\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$ (rejected $\because x>0$)

Sub $x=\sqrt{\frac{1}{2}}$ into $y=3\sqrt{6}-\sqrt{3}x$:

$y=3\sqrt{6}-\sqrt{3}\left(\frac{1}{\sqrt{2}}\right)=3\sqrt{6}-\frac{\sqrt{3}}{\sqrt{2}}$

Coordinates of the point

$\left(\frac{1}{\sqrt{2}},3\sqrt{6}-\frac{\sqrt{3}}{\sqrt{2}}\right)$ or $(0.707,6.12)$

(iv) Method 2: Parametric Form

We have $-\operatorname{cosec} p=\sqrt{3}\Rightarrow \sin p=-\frac{1}{\sqrt{3}}$

- Since $\sin p=-\frac{1}{\sqrt{3}}$, p is in the 3rd or 4th quadrant.
- Since we are looking at the lower arc of D, $\frac{\pi}{2}<p\le\pi$ or $-\pi<p<-\frac{\pi}{2}$ i.e. p lies in the 2nd or 3rd quadrant.

Hence, p lies in the 3rd quadrant.

Method 2A

Since p lies in the 3rd quadrant (only tangent is positive in this quadrant),

- $\cos p = -\frac{\sqrt{2}}{\sqrt{3}} \Rightarrow \sec p = -\frac{\sqrt{3}}{\sqrt{2}}$
- $\tan p = \frac{\sin p}{\cos p} = \frac{-\frac{1}{\sqrt{3}}}{-\frac{\sqrt{2}}{\sqrt{3}}} = \frac{1}{\sqrt{2}}$

Using the parametric form $\left(\tan p, \sec p + 3\sqrt{6}\right)$ and $\sec p = -\frac{\sqrt{3}}{\sqrt{2}}$ and $\tan p = \frac{1}{\sqrt{2}}$,

coordinates of the intersection of r and lower piece of D are:

$$\left(\frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{\sqrt{2}} + 3\sqrt{6}\right)$$

Method 2B

We have $-\operatorname{cosec} p = \sqrt{3} \Rightarrow \sin p = -\frac{1}{\sqrt{3}}$

Basic angle $= \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) = 0.61547970867 = 0.615480$ (6 s.f.)

Since p is on the 3rd quadrant, $p = 0.615480 + \pi = 3.75707$ (6 s.f.)

$\tan p = \tan 3.75707 = 0.707$ (3 s.f.)

$\sec p + 3\sqrt{6} = \sec 3.75707 + 3\sqrt{6} = 6.12373 = 6.12$ (3 s.f.)

Coordinates of the intersection of r & lower piece of D: $(0.707, 6.12)$