NEW TOWN SECONDARY SCHOOL
Preliminary Examination
Secondary 4 Express

NAME	Mark Scheme
CLASS	
INDEX NUMBER	

Additional Mathematics
Paper 1

4049/01
28 July 2021
10:35 – 12:50
2 hours 15 min

READ THESE INSTRUCTIONS FIRST

Write your name, register number and class in the spaces provided above and on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 90.

For Examiner's Use

This document consists of **17** printed pages.

Setter: Mr Alan Cheng

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} bc \sin A$$

1 The curve $\frac{x^2}{4}-\frac{y}{a}=x+b$, where a and b are constants, intersects the y-axis at A and the x-axis at B and C. The coordinates of B are $(-2,0)$. Given that the gradient of AB is -3, find the value of a and of b. [5]

Sub $B(-2,0)$, $\frac{(-2)^2}{4}-\frac{0}{a}=-2+b$ [M1]

$b=3$ [A1]

Sub $x=0$, $b=3$, $-\frac{y}{a}=b$

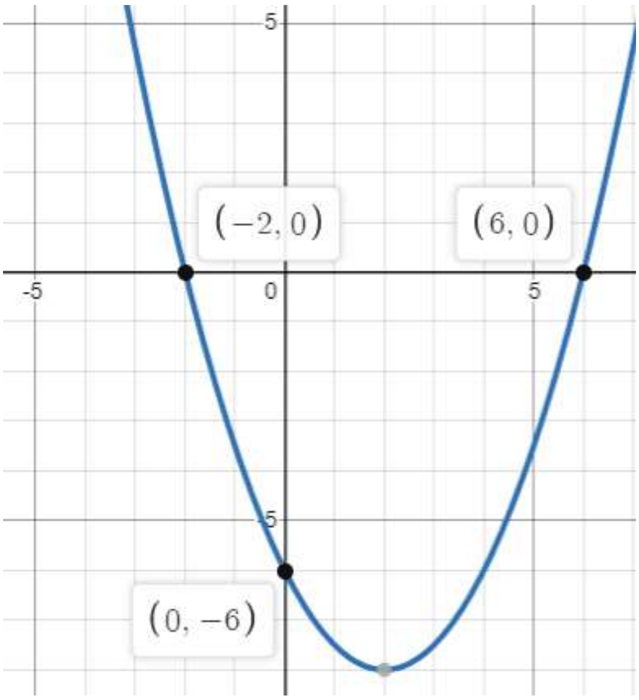
$y=-3a$ --- (1) [M1]

$A(0,-3a)$

Gradient of AB $=\frac{0-(-3a)}{-2-0}=-3$ [M1]

$\frac{3a}{-2}=-3$

$a=2$ [A1]



2 The equation of a curve is $y = c - \frac{3}{2}\cos 2x$ where c is a constant. The curve passes through the point $\left(\frac{\pi}{6}, \frac{1}{4}\right)$.

(a) Find the value of c. [2]

Sub $\left(\frac{\pi}{6}, \frac{1}{4}\right)$

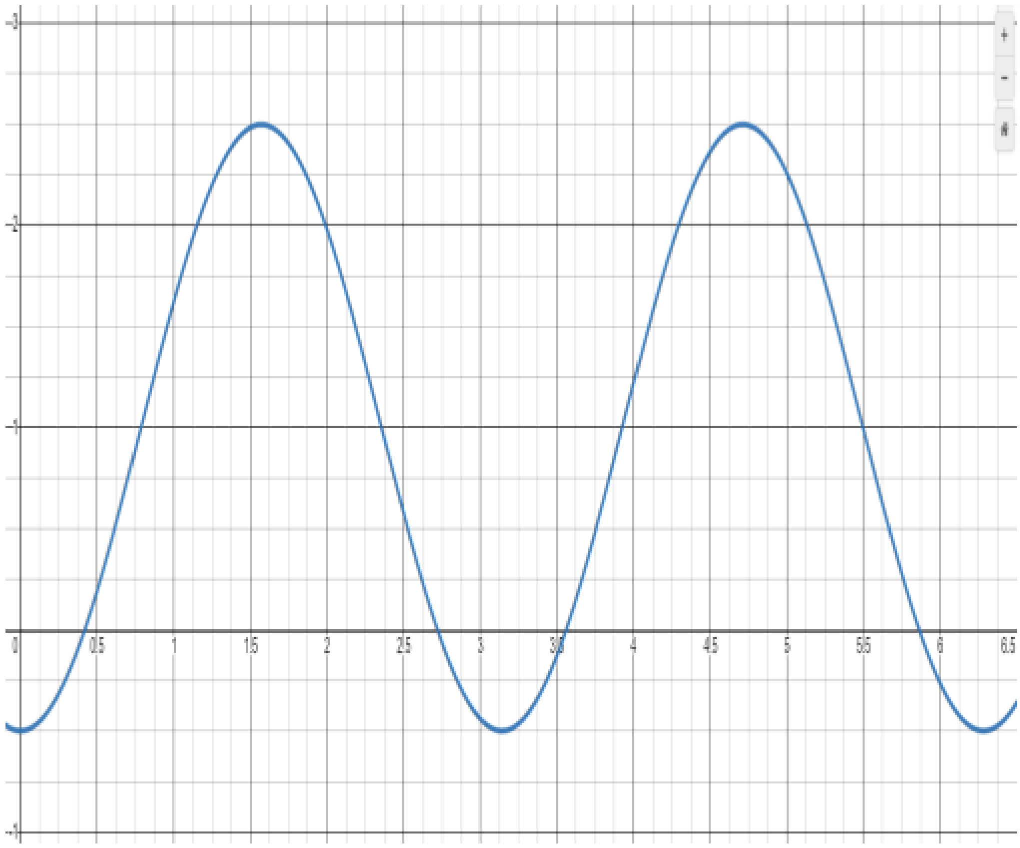
$\frac{1}{4} = c - \frac{3}{2}\cos 2\left(\frac{\pi}{6}\right)$ [M1]

$\frac{1}{4} = c - \frac{3}{2}\left(\frac{1}{2}\right)$

$c = 1$ [A1]

(b) Using the value of c found in part **(a)**, sketch the graph of $y = c - \frac{3}{2}\cos 2x$ for $0 \le x \le 2\pi$. [3]

[B1 – 2 complete cycles]
[B1 – start and end at -0.5]
[B1 – fully correct curve with correct Max value @ 2.5 or Min value @ -0.5]



3 **(a)** Write down, and simplify, the first 4 terms in the expansion of $\left(1-\frac{x}{2}\right)^8$ in ascending powers of x. [3]

$$\left(1-\frac{x}{2}\right)^8 = (1)^8 + \binom{8}{1}(1)^7\left(-\frac{x}{2}\right)^1 + \binom{8}{2}(1)^6\left(-\frac{x}{2}\right)^2 + \binom{8}{3}(1)^5\left(-\frac{x}{2}\right)^3 + \ldots$$
$$= 1 - 4x + 7x^2 - 7x^3 + \ldots$$

[B1 – 1st and 2nd term correct]
[B1 – 3rd term correct]
[B1 – 4th term correct]

(b) Find the coefficient of x in the expansion of $\left(1-\frac{x}{2}\right)^8\left(\frac{2}{x}+3x\right)^2$. [3]

$$\left(\frac{2}{x}+3x\right)^2 = \frac{4}{x^2} + 12 + 9x^2$$ [M1]

$$\left(1-\frac{x}{2}\right)^8\left(\frac{2}{x}+3x\right)^2 = \left(1-4x+14x^2-7x^3+\ldots\right)\left(\frac{4}{x^2}+12+9x^2\right)$$

Terms with $x = (-4x)(12) + (-7x^3)\left(\frac{4}{x^2}\right)$ [M1]
$= -48x - 28x = -76x$
Coefficient of $x = -76$ [A1]

4 **(a)** Given that $y = \frac{e^x}{x^2+1}$, $x \neq 1$, explain, with working, whether y is an increasing or decreasing function. [3]

$$\frac{dy}{dx} = \frac{\left(x^2+1\right)e^x - (2x)e^x}{\left(x^2+1\right)^2}$$ [M1]

$$= \frac{e^x\left(x^2+1-2x\right)}{\left(x^2+1\right)^2}$$

$$= \frac{e^x(x-1)^2}{\left(x^2+1\right)^2}$$ [M1]

Since $\left(x^2+1\right)^2 > 0$, $(x-1)^2 > 0$ and $e^x > 0$, therefore $\frac{dy}{dx} > 0$

y is increasing function. [B1]

(b) Air is escaping from a hole in a spherical balloon of radius r cm in such a way that the total volume, V cm^3, is decreasing at a constant rate of 25π cm^3/s. Assuming that the balloon retains its shape, calculate the rate of change of r when $r = 5$. [3]

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$ [M1]

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$-25\pi = 4\pi(5)^2 \times \frac{dr}{dt}$$ [M1]

$$\frac{dr}{dt} = -\frac{1}{4} \text{ cm/s}$$ [A1]

5 The number of fishes, F in a fish farm after t days can be modelled by the formula $F = 6000 + Ae^{kt}$ where A and k are constants. Initially, there were 2500 fishes in the fish farm and 5 days later, there were 3500 fishes.

(a) Show that $A = -3500$ [1]

Sub $t = 0, F = 2500$

$2500 = 6000 + Ae^{k(0)}$ [M1]

$A = 2500 - 4000 = -3500$

(b) Find the number of days required for the fishes to increase its population by 80%. [3]

$3500 = 6000 - 3500e^{k(5)}$ [M1]

$k = -0.067294$

$\frac{180}{100} \times 2500 = 6000 - 3500e^{-0.067294t}$ [M1]

$t = 12.6$ days [A1]

(c) Explain why the number of fishes in the fish farm cannot be 6000. [2]

Suppose the no. of fishes is 6000, then

$6000 = 6000 - 3500e^{-0.0673t}$

$e^{-0.0673t} = 0$ [B1]

But $e^{-0.0673t} > 0$ for all real values of t

Hence the no. of fishes cannot be 6000. [B1]

6 A curve is such that $\frac{d^2y}{dx^2} = 2\sin x - 3\cos 2x$ and the point $A(\pi, 5)$ lies on the curve. The gradient of the curve at A is -3. Find the equation of the curve. [6]

$\frac{d^2y}{dx^2} = 2\sin x - 3\cos 2x$

$\frac{dy}{dx} = -2\cos x - \frac{3}{2}\sin 2x + c$ [M1]

Sub $x = \pi, \frac{dy}{dx} = -3$

$-3 = -2\cos(\pi) - \frac{3}{2}\sin 2(\pi) + c$ [M1]

$-3 = -2(-1) - 0 + c$

$c = -5$

$\frac{dy}{dx} = -2\cos x - \frac{3}{2}\sin 2x - 5$ [A1]

$y = -2\sin x + \frac{3}{4}\cos 2x - 5x + d$ [M1]

Sub $x = \pi, y = 5$

$5 = -2\sin(\pi) + \frac{3}{4}\cos 2(\pi) - 5(\pi) + d$ [M1]

$5 = 0 + \frac{3}{4}(1) - 5(\pi) + d$

$d = \frac{17}{4} + 5\pi$

$y = -2\sin x + \frac{3}{4}\cos 2x - 5x + \frac{17}{4} + 5\pi$ [A1]

7 **(a)** Express each of $8-4x+x^2$ and $-2x^2-12x-14$ in the form $a(x+b)^2+c$, where *a*, *b* and *c* are constants. [4]

$$8-4x+x^2 = x^2-4x+8$$
$$=(x-2)^2-\left(\frac{4}{2}\right)^2+8 \quad \text{[M1]}$$
$$=(x-2)^2+4 \quad \text{[A1]}$$

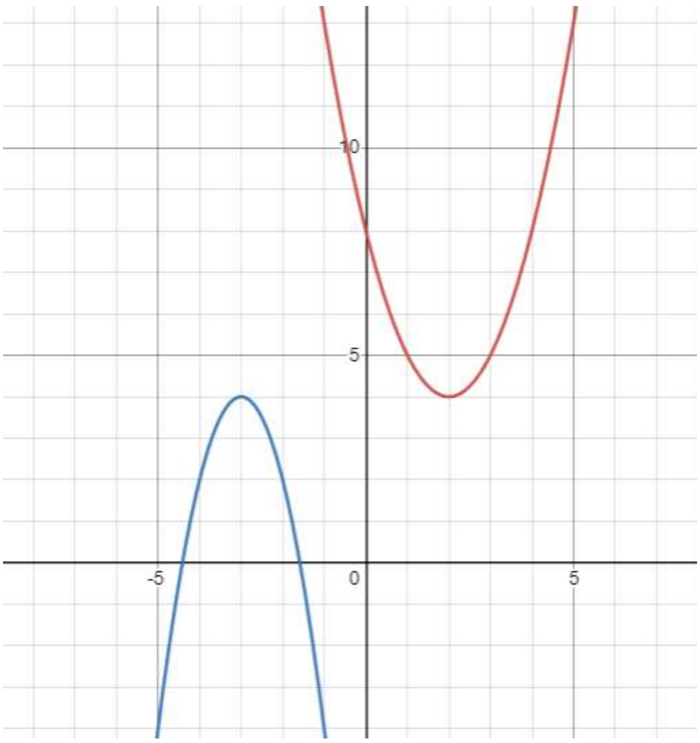
$$-2x^2-12x-14=-2\left(x^2+6x+7\right)$$
$$=-2\left[(x+3)^2-\left(\frac{6}{2}\right)^2+7\right] \quad \text{[M1]}$$
$$=-2\left[(x+3)^2-2\right]$$
$$=-2(x+3)^2+4 \quad \text{[A1]}$$

(b) Use your answers from part **(a)** to explain if the curves with equations $y=8-4x+x^2$ and $y=-2x^2-12x-14$ will intersect. [3]

Min point of $y=8-4x+x^2$ is $(2,4)$ [B1]

Max point of $y=-2x^2-12x-14$ is $(-3,4)$ [B1]

Since $8-4x+x^2 \geq 4$ and $-2x^2-12x-14 \leq 4$, the only possible intersection is at the turning point, however, the **turning points are not the same** and **hence both curves will not intersect**. [B1]



8 **Without using a calculator,**

(a) show that $\sin 15^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$, [3]

$$\sin 15^\circ = \sin(60^\circ - 45^\circ)$$

$$= \sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ \qquad \text{[M1]}$$

$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \qquad \text{[M1]}$$

$$= \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{2\sqrt{2}\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \qquad \text{[A1]}$$

(b) express $\operatorname{cosec}^2 15^\circ$ in the form $p + q\sqrt{3}$, where p and q are integers. [4]

$$\operatorname{cosec}^2 15^\circ = \frac{1}{\sin^2 15^\circ} \qquad \text{[M1]}$$

$$= \frac{1}{\left(\dfrac{\sqrt{3}-1}{2\sqrt{2}}\right)^2}$$

$$= \frac{\left(2\sqrt{2}\right)^2}{\left(\sqrt{3}-1\right)^2} \qquad \text{[M1]}$$

$$= \frac{4(2)}{3-2\sqrt{3}+1}$$

$$= \frac{8}{4-2\sqrt{3}}$$

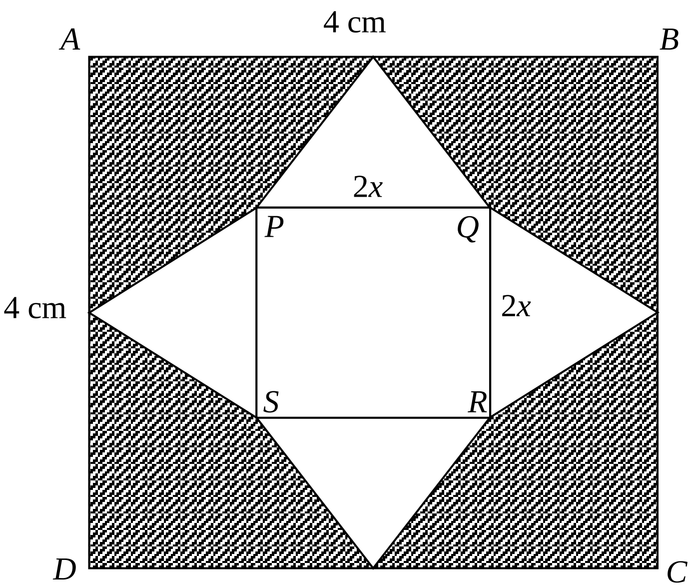
$$= \frac{4}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}} \qquad \text{[M1]}$$

$$= \frac{8+4\sqrt{3}}{4-3}$$

$$= 8+4\sqrt{3} \qquad \text{[A1]}$$

9 In the figure, $ABCD$ is a square plastic plate of side 4 cm and $PQRS$ is a square whose centre coincides with that of $ABCD$. The shaded regions are to be cut off and the remaining plastic is folded to form a right pyramid with base $PQRS$.

Given that $PQ = 2x$ cm and V is the volume of the pyramid.



(a) Show that the height of the pyramid is $2\sqrt{1-x}$ cm. [2]

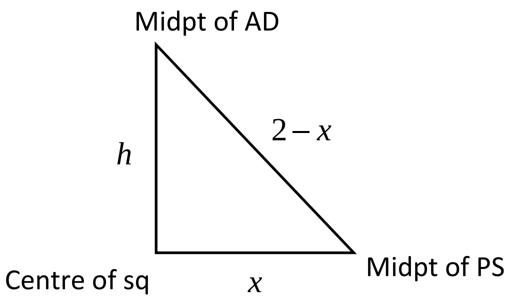
By Pythagoras Thm,

$h^2 = (2-x)^2 - x^2$ [M1]

$h^2 = 4 - 4x + x^2 - x^2$

$h^2 = 4 - 4x$

$h = 2\sqrt{1-x}$ [A1]



(b) Given that x can vary, find the value of x for which V has a stationary value and determine if it is a maximum or a minimum. [5]

$$V = \frac{1}{3}(2x)^2\left[2\sqrt{1-x}\right]$$

$$V = \frac{8}{3}x^2\sqrt{1-x}$$ [B1]

$$\frac{dV}{dx} = \frac{8}{3}(2x)(1-x)^{\frac{1}{2}} + \frac{8}{3}x^2\left[\frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)\right]$$ [M1]

$$= \frac{4}{3}x(1-x)^{-\frac{1}{2}}\left[4(1-x)-x\right]$$

$$= \frac{4x(4-5x)}{3\sqrt{1-x}}$$

$$\frac{dV}{dx} = 0$$

$$\therefore \frac{4x(4-5x)}{3\sqrt{1-x}} = 0$$ [M1]

$$\Rightarrow x(4-5x) = 0$$

$x = 0$ or $x = \frac{4}{5}$

$$\frac{dV}{dx} = \frac{1}{3}(16x-20x^2)(1-x)^{-\frac{1}{2}}$$

$$\frac{d^2V}{dx^2} = \frac{1}{3}\left\{(16-40x)(1-x)^{-\frac{1}{2}} + (16x-20x^2)\left[-\frac{1}{2}(1-x)^{-\frac{3}{2}}(-1)\right]\right\}$$

[M1]

When $x = \frac{4}{5}$, $\frac{d^2V}{dx^2} < 0$

V has a maximum value at $x = \frac{4}{5}$. [A1]

[Award M1A1 for correct 1st derivative test]

10 **(a)** The function $f(x)=2x^3-3x^2+ax+b$, where *a* and *b* are constants, is exactly divisible by $x-1$. Given that $f(x)$ leaves a remainder of 2 when divided by $x+1$, find the value of *a* and *b* and hence solve the equation $f(x)=0$ [6]

By Factor thm,

$f(1)=2(1)^3-3(1)^2+a(1)+b=0$

$a+b=1$ --- (1) [M1]

By Remainder thm,

$f(-1)=2(-1)^3-3(-1)^2+a(-1)+b=2$

$-a+b=7$ --- (2) [M1]

(1)+(2): $2b=8$

$b=4$ [A1]

Sub $b=4$ to (1): $a=-3$ [A1]

$\therefore f(x)=2x^3-3x^2-3x+4$

$f(x)=(x-1)(Ax^2+Bx+C)$

By inspection,

for x^2: $A=2$

for x^0: $C=-4$

for x: $C-B=-3$

$-4-B=-3$

$B=-1$

$\therefore f(x)=(x-1)(2x^2-x-4)$ [M1]

$(x-1)(2x^2-x-4)=0$

$x=1$ or $x=\dfrac{1\pm\sqrt{(-1)^2-4(2)(-4)}}{2(2)}$

$x=1.69,-1.19$ [A1]

(b) It is given that $x-3$ is a factor of $g(x)+2$, where $g(x)$ is a polynomial. Find the remainder when $h(x)=4x^3+g(x)+3$ is divided by $x-3$. [2]

$h(x)=4x^3+g(x)+3$

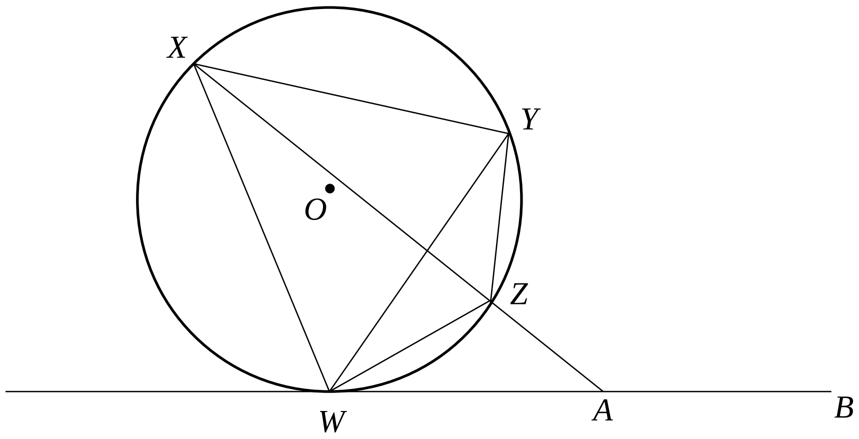
$=4x^3+1+g(x)+2$ [M1]

By Remainder Thm,

$h(3)=4(3)^3+1+0$ [since $x-3$ is a factor of $g(x)+2$]

$=109$ [A1]

11 In the figure below, $YZ = WZ$ and the line WAB is tangent to the circle at the point W. Line AX bisects angle WXY and cuts the circle at point Z.



(a) Show that $\angle AWZ = \angle ZWY$ [3]

$\angle AWZ = \angle WXZ$ (alternate segment theorem) [B1]
$\angle WXZ = \angle YXZ$ (AX bisects $\angle WXY$) [B1]
$\angle ZWY = \angle YXZ$ ($\angle$s in the same segment) [B1]
Hence $\angle AWZ = \angle ZWY$ (shown)

(b) Show that $AW \times WX = AX \times YZ$ [5]

$\angle AWZ = \angle AXW$ (alternate segment theorem) [B1]
$\angle WAZ = \angle WAX$ (common $\angle$) [B1]
ΔZWA is similar to ΔWXA (AA) (shown)

Since ΔZWA is similar to ΔWXA,
$\frac{ZW}{WX} = \frac{AW}{XA}$ [B1]
Since $YZ = WZ$,
$\frac{YZ}{WX} = \frac{AW}{AX}$ [B1]
Hence $AW \times WX = AX \times YZ$ [B1]

12 **(a)** Show that $\dfrac{5\sin 2x+2\cos 2x-2}{1+\cos 2x}=5\tan x-2\tan^2 x$. [5]

LHS $=\dfrac{5(2\sin x\cos x)+2(1-2\sin^2 x)-2}{1+2\cos^2 x-1}$ [B1 – correct $2\sin x\cos x$]

[B1 – correct $1-2\sin^2 x$]

[B1 – correct $2\cos^2 x-1$]

$=\dfrac{10\sin x\cos x+2-4\sin^2 x-2}{2\cos^2 x}$

$=\dfrac{10\sin x\cos x}{2\cos^2 x}-\dfrac{4\sin^2 x}{2\cos^2 x}$

$=\dfrac{5\sin x}{\cos x}-2\tan^2 x$ [M1]

$=5\tan x-2\tan^2 x$ [A1]

$=$RHS

(b) Solve the equation $\dfrac{5\sin 2x+2\cos 2x-2}{1+\cos 2x}=\dfrac{1}{2}\tan x$ for $0°\le x\le 360°$. [4]

$5\tan x-2\tan^2 x=\dfrac{1}{2}\tan x$ [M1]

$2\tan^2 x-5\tan x+\dfrac{1}{2}\tan x=0$

$2\tan^2 x-4.5\tan x=0$

$\tan x(2\tan x-4.5)=0$ [M1]

$\tan x=0$ or $2\tan x-4.5=0$

Basic angle $=0°$ or $\tan x=\dfrac{9}{4}$

$x=0°,180°,360°$ [A1] or Basic angle $=66.038°$

or $x=66.0°,180+66.0°$ [A1]

$x=0°,66.0°,180°,246.0°,360°$

13 Two bicycles, *A* and *B* leave a point *O* at the same time and travel along the same straight line. Bicycle *A* starts from rest and travels with a uniform velocity of 1.5 m/s. The velocity of Bicycle *B*, *t* seconds after leaving *O*, is given by $V_B = 10 + t - 3t^2$ m/s .

(a) Find the value of *t* when Bicycle *B* is instantaneously at rest. [2]

$10 + t - 3t^2 = 0$ [M1]

$3t^2 - t - 10 = 0$

$(3t+5)(t-2) = 0$

$t = 2$ or $t = -\frac{5}{3}$ (rej) [A1]

(b) Find the distance travelled by Bicycle *B* in the first 3 seconds. [3]

$\int_0^2 10 + t - 3t^2 \; dt = \left[10t + \frac{1}{2}t^2 - t^3\right]_0^2$ [M1]

$= 14$

$\int_2^3 10 + t - 3t^2 \; dt = \left[10t + \frac{1}{2}t^2 - t^3\right]_2^3$ [M1]

$= \left[\frac{15}{2} - 14\right] = -6.5$

Distance travelled $= 14 + 6.5 = 20.5$ m [A1]

(c) Find the distance from O when Bicycle B first meet with Bicycle A again. [5]

$S_A = \int 1.5 \; dt = 1.5t + c$

When $t = 0, s = 0, \therefore c = 0$

$S_A = 1.5t$ [B1]

$S_B = \int 10 + t - 3t^2 \; dt = 10t + \frac{1}{2}t^2 - t^3 + c$

When $t = 0, s = 0, \therefore c = 0$

$S_B = 10t + \frac{1}{2}t^2 - t^3$ [B1]

At collision, $1.5t = 10t + \frac{1}{2}t^2 - t^3$ [M1]

$t^3 - \frac{1}{2}t^2 - 8.5t = 0$

$t\left(t^2 - \frac{1}{2}t - 8.5\right) = 0$

$t = 0$ or $t^2 - \frac{1}{2}t - 8.5 = 0$

$$t = \frac{0.5 \pm \sqrt{(-0.5)^2 - 4(1)(-8.5)}}{2(1)}$$

$t = 3.1762, -2.6762$ (rej) [A1]

Sub $t = 3.1762$

$S_A = 1.5(3.1762) = 4.76\,\text{m}$ [A1]

End of Paper