F07: Matrices and Linear Spaces

1 **(a)** A square matrix $\mathbf{A}$ is called *orthogonal* if

$$\mathbf{A}\mathbf{A}^T = \mathbf{I} \quad \text{and} \quad \mathbf{A}^T\mathbf{A} = \mathbf{I}.$$

Show that if $\mathbf{A}$ and $\mathbf{B}$ are orthogonal matrices, then $\mathbf{AB}$ is an orthogonal matrix. [2]

(b) Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_k$ be vectors in a vector space V such that

$$\text{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}\} = V.$$

Show that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_k$ are linearly dependent. [2]

(c) **(i)** Let $\mathbf{A}$ be a square matrix and λ is an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{u}$. Show that λ^2 is an eigenvalue of $\mathbf{A}^2$ with corresponding eigenvector $\mathbf{u}$. [2]

(ii) Suppose further that $\mathbf{A}^2 = \mathbf{A}$. Show that if λ is an eigenvalue of $\mathbf{A}$, then $\lambda = 0$ or 1. [2]

(2016 NYJC / JC1 / MYE / Q5)

2 It is given that the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the matrix

$$\mathbf{M} = \begin{pmatrix} a+\frac{3}{8} & -\frac{1}{8} & \frac{1}{8} \\ -\frac{1}{4} & a+\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{8} & \frac{1}{8} & a+\frac{5}{8} \end{pmatrix},$$

are the roots of the equation

$$32(\lambda-a)^3 - 48(\lambda-a)^2 + 22(\lambda-a) - 3 = 0.$$

Find $\lambda_1, \lambda_2, \lambda_3$ in terms of a. [3]

Find matrices $\mathbf{Q}$ and $\mathbf{D}$ such that $\mathbf{M} = \mathbf{QDQ}^{-1}$, where the elements of $\mathbf{Q}$ are independent of a, and $\mathbf{D}$ is a diagonal matrix. [The evaluation of $\mathbf{Q}^{-1}$ is not required. [5]

Find the set of values of a such that all the elements of $\mathbf{M}^n$ tend to zero as $n \to \infty$. [3]

(2016 NYJC / JC1 / MYE / Q6)

3 A linear transformation $L:\mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-2z \\ x+y \\ y+az \end{pmatrix},$$

where a is a real constant.

(a) Determine the conditions for a and b such that the point $(2, -3, b)$ is in the range of L. [3]

(b) Find the value of a such that the null space of L is non-trivial. Find a basis for the range of L and a basis for the null space of L for this value of a. [5]

For this value of a,

(i) show that $(0, 0, 0)$ is the only invariant point under L, [2]

(ii) find, in the form $\mathbf{r} = \mathbf{a} + \mu\mathbf{b}$, an equation for the line whose image under L is the point with position vector $\begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$, [3]

(iii) show that the image under L of the plane $x - y + z = 2$ is the plane $x - y + z = 0$. [3]

(2016 NYJC / JC1 / MYE / Q7)

4 Let P_n be the set of polynomials with real coefficients of the form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n .$$

(i) Assume that P_n is a vector space over $\mathbb{R}$, write down the standard basis for this vector space. [1]

(ii) Let $V = \{p(x) \in P_n : p(0) = 0\}$. Show that V is a subspace of P_n. [3]

Every polynomial with real coefficients of the form $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ can be expressed as a $(n+1)\times 1$ column vector $\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}$. The I-transformation, which acts on elements of P_n, is defined as $I(p) = \int_0^x p(t)\,dt$.

(iii) Show that $I : P_n \mapsto P_{n+1}$ is a linear transformation. Write down the matrix representing I and find a basis for the range space of I. [5]

(2016 NYJC / JC1 / Promo / Q4)

5 **(i)** Find the eigenvalues and eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ of the matrix

$$\mathbf{A}=\begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix}.$$ [4]

(ii) Show that the eigenvectors are mutually perpendicular. [2]

(iii) Let $\mathbf{b}=\begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix}$. By taking scalar products with each of the eigenvectors, find p_1, p_2 and p_3 such that $\mathbf{b}=p_1\mathbf{e}_1+p_2\mathbf{e}_2+p_3\mathbf{e}_3$. [3]

(iv) By writing $\mathbf{x}=q_1\mathbf{e}_1+q_2\mathbf{e}_2+q_3\mathbf{e}_3$ and using the decomposition in **(iii)**, solve the equation $\mathbf{A}\mathbf{x}=\mathbf{b}$. [3]

(2016 NYJC / JC1 / Promo / Q9)

6 Determine the values of a and b for which the linear system below has

$$x+y+7z=-7$$
$$2x+3y+17z=a$$
$$2x+2y+(a^2+1)z=3b$$

(a) no solution,
(b) exactly one solution,
(c) infinitely many solutions.

Interpret the geometrical meaning of the system in part **(a)**. [5]

(2016 HCI / JC1 / Promo / Q3)

7 Let $M_{33}(\mathbb{R})$ denote the linear space of all 3×3 real matrices under the usual addition and multiplication of matrices. Identify which of the following collection of 3×3 matrices form a subspace of $M_{33}(\mathbb{R})$.

(a) All matrices of rank at most 1.
(b) All non-invertible matrices.
(c) Any matrix A satisfying $2A-A^T=O$. [8]

(2016 HCI / JC1 / Promo / Q5)

8 The transformations $\mathrm{T}_1:\mathbb{R}^4\to\mathbb{R}^4$ and $\mathrm{T}_2:\mathbb{R}^4\to\mathbb{R}^4$ are represented by the matrices

$$M_1=\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } M_2=\begin{pmatrix} 1 & 1 & 2 & 0 \\ 2 & 0 & 3 & 1 \\ -1 & -1 & 0 & 2 \\ 3 & 1 & 1 & -3 \end{pmatrix} \text{ respectively.}$$

The null space of T_1 is denoted by K_1 and the range space of T_2 is denoted by R_2.

(i) Find dim(K_1) and a basis for K_1. [3]

(ii) By using a graphing calculator, find dim(R_2). [1]

(iii) With the aid of the result in part **(i)**, find a non-zero vector $\mathbf{x}$ in $\mathbb{R}^4$ such that $M_1M_2\mathbf{x}=\mathbf{0}$. [4]

(2016 HCI / JC1 / Promo / Q6)

9 It is given that the eigenvalues of matrix $A=\begin{pmatrix}1 & 2 & -2\\ 6 & 4 & -6\\ 6 & 5 & -7\end{pmatrix}$ are $-2,-1$ and 1 with corresponding eigenvectors $\mathbf{j}+\mathbf{k}$, $\mathbf{i}+\mathbf{k}$ and $\mathbf{u}$ respectively.

Let the linear transformation $\mathrm{L}:\mathbb{R}^3\to\mathbb{R}^3$ be $\mathrm{L}(\mathbf{x})=A(\mathbf{x})$.

(i) Find $\mathbf{u}$. [2]

(ii) Find the Cartesian equation of the line through the origin, perpendicular to $\mathbf{j}$ and is invariant under L. [2]

(iii) Find the set of points which are invariant under L. [2]

(iv) Show that the plane $2x+y-2z=0$ is invariant under L. [3]

(2016 HCI / JC1 / Promo / Q7)

10 Show that, for all real values of a, the rank of the matrix

$$\mathbf{M}=\begin{pmatrix}1 & -2 & -3 & a\\ -1 & 3 & a+3 & -a+1\\ 1 & -1 & a-3 & a+1\\ 2 & -3 & a-6 & 2a+1\end{pmatrix},$$

is equal to 2. [2]

The null space of the linear transformation represented by $\mathbf{M}$ is denoted by K. The set $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for K, and the vectors $\mathbf{x}_0$ and $\mathbf{b}$ are such that $\mathbf{M}\mathbf{x}_0=\mathbf{b}$.

(i) Show that if $\mathbf{x}=\mathbf{x}_0+\lambda\mathbf{e}_1+\mu\mathbf{e}_2$, with $\lambda,\ \mu\in\mathbb{R}$, then $\mathbf{M}\mathbf{x}=\mathbf{b}$. [2]

(ii) Show that if $\mathbf{M}\mathbf{x}=\mathbf{b}$, then $(\mathbf{x}-\mathbf{x}_0)\in K$, and deduce that $\mathbf{x}$ is of the form $\mathbf{x}_0+\lambda\mathbf{e}_1+\mu\mathbf{e}_2$. [4]

(iii) Find the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ which are of the forms

$$\begin{pmatrix}r\\ s\\ 1\\ 0\end{pmatrix} \text{ and } \begin{pmatrix}t\\ u\\ 0\\ 1\end{pmatrix}$$

respectively, where r, s, t and u may depend on a. [3]

(2017 NYJC / JC2 / BT / Q7)

11 Find the value of a for which the simultaneous equations

$$3x+2y-z=10,$$
$$5x-y-4z=17,$$
$$x+5y+az=b,$$

do not have a unique solution for x, y and z. [4]

The equations represent three planes having a common line of intersection, L, find equations for L giving your answer in the form $\dfrac{x-p}{l}=\dfrac{y-q}{m}=\dfrac{z-r}{n}$. [3]

(2017 TJC / JC2 / BT / Q3)

12 **(i)** A square matrix is said to be *Markov* if the sum of entries for each column is equal to 1. Show that the product of two 2×2 Markov matrices is also a Markov matrix. [2]

(ii) Using **(i)**, prove that if $\mathbf{A}$ is a 2×2 Markov matrix, then $\mathbf{A}^n$ is a Markov matrix for all $n\in\mathbb{Z}^+$. [3]

A Markov matrix is used when a situation can be modelled using *Markov chain*. Consider the following problem. Each year 100α % of the urban population of a country moves to the rural district while 100β % of the rural population moves to the urban district where α and β are non-zero. Let a_n and b_n be the urban and rural population respectively after *n* years. If we let $\mathbf{x}_n=\begin{pmatrix}a_n\\ b_n\end{pmatrix}$, then $\mathbf{x}_n=\mathbf{A}^n\mathbf{x}_0$ where $\mathbf{A}$ is a 2×2 Markov matrix with $\mathbf{A}=\begin{pmatrix}1-\alpha & \beta\\ \alpha & 1-\beta\end{pmatrix}$ and $\mathbf{x_0}$ represents the initial population. Denote $\mathbf{A}^n=\begin{pmatrix}a_{11}^{(n)} & a_{12}^{(n)}\\ a_{21}^{(n)} & a_{22}^{(n)}\end{pmatrix}$.

(iii) Show that $a_{11}^{(n)}=\beta+(1-\alpha-\beta)a_{11}^{(n-1)}$. Hence show that

$$a_{11}^{(n)}=\frac{\beta}{\alpha+\beta}+\frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n.$$ [5]

(iv) Show that $\mathbf{A}$ has two eigenvalues 1 and λ where $\lambda<1$. [3]

(v) If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1, show that $\mathbf{A}^n\mathbf{x}=\mathbf{x}$ for all $n\in\mathbb{Z}^+$. Explain the significance (in context) of this result. [2]

(2017 NYJC / JC2 / BT / Q10)

13 The matrix $\mathbf{M}$ is given by

$$\mathbf{M}=\begin{pmatrix}1 & \frac{1}{2} & 0\\ 0 & \frac{1}{2} & 1\\ 0 & 0 & 0\end{pmatrix}.$$

(i) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{M}=\mathbf{PDP}^{-1}$. [6]

(ii) Hence find $\mathbf{M}^n$, where *n* is a positive integer. [2]

(iii) The vector $\mathbf{x}$ is given by $\mathbf{x}=\begin{pmatrix}a\\ b\\ c\end{pmatrix}$, where *a*, *b* and *c* are constants. Find the vector $\mathbf{M}^n\mathbf{x}$ when *n* tends to infinity. [1]

(2017 ACJC / JC2 / MYE / Q1)

14 Consider the following simultaneous equations where *z* and *w* are complex variables, λ and μ are real constants:

$$\lambda z+(1+i)w^*=1+\lambda\,\mathrm{Re}(z)$$
$$z^*+w=\mu i$$

Find the conditions to be satisfied by λ and by μ given that the system of simultaneous equations has no solution in *z* and *w*. [6]

(2017 ACJ / JC2 / MYE / Q2)

15 A linear transformation $\mathrm{T}:\mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 8 \\ 2 & 2 & 5 \end{pmatrix}.$$

(i) Find bases for the range space and null space of T. [3]

Let V_1 be the set of vectors of the form $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 6 \\ -2 \end{pmatrix}$, where λ is a real number.

(ii) Find W_1, the set of images of vectors in V_1 under the transformation T, and describe W_1 geometrically. [3]

Let V_2 be the set of vectors $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $2x + y + 2z = 2$.

(iii) Find W_2, the set of images of vectors in V_2 under the transformation T, and describe W_2 geometrically. [3]

(iv) Determine if W_2 is a subspace of $\mathbb{R}^3$. [1]

(2017 ACJC / JC2 / MYE / Q3)

16 The linear transformation T : $\mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix **M**, where

$$\mathbf{M} = \begin{pmatrix} 2 & 4 & 1 \\ 1 & -3 & -2 \\ 3 & 1 & -1 \end{pmatrix}.$$

The range space and null space of T is denoted by *R* and *N* respectively. Find a basis for *R* and determine the dimension of *N*. [2]

The plane π in $\mathbb{R}^3$ has equation $\mathbf{r} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ b \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ c \end{pmatrix}, \lambda, \mu \in \mathbb{R}$. The set *S* denotes the set of positions vectors of all the points on π.

(i) If $b = c = 0$, show that $R \cup S$ is not a vector space for all real values of *a*. [4]

(ii) Given that $R \cap S$ is a vector space, explain why it is necessary and sufficient for π to contain the origin. Hence, find a condition to be satisfied by *a*, *b* and *c*. [4]

(2017 AJC / JC2 / MYE / Q3)

17 Given that **e** is an eigenvector of the non-singular matrix **A** with corresponding eigenvalue λ, show that **e** is an eigenvector of the matrix $\mathbf{A}^{-1}$ with corresponding eigenvalue λ^{-1}.

Find the eigenvalues and corresponding eigenvectors of the matrix **B**, where

$$\mathbf{B} = \begin{pmatrix} -2 & 5 & 5 \\ -4 & 7 & 5 \\ 4 & -4 & -2 \end{pmatrix}.$$

Hence, evaluate $\sum_{r=1}^{\infty} \left(\mathbf{B}^{-1}\right)^r$. [11]

(2017 AJC / JC2 / MYE / Q4)

18 The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -2 \\ 2 & -1 & -1 \\ 3 & -2 & -3 \end{pmatrix}.$$

(i) Find the rank of $\mathbf{A}$ and a basis for the null space of T. [4]

(ii) A plane π in $\mathbb{R}^3$ has equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = 1$. $T(\pi)$ is the image of the plane π under the linear transformation T.

Find a vector equation of $T(\pi)$ and describe $T(\pi)$ geometrically. [4]

(2017 CJC / JC2 / MYE / Q4)

19 A sequence $a_0, a_1, a_2, \cdots$ is defined by $a_n = pa_{n-1} + q^2 a_{n-2} - pq^2 a_{n-3}$, $n \geq 3$, and $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, where p and q are distinct positive constants. Consider the vector equation

$$\begin{pmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{pmatrix} = \mathbf{A} \begin{pmatrix} a_{n-1} \\ a_{n-2} \\ a_{n-3} \end{pmatrix},$$

where $\mathbf{A}$ is a 3×3 matrix.

(i) Verify that $\mathbf{A} = \begin{pmatrix} p & q^2 & -pq^2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ satisfies the vector equation. [1]

(ii) Find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$. [8]

(iii) Hence, by considering $\mathbf{A}^n \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix}$, or otherwise, find the general formula for a_n in terms of n when $p = 2$ and $q = 1$. [4]

(2017 CJC / JC2 / MYE / Q6)

20 Let V and W be subspaces of $\mathbb{R}^n$.

(i) Show that $V \cap W$ is also a subspace of $\mathbb{R}^n$. [2]

(ii) Find a basis for $V \cap W$ where:

$$V = \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \mathbf{A}\boldsymbol{x} = \mathbf{0}, \text{ where } \mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \right\},$$

$$W = \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \mathbf{B}\boldsymbol{x} = \mathbf{0}, \text{ where } \mathbf{B} = \begin{pmatrix} 2 & 1 & -7 \\ -4 & -2 & 14 \end{pmatrix} \right\}.$$ [2]

(2017 DHS / JC2 / MYE P1 / Q2)

21 **(i)** Show that the eigenvalues of the matrix **A** are -1, 1 and 2 where

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ -2 & 5 & 1 \\ 2 & -12 & -2 \end{pmatrix}.$$ [2]

Hence find the corresponding eigenvector for the eigenvalue -1. [2]

(ii) The corresponding eigenvectors of **A** for eigenvalue 1 and 2 are $\begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$ respectively. **P** is a 3x3 matrix where its columns are the three eigenvectors. Using the fact that the eigenvectors are linearly independent, explain why **P** is invertible. Hence state **P** and **D** such that $\mathbf{A} = \mathbf{PDP}^{-1}$. [3]

(iii) Hence show that $\sum_{r=1}^{21} \left[(-1)^{r+1} \mathbf{A}^r \right]$ can be expressed as $\mathbf{PVP}^{-1}$, where the matrix **V** is to be determined. [4]

(2017 DHS / JC2 / MYE P1 / Q7)

22 Let $\mathrm{T} : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$\mathrm{T}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \ \mathrm{T}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \text{ and } \mathrm{T}\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ c \end{pmatrix}, \ c \in \mathbb{R}.$$

The matrix **A** is defined such that $\mathbf{T}(\mathbf{x}) = \mathbf{Ax}$ for $\boldsymbol{x} \in \mathbb{R}^3$.

(i) Show that the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 3 & 1 & 2 \end{pmatrix}$ is non-singular. [2]

(ii) Given that $\mathbf{AX} = \mathbf{B}$, where $\mathbf{X} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 3 & 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 1 & -1 \\ 5 & 3 & 2 \\ 6 & 5 & c \end{pmatrix}$, and **A** is non-singular, find the possible values of c. [2]

(iii) If the rank of **A** is 2, state the dimension of Ker(T). Explain why there is only one possible value of c. (Ker(T) refers to the null space of T) [2]

(iv) Using the value of c found in **(iii)**, find the null space of **B**. Hence or otherwise, find the basis for Ker(T). [4]

(2017 DHS / JC2 / MYE P1 / Q8)

23 The linear transformation $\mathrm{T}:\mathbb{R}^4 \to \mathbb{R}^3$ is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & -2 & a \\ 2 & -1 & 3 & -5 \\ -3 & -3 & 0 & 3 \end{pmatrix},$$

where a is a real constant.

(i) It is given that the dimension of the null space of T is 2. Find the value of a. Hence find a basis for the null space of T. [5]

(ii) Show that R, the range space of T, is a plane, and find the Cartesian equation of R. [4]

(iii) Let V be a vector space spanned by $\mathbf{v}$ where $\mathbf{v} = \begin{pmatrix} 0 \\ b \\ c \end{pmatrix}$, $b, c \in \mathbb{R}$. If $R \cup V$ is a vector space, find a relationship between b and c. [3]

(2017 HCI / JC2 / MYE P1 / Q8)

24 A matrix $\mathbf{M} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ k & -2 & 3 \end{pmatrix}$ has an eigenvalue equal to -1.

(i) Find the value of k, and find the other eigenvalues of $\mathbf{M}$. [4]

(ii) Find the eigenvectors of $\mathbf{M}$ corresponding to the eigenvalue -1 and eigenvalues as found in **(i)**. [4]

(iii) Find a non-singular matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$(\mathbf{M} + 3\mathbf{I})^3 = \mathbf{P}^{-1}\mathbf{D}\mathbf{P},$$

where $\mathbf{I}$ is the 3×3 identity matrix. [2]

(2017 HCI / JC2 / MYE P2 / Q2)