2017 GCE A Level H2 FM 9649 Paper 1 Solutions

1 Explain how the six roots of the equation $z^6 + z^3 + 1 = 0$ are related to the nine 9th roots of unity. Sketch all of these roots on an Argand diagram, indicating clearly the roots that satisfy the given equation. [3]

[Solution]

Let $w = \sqrt[9]{1}$

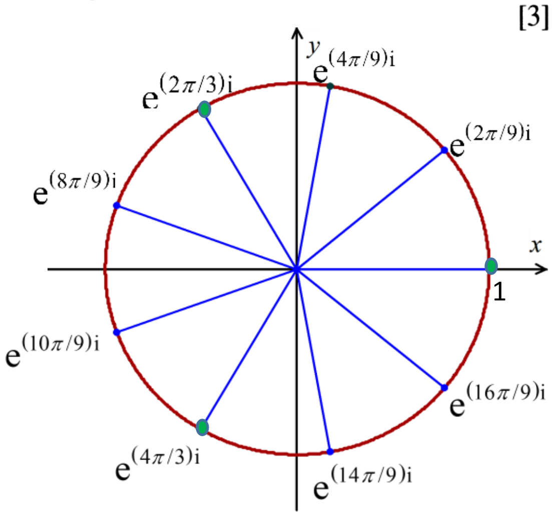
$\Rightarrow \quad w^9 - 1 = 0$

$\Rightarrow \quad (w^3)^3 - 1 = 0$

$\Rightarrow \quad (w^3 - 1)\left[(w^3)^2 + w^3 + 1\right] = 0$

$\Rightarrow \quad (w^3 - 1)(w^6 + w^3 + 1) = 0$

$\Rightarrow \quad w^3 - 1 = 0 \quad \text{or} \quad w^6 + w^3 + 1 = 0$



$\therefore$ 6 roots of $z^6 + z^3 + 1 = 0$ are among the nine 9th roots of unity.

Since $w^9 - 1 = 0$ has the 9 roots as shown on the Argand diagram

and $w^3 - 1 = 0$ has roots $e^{(2\pi/3)i}$, $e^{(4\pi/3)i}$ and 1.

Hence, 6 roots of $z^6 + z^3 + 1 = 0$ are $e^{(2\pi/9)i}, e^{(4\pi/9)i}, e^{(8\pi/9)i}, e^{(10\pi/9)i}, e^{(14\pi/9)i}$ & $e^{(16\pi/9)i}$.

2 The matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is such that the sum of the entries in each column is equal to s. Show that s is an eigenvalue of $\mathbf{A}$ and determine the second eigenvalue in terms of a and b. [5]

[Solution]

$s = a + c = b + d$ -------- (1)

$\mathrm{A} - \lambda \mathrm{I} = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$

λ is an eigenvalue of A $\Leftrightarrow \det(\mathrm{A} - \lambda \mathrm{I}) = 0$

$\Rightarrow (a-\lambda)(d-\lambda) - bc = 0$

$\Rightarrow \lambda^2 - (a+d)\lambda + ad - bc = 0$

$\Rightarrow \lambda^2 - (a+d)\lambda + ad - (s-d)(s-a) = 0$ from (1)

$\Rightarrow \lambda^2 - (a+d)\lambda + ad - \left(s^2 - (a+d)s + ad\right) = 0$

$\Rightarrow \lambda^2 - (a+d)\lambda - s^2 + (a+d)s = 0$

$\Rightarrow (\lambda - s)[\lambda + (s - a - d)] = 0$ [by long division]

$\Rightarrow \lambda = s \quad \text{or} \quad \lambda = a + d - s$

$\therefore$ s is an eigenvalue of A and the second eigenvalue $= a + d - s$

$= a - b$

3 The nth derivative of the function $f(x)$ is denoted by $f^{(n)}(x)$ for all positive integers n.

For $f(x) = xe^{2x}$,

(i) find $f^{(n)}(x)$ in the form $(a_n x + b_n)e^{2x}$ for $n = 1, 2, 3$ and 4, and hence conjecture an expression for $f^{(n)}(x)$ of this form; [4]

(ii) use mathematical induction to prove the correctness of your conjecture. [4]

[Solution]

$$f(x) = xe^{2x}$$

(i) $f^{(1)}(x) = 2xe^{2x} + e^{2x} = (2x+1)e^{2x}$

$f^{(2)}(x) = 2^2 xe^{2x} + 2e^{2x} + 2e^{2x} = \left(2^2 x + 2(2)\right)e^{2x}$

$f^{(3)}(x) = 2^3 xe^{2x} + \left(2^2\right)e^{2x} + 2^2 e^{2x} + 2^2 e^{2x} = \left(2^3 x + 3\left(2^2\right)\right)e^{2x}$

$f^{(4)}(x) = 2^4 xe^{2x} + 2^3 e^{2x} + 2^3 e^{2x} + 2^3 e^{2x} + 2^3 e^{2x} = \left(2^4 x + 4\left(2^3\right)\right)e^{2x}$

Conjecture: $f^{(n)}(x) = \left(2^n x + n2^{n-1}\right)e^{2x}$, for all $n \in \mathbb{Z}^+$

(ii) Let Let P_n be the statement: $f^{(n)}(x) = \left(2^n x + n2^{n-1}\right)e^{2x}$, for all $n \in \mathbb{Z}^+$

When $n = 1$, LHS = $f^{(1)}(x) = (2x+1)e^{2x}$;

RHS $= \left(2^1 x + 2^0\right)e^{2x} = (2x+1)e^{2x}$ = LHS

Hence P_1 is true.

Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e., $f^{(k)}(x) = \left(2^k x + k2^{k-1}\right)e^{2x}$

Need to show that P_{k+1} is true, i.e., $f^{(k+1)}(x) = \left[2^{k+1} x + (k+1)2^k\right]e^{2x}$

$$\begin{aligned}
\text{LHS} = f^{(k+1)}(x) &= \frac{d}{dx} f^{(k)}(x) \\
&= \frac{d}{dx}\left[\left(2^k x + k2^{k-1}\right)e^{2x}\right] \\
&= 2\left(2^k x + k2^{k-1}\right)e^{2x} + 2^k e^{2x} \\
&= \left(2^{k+1} x + k2^k\right)e^{2x} + 2^k e^{2x} \\
&= \left[2^{k+1} x + (k+1)2^k\right]e^{2x} = \text{RHS}
\end{aligned}$$

Thus P_k is true $\Rightarrow P_{k+1}$ is also true.

Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.

4 The curve T has polar equation $r = 2 + \cos 3\theta,\ 0 \leqslant \theta \leqslant 2\pi$.

(i) Sketch T, indicating clearly all key features and symmetries of the curve. [3]

(ii) Use calculus to evaluate the exact area enclosed by T. [5]

[Solution]

T: $r = 2 + \cos 3\theta, \quad 0 \le \theta \le 2\pi$

(i) $r = 0 \quad \Rightarrow$ No solution.

Max $r = 3$, when $\cos 3\theta = 1$

$\Rightarrow \quad 3\theta = 0, 2\pi, 4\pi$

$\Rightarrow \quad \theta = 0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

Min $r = 1$, when $\cos 3\theta = -1$

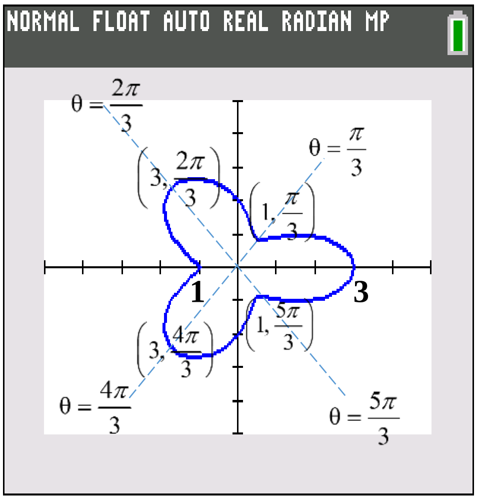
$\Rightarrow \quad 3\theta = \pi, 3\pi, 5\pi$

$\Rightarrow \quad \theta = \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$

Lines of symmetry: x-axis ($\theta = 0, \pi$),

The line $\theta = \dfrac{\pi}{3}, \dfrac{4\pi}{3}$

The line $\theta = \dfrac{2\pi}{3}, \dfrac{5\pi}{3}$



(ii) Area enclosed by T

$$= 2\int_0^{\pi} \left(\frac{1}{2} r^2\right) \mathrm{d}\theta$$

$$= \int_0^{\pi} (2 + \cos 3\theta)^2 \mathrm{d}\theta$$

$$= \int_0^{\pi} (4 + 4\cos 3\theta + \cos^2 3\theta) \mathrm{d}\theta$$

$$= \left[4\theta + \frac{4}{3}\sin 3\theta\right]_0^{\pi} + \frac{1}{2}\int_0^{\pi} (1 + \cos 6\theta) \mathrm{d}\theta$$

$$= 4\pi + \frac{1}{2}\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi}$$

$$= 4\pi + \frac{\pi}{2} = \frac{9\pi}{2}$$

5 **(i)** The function f is such that $f(a)f(b) < 0$, where $a < b$. A student concludes that the equation $f(x) = 0$ has exactly one root in the interval (a, b). Draw sketches to illustrate two distinct ways in which the student could be wrong. [2]

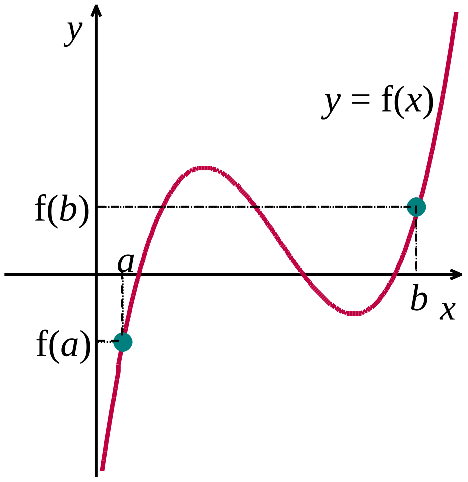
(ii) The equation $\sec^2 x - e^x = 0$ has a root α in the interval [1.5, 2.5]. A student uses linear interpolation once on this interval to find an approximation to α. Find the approximation to α given by this method and comment on the suitability of the method in this case. [3]

(iii) The equation $\sec^2 x - e^x = 0$ also has a root β in the interval (0.1, 0.9). Use the Newton-Raphson method, with $f(x) = \sec^2 x - e^x$ and initial approximation 0.5, to find a sequence of approximations $\{x_1, x_2, x_3, \ldots\}$ to β. Describe what is happening to x_n for large n, and use a graph of the function to help you explain why the sequence is not converging to β. [5]

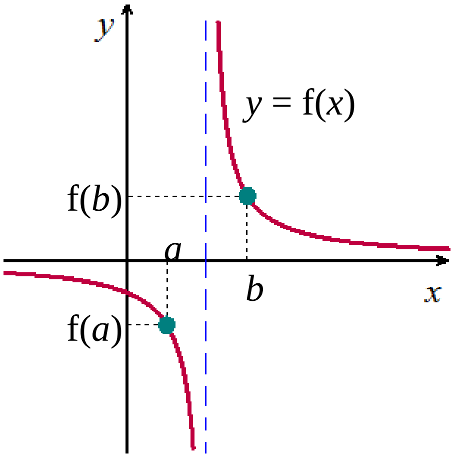
[Solution]

Given that $f(a)f(b) < 0$, $a < b$.

(i) **Case (1):** 3 distinct real roots in (a, b).

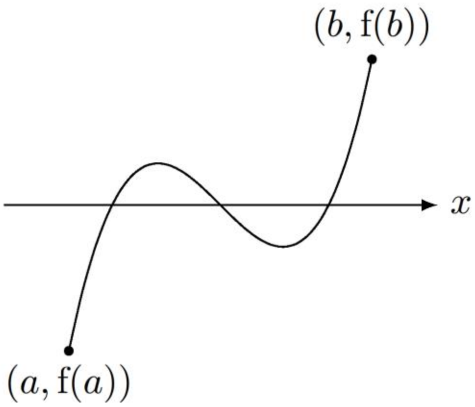


Case (2): No real root in (a, b).

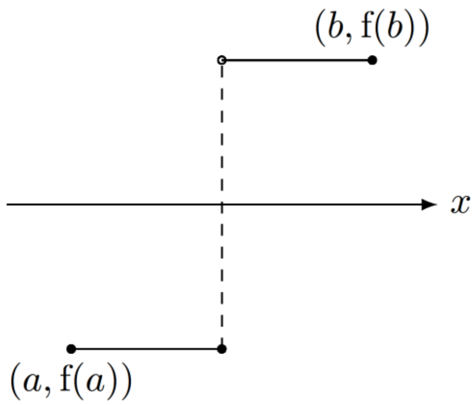


In both cases above, $f(a)f(b) < 0$ but $f(x) = 0$ does not have exactly one real root.

Alternatively: **Case 1**



Case 2



(ii) Let $\mathrm{f}(x)=\sec^2 x-\mathrm{e}^x$. α is a root of $\mathrm{f}(x)=0$, $\alpha\in[1.5,\ 2.5]$.

By linear interpolation, $x_1=\dfrac{1.5|\mathrm{f}(2.5)|+2.5|\mathrm{f}(1.5)|}{|\mathrm{f}(2.5)|+|\mathrm{f}(1.5)|}$

$$=\frac{1.5|-10.62445|+2.5|195.36835|}{10.62445+195.36835}\approx 2.4484$$

$\therefore\ \alpha\approx 2.45$ (to 3sf)

Since $\mathrm{f}(x_1)=\mathrm{f}(2.4484)=-9.8797<0$ and $\mathrm{f}(1.5)=195.36835>0$,

the root $\alpha\in[1.5,\ 2.45]$.

Since f is a continuous function in the interval [1.5, 2.5], linear interpolation is a suitable method to find the root. However, the first approximation lies in the interval $[1.5,\ 2.45]$ which is only slightly shorter than the interval $[1.5,\ 2.5]$. Hence this method of approximating the root is slow, not an efficient method.

(iii) β is a root of $\mathrm{f}(x)=0$, $\beta\in(0.1,\ 0.9)$.

By Newton-Raphson method, $x_0=0.5$, $\mathrm{f}'(x)=2\sec^2 x\tan x-\mathrm{e}^x$

$$x_{n+1}=x_n-\frac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$$

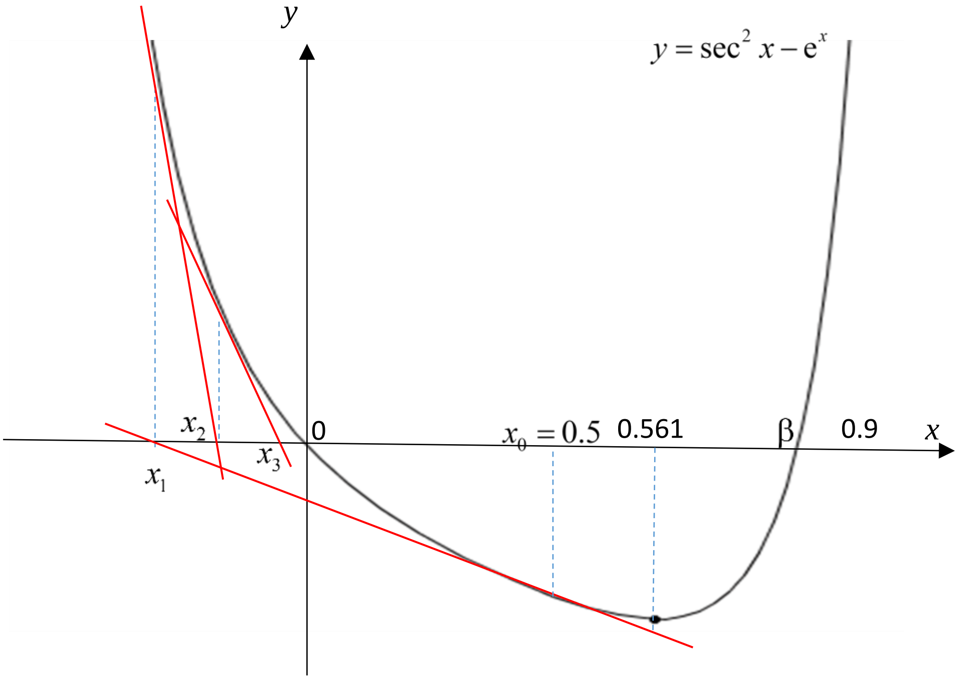
$$=x_n-\frac{\sec^n x_n-e^{x_n}}{2\sec^2 x_n\tan x_n-e^{x_n}}$$

$$x_1=0.5-\frac{\mathrm{f}(0.5)}{\mathrm{f}'(0.5)}=0.5-\frac{-0.35027486}{-0.2300322}\approx-1.0227$$

$x_2\approx-0.75526$

$x_3\approx-0.40306$

As $n \to \infty$, $x_n \to 0$. x_n converges to the other root 0 instead of β.



As shown in the diagram, since the initial approximation $x_0 = 0.5$ lies on the LHS of the minimum point. The tangent line has negative gradient and cuts the x-axis to the left of origin. All subsequent approximations will converge to the other root 0 instead of β.

6 The cartesian equation for a hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. In this question, the constants a and b, along with all distances, are positive and represent lengths in *astronomical units* (AUs), where 1 AU is the radius of the Earth's orbit (assumed circular) around the Sun.

In 2002, comet C2002/Q3A passed through the solar system on a hyperbolic orbit, with the Sun at a focus.

(i) **(a)** The comet's perihelion is that point on its orbital path at which it is closest to the Sun. The perihelion distance from the Sun is p AU and the eccentricity of the orbit is e. Show that $p = a(e - 1)$. [2]

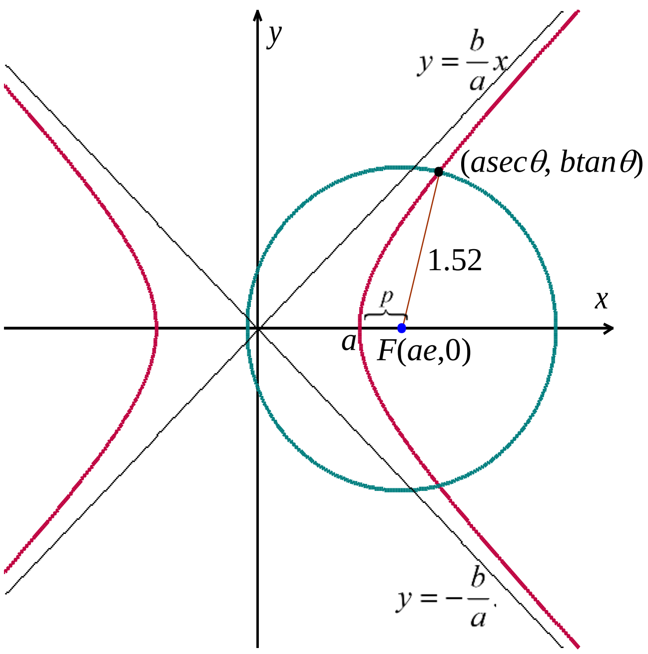
(b) Given that, for comet C2002/Q3A, $e = 1.04$ and $p = 1.312$, determine the values of a and b to 3 significant figures. [2]

(ii) Mars has an approximately circular orbit around the Sun with radius 1.52 AU. It is to be assumed that the motions of Mars and the comet take place in the same plane.

The orbit of comet C2002/Q3A can be written in the parametric form $x = a\sec\theta$, $y = b\tan\theta$, for some suitably chosen angle θ. Write down, in terms of θ, a definite integral that gives the length of the part of the comet's orbit that lies within the orbit of Mars. Evaluate this integral numerically. [7]

[Solution]

Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $a > 0, b > 0$.



(i) $x = a\sec\theta$, $y = b\tan\theta$

Sun at focus $(ae, 0)$.

(a) Point on the orbital path at which it is closest to the sun is the point $(a, 0)$.

$\therefore\ p = ae - a$

$= a(e-1)$

(b) Given that $e = 1.04$, $p = 1.312$,

$\therefore\ 1.312 = a(0.04)$

$a = 32.8$

Since, $a^2 + b^2 = (ae)^2$

$b^2 = 87.788544$

$b = 9.37$ (3s.f)

(ii) Circle: $(x-ae)^2 + (y-0)^2 = 1.52^2$

$(a\sec\theta - ae)^2 + (b\tan\theta)^2 = 1.52^2$

$a^2\sec^2\theta - 2a^2e\sec\theta + a^2e^2 + b^2(\sec^2\theta - 1) = 1.52^2$

$(a^2+b^2)\sec^2\theta - 2a^2e\sec\theta + (a^2e^2 - b^2) = 1.52^2$

$a^2e^2\sec^2\theta - 2a^2e\sec\theta + a^2 = 1.52^2$

$(ae\sec\theta - a)^2 = 1.52^2$

Since $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\sec\theta > 0$. $\therefore\ \sec\theta = \frac{1.52+a}{ae} \Rightarrow \cos\theta = \frac{ae}{1.52+a} \approx 0.994$

$\theta \approx 0.110$

$$\text{Arc length} = 2\int_0^{0.11}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{dt}}\right)^2+\left(\frac{\mathrm{dy}}{\mathrm{dt}}\right)^2}\ dt$$

$$= 2\int_0^{0.11}\sqrt{(32.8\sec\theta\tan\theta)^2+(9.37\sec^2\theta)^2}\ dt$$

$$=1.0599 = 2.11986 \approx 2.12$$

7 In order to control fluctuations in the price of grain, the Government operates a '*buffer stock scheme*'. In years of plenty, the Government purchases excess grain and places it in long-term storage in order to prevent prices falling too low. The scheme runs according to the Government's Seven-Year Plan (SYP).

(i) One SYP is about to begin now, and there is 100 000 tonnes of grain currently in storage. In this SYP, the Government will purchase 33 000 tonnes of grain at the end of each year. At the end of each year, $7\frac{1}{2}\%$ of the grain in storage is destroyed as a result of contamination and decay and an additional 5% is given away for famine relief abroad.

Express this situation as a recurrence relation, using g_n to denote the tonnage of grain held in storage at the end of year n of the SYP. Solve this recurrence relation and **hence** show that there will be approximately 200 000 tonnes of grain in storage at the end of this SYP. [7]

(ii) A forward-planning Government department prepares for the next SYP, which will begin with an initial 200 000 tonnes of grain in storage. One of the possible scenarios is that, following the present SYP, there will be a period of seven years of poor grain harvests. In this case, to prevent prices from soaring, the next SYP will involve, at the end of each year, the release of a fixed amount, t tonnes, into the market, the destruction of $7\frac{1}{2}\%$ of the grain and a reduction to $2\frac{1}{2}\%$ in the amount of grain given in foreign aid.

Find the value of t for which the grain reserve would be reduced to zero at the end of the next SYP. [5]

[Solution]

(i) $g_1 = 33000 + (1-0.075-0.05)100000 = 33000 + (0.875)100000$

$g_2 = 33000 + 0.875g_1 = 33000(1+0.875) + 0.875^2(100000)$

$g_3 = 33000 + 0.875g_2 = 33000(1+0.875+0.875^2) + 0.875^3(100000)$

$\vdots$

$\therefore\ g_n = 33000 + 0.875g_{n-1}$ is the recurrence relation.

Also, $g_n = 33000(1+0.875+0.875^2+\cdots+0.875^{n-1}) + 0.875^n(100000)$

$$= 33000\left(\frac{1-0.875^n}{1-0.875}\right) + 0.875^n(100000)$$

$$= 264000(1-0.875^n) + 0.875^n(100000)$$

$\therefore\ g_7 = 199597.872 \approx 200000$ (to 3 sf)

(ii) $g_0 = 200000$

$g_1 = [1-(0.075+0.025)]g_0 - t = 0.9g_0 - t$

$g_2 = 0.9g_1 - t = 0.9(0.9g_0 - t) - t = 0.9^2 g_0 - t(1+0.9)$

$\vdots$

So, $g_7 = 0.9^7 g_0 - t(1+0.9+0.9^2+\cdots+0.9^6) = 0$

$$\Rightarrow\ t\left(\frac{1-0.9^7}{1-0.9}\right) = 0.9^7(200000)$$

$$\Rightarrow\ t = 18335.98$$

8 The number of plaice (a type of fish) in a region of the North Sea is denoted by P (in millions). The chosen population model for P is

$$\frac{dP}{dt} = \tfrac{1}{3}P(1 - \tfrac{1}{4}P) - h,$$

where $h \geqslant 0$ is a harvesting term.

(i) In the case when $h = 0$, describe what this model predicts for the population of plaice, justifying fully your conclusions. [6]

(ii) **(a)** Determine the values of h for which there are two (distinct and positive) equilibrium population values. State these values in terms of h and say whether they are stable or unstable. [3]

(b) On a single diagram, illustrate the behaviour of P for $t \geqslant 0$ in each of the possible cases that can arise for different values of the initial population P_0 relative to these equilibrium values. [4]

[Solution]

$$\frac{dP}{dt} = \frac{1}{3}P\left(1 - \frac{1}{4}P\right) - h$$

(i) When $h = 0$, $\quad \dfrac{dP}{dt} = \dfrac{1}{3}P\left(1 - \dfrac{1}{4}P\right)$

$$\int \frac{1}{P(4-P)}\,dP = \int \frac{1}{12}\,dt$$

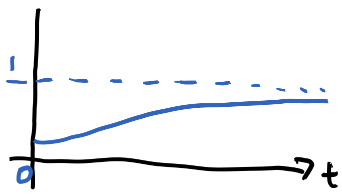
$$\frac{1}{4}\int \left(\frac{1}{P} + \frac{1}{4-P}\right) dP = \frac{1}{12}t + c$$

$$\frac{1}{4}\left[\ln|P| - \ln|4-P|\right] = \frac{1}{12}t + c$$

$$\ln\left|\frac{P}{4-P}\right| = \frac{1}{3}t + c'$$

$$\frac{P}{4-P} = Ae^{t/3} \quad \text{where} \quad A = \pm e^{c'}$$

$$P = \frac{4Ae^{t/3}}{1 + Ae^{t/3}} = \frac{4A}{e^{-t/3} + A}$$

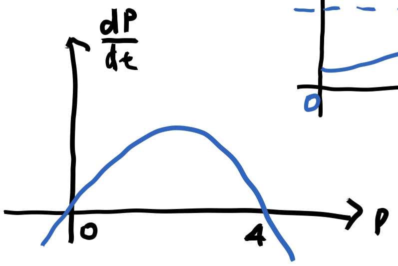


As $t \to \infty$, $\quad e^{-t/3} \to 0$. $\quad \therefore \quad P \to 4$ (millions).

OR $\dfrac{dP}{dt} = \dfrac{1}{3}P\left(1 - \dfrac{1}{4}P\right)$

Let $\dfrac{dP}{dt} = \dfrac{1}{3}P\left(1 - \dfrac{1}{4}P\right) = 0$

$\Rightarrow$ equilibrium values : $P = 0$ or $P = 4$.



For $0 < P < 4$, $\dfrac{dP}{dt} > 0$, population of the fish increases and stabilizes at 4 millions.

For $P = 4$, $\dfrac{dP}{dt} = 0$, the population of the fish remains constant at 4 millions.

For $P > 4$, $\dfrac{dP}{dt} < 0$, the population of the fish decreases and stabilizes at 4 millions.

(ii) (a) At equilibrium, $\frac{\mathrm{d}P}{\mathrm{d}t}=0 \quad \Rightarrow \quad \frac{1}{3}P\left(1-\frac{1}{4}P\right)-h=0$

$$\Rightarrow \quad P^2-4P+12h=0$$

$$P=\frac{4\pm\sqrt{16-4(12h)}}{2}=2\pm2\sqrt{1-3h}$$

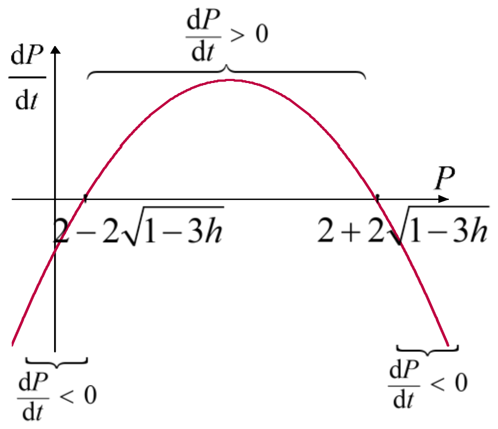
For 2 distinct positive equilibrium values of P,

$1-3h>0 \quad \Rightarrow \quad h<\frac{1}{3}$

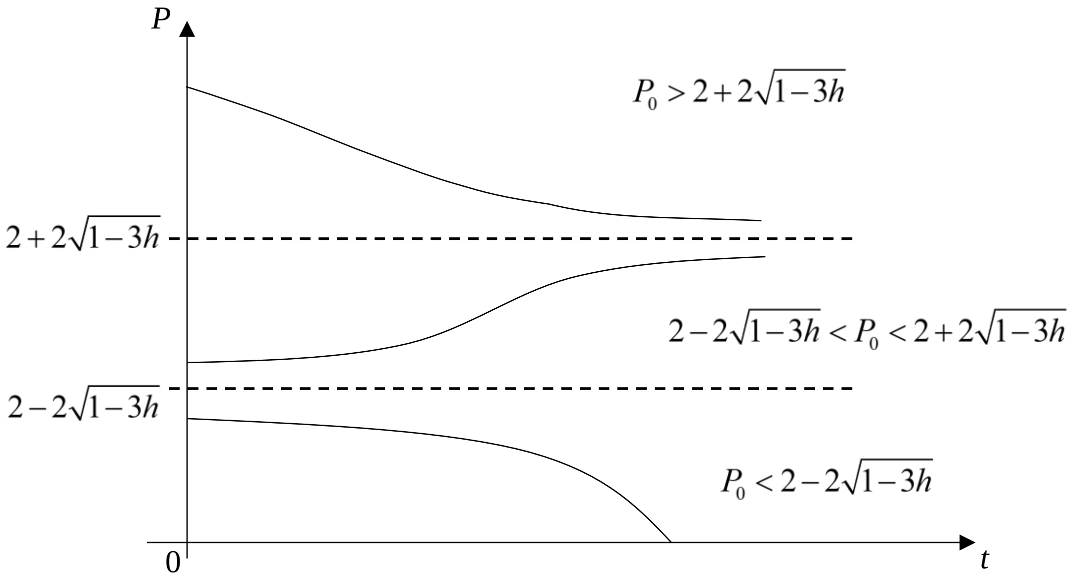
From the diagram,

$P=2-2\sqrt{1-3h}$ is unstable

but $P=2+2\sqrt{1-3h}$ is stable.



(b)



9 A particle P is moving along a straight line which passes through a fixed point O. A student is recording the motion of P and starts timing at the instant when P is 1 metre from O and travelling at 1 m s^{-1} away from O. The student models the acceleration of P as $\frac{1}{x^3}$, where x metres is the distance OP.

(i) Explain why P is always moving in the direction away from O. [2]

(ii) Show that the acceleration of P can be written as $v\frac{\mathrm{d}v}{\mathrm{d}x}$, where $v\text{ m s}^{-1}$ is the speed of P at time t. [1]

(iii) Determine the relationship between x and t. Hence find the distance travelled by P during the first 20 seconds of its motion. [11]

[Solution]

Given that acceleration of P = $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{1}{x^3}$

(i) $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{1}{x^3} > 0$ where $x > 0$, since x is the distance OP.

i.e. $v = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)$ is an increasing function with time t. Hence, P does not turn back and keeps moving in the direction away from O.

(ii) Since $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{x^3}$

$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right) = \frac{1}{x^3}$

$\Rightarrow v\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right) = \frac{1}{x^3}$

$\therefore$ acceleration of P = $v\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)$

(iii) $v\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right) = \frac{1}{x^3}$

$\Rightarrow \int v\,\mathrm{d}v = \int \frac{1}{x^3}\,\mathrm{d}x$

$\Rightarrow \frac{v^2}{2} = -\frac{1}{2x^2} + c$

$\Rightarrow v^2 = A - \frac{1}{x^2}$ where $A = 2c$

When $x = 1$, $v = 1$ $\Rightarrow$ $1 = A - 1$ $\Rightarrow$ $A = 2$

$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{2 - \frac{1}{x^2}} = \frac{\sqrt{2x^2-1}}{x^2}$

$\Rightarrow \int \frac{x}{\sqrt{2x^2-1}}\,\mathrm{d}x = \int 1\,\mathrm{d}t$

$\Rightarrow \frac{1}{4}\int 4x\left(2x^2-1\right)^{-\frac{1}{2}}\,\mathrm{d}x = t + B$

$$\Rightarrow \quad \frac{1}{4}\frac{\left(2x^2-1\right)^{\frac{1}{2}}}{\frac{1}{2}} = t + B$$

$$\Rightarrow \quad \sqrt{\left(2x^2-1\right)} = 2t + 2B$$

When $t = 0$, $x = 1$ $\Rightarrow \quad 1 = 2B \quad \Rightarrow \quad B = \frac{1}{2}$

$$\Rightarrow \quad 2x^2 = \left(2t+1\right)^2 + 1$$

When $t = 20$, $2x^2 = 41^2 + 1 \quad \Rightarrow x = 29$ as $x > 0$

Hence distance travelled by P = 29 – 1 = 28 m.

10 The function $y = y(x)$ satisfies $\frac{dy}{dx} = \frac{1}{10}(\sin x - xy)$.

(i) The value of $y(h)$ is to be found, where h is a small positive number, and $y(0) = 0$.

(a) Use two steps of Euler's method to determine an approximation to $y(h)$ in terms of h. [3]

(b) Use one step of the improved Euler formula to find an alternative approximation to $y(h)$ in terms of h. [2]

(ii) Show that $y = y(x)$ satisfies $e^{0.05h^2} y(h) = \int_0^h 0.1e^{0.05x^2} \sin x \, dx$. [4]

(iii) Use the fact that h is small to estimate $\int_0^h 0.1e^{0.05x^2} \sin x \, dx$. Hence find another approximation to $y(h)$ in terms of h. [5]

(iv) Discuss the relative merits of the three methods employed to obtain these approximations. [2]

[Solution]

Let $f(x, y) = \frac{dy}{dx} = \frac{1}{10}(\sin x - xy)$.

(i)(a) By Euler's Method, $y\left(\frac{h}{2}\right) \approx y(0) + \frac{h}{2} f(0, 0) = 0$

$$y(h) \approx y\left(\frac{h}{2}\right) + \frac{h}{2} f\left(\frac{h}{2}, 0\right)$$

$$= 0 + \frac{h}{2} \times \frac{1}{10}\left(\sin\frac{h}{2} - 0\right) = \frac{h}{20}\sin\frac{h}{2}$$

(b) $u_1 \approx y(0) + hf(0, 0) = 0$

By Improved Euler's Method, $y(h) \approx y(0) + \frac{h}{2}\left[f(0, y(0)) + f(h, u_1)\right]$

$$= 0 + \frac{h}{2}\left[0 + \frac{1}{10}(\sin h - 0)\right]$$

$$= \frac{h}{20}\sin h$$

(ii) $\frac{dy}{dx} = \frac{1}{10}(\sin x - xy) \Rightarrow \frac{dy}{dx} + 0.1xy = 0.1\sin x$

Integrating Factor = $e^{\int 0.1x \, dx} = 10e^{0.05x^2}$

Multiply integrating factor, we get: $10e^{0.05x^2}\frac{dy}{dx} + 10e^{0.05x^2} 0.1xy = 10e^{0.05x^2} 0.1\sin x$

$$\frac{d}{dx}\left(10e^{0.05x^2} y\right) = 10e^{0.05x^2} 0.1\sin x$$

$$\left[e^{0.05x^2} y\right]_0^h = \int_0^h 0.1e^{0.05x^2} \sin x \, dx$$

$$e^{0.05h^2} y(h) - y(0) = \int_0^h 0.1e^{0.05x^2} \sin x \, dx \quad \text{(Shown)}$$

(iii) Since h is small, $\therefore x$ small, assuming terms in x^3 above are negligible,

$$\int_0^h 0.1\mathrm{e}^{0.05x^2}\sin x\,\mathrm{d}x \approx \int_0^h 0.1\left(1+0.05x^2\right)(x)\,\mathrm{d}x$$

$$= 0.1\left[\frac{1}{2}x^2+\frac{0.05x^4}{4}\right]_0^h$$

$$= 0.05h^2+0.00125h^4$$

$$\mathrm{e}^{0.05h^2}y(h)-y(0)\approx 0.05h^2+0.00125h^4$$

$$y(h)\approx\frac{0.05h^2+0.00125h^4}{\mathrm{e}^{0.05h^2}}$$

$$\approx\left(0.05h^2+0.00125h^4\right)\left(1-0.05h^2\right)$$

$$=\frac{h^2}{20}-\frac{h^4}{800}\approx\frac{h^2}{20}$$

(iv) When h is small, apply $\sin h\approx h-\frac{h^3}{3!}$

(a) $y(h)\approx\frac{h}{20}\sin\frac{h}{2}\approx\frac{h}{20}\left[\frac{h}{2}-\frac{\left(\frac{h}{2}\right)^3}{3!}\right]=\frac{h^2}{40}-\frac{h^4}{960}$ [Euler's method]

(b) $y(h)\approx\frac{h}{20}\sin h\approx\frac{h}{20}\left(h-\frac{h^3}{3!}\right)=\frac{h^2}{20}-\frac{h^4}{120}$ [Improved Euler's method]

(c) $y(h)\approx\frac{h^2}{20}-\frac{h^4}{800}$ [Maclaurin's series approximation]

Accuracy of method **(a)** can be easily improved by increasing the number of intervals in [0, *h*].

Method **(b)** is more accurate as compared to method **(a)** as it takes the corrected value $\frac{1}{2}\left[\mathrm{f}\left(0,y(0)\right)+\mathrm{f}\left(h,u_1\right)\right]$ to be the estimated gradient at $y(0)$, reducing the error incurred.

However, it will be tedious to perform the calculation using method **(b)** if the number of intervals is increased.

Method **(c)** gives a good estimation, similar to method **(b).**

The order of approximation when method **(c)** is used, can be easily increased by keeping more higher ordered terms in the Maclaurin's series expansion of $\mathrm{e}^{0.05x^2}$ and $\sin x$.

However method **(c)** is useful only in this question because *h* is given as small and so is close to 0. Method **(c)** requires *h* to be small unlike method (a) and **(b)** which do not require *h* to be small.