1 **(i)** Find the derivative of $\sqrt{4-x^2}$ with respect to x. [1]

(ii) Given the differential equation $\sqrt{4-x^2}\,\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=1$, find y in terms of x. [4]

[Solution]

(i) $$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2}\left(4-x^2\right)^{-\frac{1}{2}}\left(-2x\right)=\frac{-x}{\sqrt{4-x^2}}$$

(ii) $$\sqrt{4-x^2}\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=1$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{\sqrt{4-x^2}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\int\frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x$$

$$=\sin^{-1}\frac{x}{2}+C$$

$$y=\int\sin^{-1}\frac{x}{2}+C\ \mathrm{d}x$$

$$=x\sin^{-1}\frac{x}{2}-\frac{1}{2}\int\frac{x}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,\mathrm{d}x+Cx \qquad =x\sin^{-1}\frac{x}{2}-\int\frac{x}{\sqrt{4-x^2}}\,\mathrm{d}x+Cx$$

$$=x\sin^{-1}\frac{x}{2}+\sqrt{4-x^2}+Cx+D$$ from part **(i)**

2 **(a)** The point A has coordinates $(3, a, b)$ where $a, b \in \square$. Given that A lies on the xy-plane and the magnitude of the position vector of A is 5, find the values of a and b. [3]

(b) The real numbers c and d are such that the vectors $\mathbf{m}=\mathbf{i}+d\mathbf{j}+c\mathbf{k}$ and $\mathbf{n}=c\mathbf{i}+d\mathbf{j}+\mathbf{k}$ are perpendicular to each other. Show that $|\mathbf{m}\times\mathbf{n}|=(c-1)^2$. [3]

[Solution]

(a) Since (3, *a*, *b*) lies on the *x*-*y* plane, *b* = 0.

$$\left|\overrightarrow{AB}\right| = 5 \quad \Rightarrow \quad \sqrt{3^2 + a^2 + 0^2} = 5$$

$$\Rightarrow \quad a^2 = 16$$

$$\Rightarrow \quad a = 4 \text{ or } -4$$

(b) Since $\begin{pmatrix} 1 \\ d \\ c \end{pmatrix}$ and $\begin{pmatrix} c \\ d \\ 1 \end{pmatrix}$ are perpendicular,

$$\Rightarrow \begin{pmatrix} 1 \\ d \\ c \end{pmatrix} \cdot \begin{pmatrix} c \\ d \\ 1 \end{pmatrix} = 0 \Rightarrow d^2 = -2c$$

$$|\mathbf{m} \times \mathbf{n}| = |\mathbf{m}||\mathbf{n}|\sin 90^o = |\mathbf{m}||\mathbf{n}|$$

$$= \sqrt{1 + d^2 + c^2}\sqrt{c^2 + d^2 + 1} = 1 + c^2 + d^2$$

$$= 1 + c^2 - 2c = \left(1 - c^2\right)$$

3 Given $\mathrm{f}\left(z\right) = pz^2 + qz + r$ where *p*, *q* and *r* are complex numbers such that $\mathrm{f}\left(1\right) = 2\mathrm{i}$. The equation $\mathrm{f}\left(z\right) = 0$ has roots $1 - \mathrm{i}$ and $1 - 2\mathrm{i}$. Find *p*, *q* and *r*. [6]

[Solution]

Since $\mathrm{f}\left(z\right) = 0$ has roots $1 - \mathrm{i}$ and $1 - 2\mathrm{i}$,

$$\therefore \mathrm{f}\left(z\right) = p\left(z - \left(1 - \mathrm{i}\right)\right)\left(z - \left(1 - 2\mathrm{i}\right)\right), \quad k \in \mathbb{C}$$

Since $\mathrm{f}\left(1\right) = 2\mathrm{i}$,

$$\therefore p\left(1 - \left(1 - \mathrm{i}\right)\right)\left(1 - \left(1 - 2\mathrm{i}\right)\right) = 2\mathrm{i}$$

$$\Rightarrow \quad p\left(\mathrm{i}\right)\left(2\mathrm{i}\right) = 2\mathrm{i}$$

$$\Rightarrow \quad -2p = 2\mathrm{i}$$

$$\Rightarrow \quad p = -\mathrm{i}$$

$$\therefore \mathrm{f}\left(z\right) = -\mathrm{i}\left(z - \left(1 - \mathrm{i}\right)\right)\left(z - \left(1 - 2\mathrm{i}\right)\right)$$

$$= -\mathrm{i}\left(z^2 - \left(1 - \mathrm{i} + 1 - 2\mathrm{i}\right)z + \left(1 - \mathrm{i}\right)\left(1 - 2\mathrm{i}\right)\right)$$

$$= -\mathrm{i}\left(z^2 + \left(-2 + 3\mathrm{i}\right)z - 1 - 3\mathrm{i}\right)$$

$$= -\mathrm{i}\,z^2 + \left(3 + 2\mathrm{i}\right)z - 3 + \mathrm{i}$$

Therefore, $p = -\mathrm{i}$, $q = 3 + 2\mathrm{i}$ and $r = -3 + \mathrm{i}$

4 Without the use of a graphic calculator, solve the inequality $2x+5\le\frac{10}{2-x}$. [3]

Hence find the solution to the inequality $2\cos\theta+5\le\frac{10}{2-\cos\theta}$, where $0\le\theta\le2\pi$. [3]

[Solution]

$$2x+5\le\frac{10}{2-x}\Rightarrow\frac{(2x+5)(2-x)-10}{2-x}\le0$$

$$\Rightarrow\frac{-2x^2-x}{2-x}\le0$$

$$\Rightarrow\frac{2x^2+x}{x-2}\le0$$

$$\therefore x\le-\frac{1}{2}\quad\text{or}\quad 0\le x<2$$

For $2\cos\theta+5\le\frac{10}{2-\cos\theta}$

Replace x by $\cos\theta$, we have

$$\therefore\cos\theta\le-\frac{1}{2}\quad\text{or}\quad 0\le\cos\theta<2$$

For $\cos\theta\le-\frac{1}{2}\Rightarrow\frac{2\pi}{3}\le\theta\le\frac{4\pi}{3}$

For $0\le\cos\theta<2\Rightarrow0\le\cos\theta\le1$

$$\therefore 0\le\theta\le\frac{\pi}{2}\text{ or }\frac{3\pi}{2}\le\theta\le2\pi$$

$$\therefore 0\le\theta\le\frac{\pi}{2}\text{ or }\frac{2\pi}{3}\le\theta\le\frac{4\pi}{3}\text{ or }\frac{3\pi}{2}\le\theta\le2\pi$$

5 A souvenir company received an order to produce a souvenir that must satisfy all of the following conditions:

(1) The souvenir is a solid cuboid with a square base.
(2) The souvenir is made using 1m^3 of superior clay.
(3) The external surface of the souvenir must be coated with a special-mixed glow paint.

Find the dimensions of the souvenir, in m, such that the amount of special paint needed is the minimum. [7]

[Solution]

Let the length of the square base be x m and the height of the cuboid be y m.

Volume of cuboid = 1 m^3

$x^2y=1 \Rightarrow \quad y=\dfrac{1}{x^2}$ ---(1)

To minimise use of paint, surface area S, has to be kept at a minimum.

$S=2x^2+4xy$ ---(2)

Sub (1) into (2):

$$S=2x^2+4x\left(\frac{1}{x^2}\right)=2x^2+\frac{4}{x}$$

$$\frac{\mathrm{d}S}{\mathrm{d}x}=4x-\frac{4}{x^2} \qquad\qquad \frac{\mathrm{d}^2S}{\mathrm{d}x^2}=4+\frac{8}{x^3}$$

For S to be minimum,

Set $\dfrac{\mathrm{d}S}{\mathrm{d}x}=0$

$\Rightarrow \quad 4x-\dfrac{4}{x^2}=0 \Rightarrow x^3-1=0$

$\Rightarrow \quad x=1$

When x = 1, $y=\dfrac{1}{x^2}=1$, $\dfrac{\mathrm{d}^2S}{\mathrm{d}x^2}=4+\dfrac{8}{x^3}=12>0$

Therefore, the required dimension is 1m by 1m by 1m.

6 A contagious disease was found to infect a village with a population of 10000 people. Let P, in thousands, be the number of infected people t days after the start of the outbreak. The disease spread at a rate that is proportional to the product of the number of infected people and the number of non-infected people. It was found that when P reaches half the initial population of the village, the disease is spreading at a rate of 1000 people per day.

Show that the spread of the disease can be modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t}=\frac{25-(P-5)^2}{25}.$$ [2]

Given that 100 people are infected by the disease initially, find P in terms of t. [3]

Explain what will happen to the village population in the long term. [2]

[Solution]

$$\frac{\mathrm{d}P}{\mathrm{d}t}=kP(10-P)$$

When P = 5,

$$\frac{\mathrm{d}P}{\mathrm{d}t}=1 \Rightarrow k(5)(10-5)=1$$

$$\therefore k=\frac{1}{25}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t}=\frac{10P-P^2}{25}=\frac{25-(P-5)^2}{25}$$

$$\int \frac{1}{5^2-(P-5)^2}\mathrm{d}P=\int \frac{1}{25}\mathrm{d}t$$

$$\frac{1}{10}\ln\left(\frac{5+P-5}{5-P+5}\right)=\frac{1}{25}t+C$$ where $P < 10$ and C is arbitrary constant

$$\ln\left(\frac{P}{10-P}\right)=\frac{2}{5}t+C'$$

$$\frac{P}{10-P}=Ae^{\frac{2}{5}t}$$ where $A=e^{C'}$

$$P=Ae^{\frac{2}{5}t}(10-P)$$

$$P=\frac{10Ae^{\frac{2}{5}t}}{1+Ae^{\frac{2}{5}t}}$$

When t = 0, $P=\frac{100}{1000}=\frac{1}{10}$,

$$\frac{1}{10}=\frac{10A}{1+A} \therefore A=\frac{1}{99}$$

$$\therefore P=\frac{\frac{10}{99}\mathrm{e}^{\frac{2}{5}t}}{1+\frac{1}{99}\mathrm{e}^{\frac{2}{5}t}}=\frac{10}{99e^{-\frac{2}{5}t}+1}$$

As $t\to\infty, \mathrm{e}^{-\frac{2}{5}t}\to 0$, $P\to 10$

Therefore, since number of infected people will eventually become 10 000, the whole village people will eventually be infected by the disease.

7 A convergent geometric sequence of positive terms, *G* has first term *a* and common ratio *r*.

Write down, in terms of *a* and *r*, an expression for the *n*th odd-numbered term of *G*. [1]

If the sum of first *n* odd-numbered terms of *G* is equal to the sum of all terms of *G* after the *n*th odd-numbered term, show that $2r^{2n}+r^{2n-1}-1=0$.

(i) Hence find the value of *r* when $n=5$. [3]

(ii) In another sequence *H*, each term is the reciprocal of the corresponding term of *G*. If the *n*th term of *G* and *H* is denoted by u_n and v_n respectively, show that a new sequence whose *n*th term is $\ln\left(\frac{u_n}{v_n}\right)$, is an arithmetic progression.

[4]

[Solution]

The *n*th odd numbered term would be $U_{2n-1}=ar^{2n-2}$.

Given that $U_1+U_3+\cdots+U_{2n-1}=U_{2n}+U_{2n+1}+\cdots$

ie $a+ar^2+\cdots+ar^{2n-2}=ar^{2n-1}+ar^{2n}+ar^{2n+1}+\cdots$

$$\Rightarrow \frac{a\left(1-\left(r^2\right)^n\right)}{1-r^2}=\frac{ar^{2n-1}}{1-r}$$

$$\Rightarrow \frac{1-r^{2n}}{(1+r)(1-r)}=\frac{r^{2n-1}}{1-r}$$

$$\Rightarrow 1-r^{2n}=r^{2n-1}(1+r)$$

$\therefore\quad 2r^{2n}+r^{2n-1}-1=0$ [Shown]

When $n=5$, we have $2r^{10}+r^9-1=0$

So $r=-1$ (NA convergent series) or 0.892

Now $V_n = \dfrac{1}{a}\left(\dfrac{1}{r}\right)^{n-1}$.

Letting the *n*th term of the new sequence be T_n, we have

$$T_n = \ln\left(\frac{U_n}{V_n}\right) = \ln(U_n) - \ln(V_n)$$

$$= \left(\ln a + (n-1)\ln r\right) - \left(-\ln a + (n-1)\ln\left(\frac{1}{r}\right)\right)$$

$$= 2\ln a + 2(n-1)\ln r$$

$$T_n - T_{n-1} = 2\ln a + 2(n-1)\ln r - 2\ln a - 2(n-2)\ln r$$

$$= 2\ln r \text{, a constant}$$

Hence the new sequence is an arithmetic progression.

Alternative:

$$T_n = \ln\left(\frac{U_n}{V_n}\right) = \ln(U_n) - \ln(V_n) = \ln(U_n) - \left[-\ln(U_n)\right] = 2\ln(U_n)$$

$$T_n - T_{n-1} = 2\ln(U_n) - 2\ln(U_{n-1}) = 2\left[\ln\left(\frac{ar^{n-1}}{ar^{n-2}}\right)\right] = 2\ln r = \text{constant}$$

8 The functions f and g are given by

$$\mathrm{f} : x \mapsto x^2 - 8x + 13, \qquad x \in \mathbb{R}, x \le 4,$$

$$\mathrm{g} : x \mapsto a - \mathrm{e}^{-x}, \qquad x \in \mathbb{R}.$$

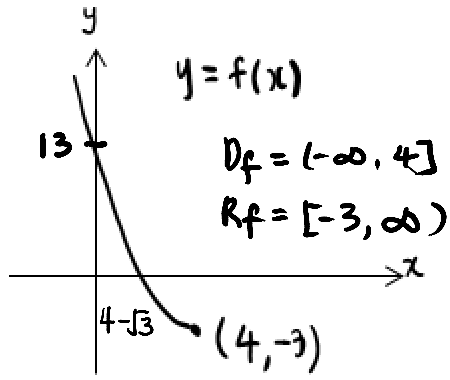
(i) Show that f^{-1} exists and express f^{-1} in a similar form, stating the domain clearly. [3]

(ii) Determine the largest integer value of a such that fg exists. [2]

(iii) For the largest value of a obtained in **(ii)**, find fg(x) and state the domain and the range of fg. [4]

[Solution]

(i) Any horizontal line $y = k$, $k \in \mathbb{R}$, cuts the graph at most once, therefore f is a one-one function. Thus, f^{-1} exists.



Let $y = x^2 - 8x + 13, \quad x \le 4$

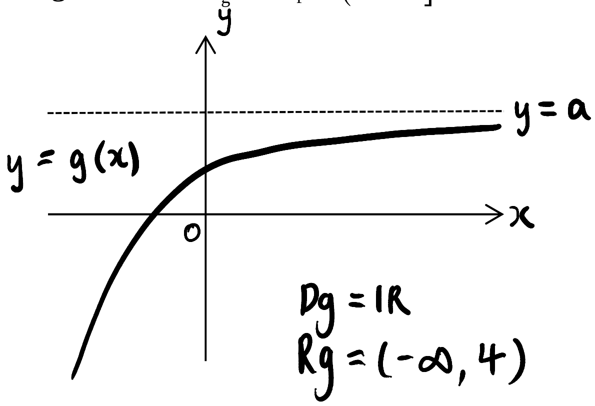
$= (x-4)^2 - 3$

$x - 4 = \pm\sqrt{y+3}$

$x = 4 - \sqrt{y+3}$ or $x = 4 + \sqrt{y+3}$ (rej $\because x \le 4$)

$\therefore\ f^{-1} : x \mapsto 4 - \sqrt{x+3}, \quad x \ge -3$

(ii) For fg to exist, $R_g \subseteq D_f = (-\infty, 4]$



Largest integer value of $a = 4$

(iii) $fg(x) = f\left(4 - e^{-x}\right)$

$= \left(4 - e^{-x}\right)^2 - 8\left(4 - e^{-x}\right) + 13$

$= 16 - 8e^{-x} + e^{-2x} - 32 + 8e^{-x} + 13$

$= e^{-2x} - 3$

OR

$= \left(4 - e^{-x} - 4\right)^2 - 3$

$= e^{-2x} - 3$

Provided you are confident with your Complete the Square.

$D_{fg} = D_g = \mathbb{R}$

$R_{fg} = (-3, \infty)$

9 Given that $\ln y = \mathrm{e}^x$, show that $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \dfrac{\mathrm{d}y}{\mathrm{d}x}\left(\mathrm{e}^x + 1\right)$. [2]

(i) Find the Maclaurin's series for $y = \mathrm{e}^{\mathrm{e}^x}$, up to and including the term in x^3. [4]

(ii) Find the first three non-zero terms of the Maclaurin series for $y = \mathrm{e}^{x+\mathrm{e}^x}$.

Hence find in terms of e, the approximate area bounded by the curve $y = \mathrm{e}^{x+\mathrm{e}^x}$, the x-axis, the y-axis and the line $x = 0.5$. [4]

[Solution]

$\ln y = \mathrm{e}^x$

Differentiate wrt x,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$$

$$\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = y\mathrm{e}^x$$

Differentiate wrt x,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^x + y\mathrm{e}^x$$

$$\Rightarrow \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^x + \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\Rightarrow \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}y}{\mathrm{d}x}\left(\mathrm{e}^x + 1\right) \qquad \text{(Shown)}$$

(i) $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \dfrac{\mathrm{d}y}{\mathrm{d}x}\left(\mathrm{e}^x + 1\right)$

Differentiate wrt x,

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\left(\mathrm{e}^x + 1\right) + \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^x$$

When $x = 0$,

$\ln y = \mathrm{e}^0 = 1 \qquad \Rightarrow \qquad y = \mathrm{e}$

$$\frac{dy}{dx}=e\left(e^{0}\right) \qquad \Rightarrow \qquad \frac{dy}{dx}=e$$

$$\frac{d^2y}{dx^2}=e\left(e^{0}+1\right) \qquad \Rightarrow \qquad \frac{d^2y}{dx^2}=2e$$

$$\frac{d^3y}{dx^3}=2e\left(e^{0}+1\right)+e\left(e^{0}\right) \quad \Rightarrow \quad \frac{d^3y}{dx^3}=4e+e=5e$$

$$\ln y=e^{x} \qquad \Rightarrow \qquad y=e^{e^{x}}$$

Thus Maclaurin's series for y is

$$y=e+ex+\frac{2e}{2!}x^2+\frac{5e}{3!}x^3+\ldots$$

$$\Rightarrow \qquad y=e\left(1+x+x^2+\frac{5}{6}x^3+\ldots\right)$$

(iii) $\quad y=e^{x+e^{x}}=e^{x}e^{e^{x}}$

Using standard expansion for e^{x},

$$y=e^{x}e^{e^{x}}=\left(1+x+\frac{x^2}{2!}+\ldots\right)e\left(1+x+x^2+\frac{5}{6}x^3+\ldots\right)$$

$$=e\left(1+x+x+x^2+x^2+\frac{x^2}{2!}+\ldots\right)$$

$$=e\left(1+2x+\frac{5}{2}x^2+\ldots\right)$$

<u>Alternative solution</u>

$$e^{e^{x}}=e\left(1+x+x^2+\frac{5}{6}x^3+\ldots\right)$$

Differentiate wrt x,

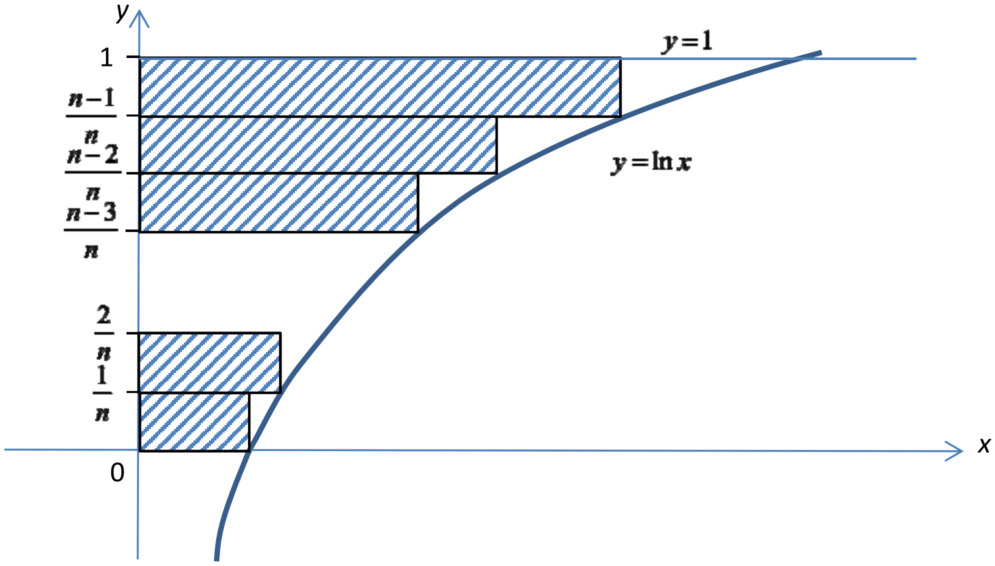
$$e^{x}e^{e^{x}}=e\left(1+2x+\frac{5}{6}\left(3x^2\right)+\ldots\right)$$

$$\Rightarrow \qquad e^{x+e^{x}}=e\left(1+2x+\frac{5}{2}x^2+\ldots\right)$$

Required area = $\int_0^{0.5} e^{x+e^x} \approx e\int_0^{0.5} 1+2x+\frac{5}{2}x^2 \, dx$

$=e\left[x+x^2+\frac{5x^3}{6}\right]_0^{0.5}=\frac{41}{48}e$

10



The region R is bounded by the x-axis, the y-axis, the line $y = 1$ and the curve $y = \ln x$ where $x \in \mathbb{R}, x > 0$.

The area of R may be approximated by the total area, A, of n rectangles each of height $\frac{1}{n}$, as shown in the above diagram.

Show that $A = \frac{1}{n}\left(\frac{1-\mathrm{e}}{1-\mathrm{e}^{\frac{1}{n}}}\right)$. [4]

Another finite region S is bounded by the x-axis, $x = \mathrm{e}$ and the curve $y = \ln x$ where $x \in \mathbb{R}, x > 0$.

Explain how A can be used to approximate the area of region S and state, with a reason, whether it is an underestimation or overestimation. [3]

Find the exact volume of the solid formed when region S is rotated completely about the y-axis. [3]

[Solution]

$$y = \ln x \Rightarrow x = e^y$$

Total area of the rectangles = $A = \frac{1}{n}\underbrace{\left(e^0 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}}\right)}_{GP\text{:common ratio } e^{\frac{1}{n}}\text{;first term1;}n\text{ terms}}$

$$= \frac{1}{n}\left(\frac{1-(e^{\frac{1}{n}})^n}{1-e^{\frac{1}{n}}}\right) = \frac{1}{n}\left(\frac{1-e}{1-e^{\frac{1}{n}}}\right)$$

Area of region $S = 1\times \mathrm{e} - \text{Area of region } R \approx \mathrm{e} - A$

From the diagram, A is an underestimation for region R. Hence using $e - A$ to approximation region S will be an overestimation.

Required volume

$$= \pi\left(e\right)^2\left(1\right) - \pi\int_0^1 \left(e^y\right)^2 dy = \pi e^2 - \pi\left[\frac{1}{2}e^{2y}\right]_0^1 = \pi e^2 - \frac{\pi}{2}\left[e^2 - 1\right] = \frac{\pi}{2}\left[e^2 + 1\right]$$

11(i) Prove by the method of induction that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n\left(n+1\right)\left(2n+1\right). \qquad [4]$$

(ii) It is given that $\mathrm{f}\left(r\right) = r^4$. Show that

$$\mathrm{f}\left(r\right) - \mathrm{f}\left(r-1\right) = ar^3 + br^2 + ar - 1,$$

for constants a and b to be determined. Hence find a formula for $\sum_{r=1}^{n} r^3$, leaving your answer in a fully factorised form. [8]

[Solution]

(i) Let P_n be the statement "$\sum_{r=1}^{n} r^2 = \frac{1}{6}n\left(n+1\right)\left(2n+1\right)$" for $n \in \mathbb{Z}^+$.

LHS $= \sum_{r=1}^{1} r^2 = 1^2 = 1$

RHS $= \frac{1}{6}(1)(2)(3) = 1 = \text{LHS}$

$\therefore$ P_1 is true.

Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. "$\sum_{r=1}^{k} r^2 = \frac{1}{6}k\left(k+1\right)\left(2k+1\right)$".

To show $\sum_{r=1}^{k+1} r^2 = \frac{1}{6}\left(k+1\right)\left(k+2\right)\left(2k+3\right)$.

LHS $= \sum_{r=1}^{k} r^2 + (k+1)^2 = \frac{1}{6}k\left(k+1\right)\left(2k+1\right) + (k+1)^2$

$= \frac{1}{6}\left(k+1\right)\left[k\left(2k+1\right) + 6(k+1)\right]$

$= \frac{1}{6}\left(k+1\right)\left[2k^2 + 7k + 6\right]$

$= \frac{1}{6}\left(k+1\right)\left(k+2\right)\left(2k+3\right) = \text{RHS of } P_{k+1}$

$\therefore$ P_k true $\Rightarrow$ P_{k+1} true

Since P_1 is true and P_k true $\Rightarrow$ P_{k+1} true, then by method of mathematical induction,

"$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$" is true for $n \in \mathbb{Z}^+$.

(ii) Given that $f(r) = r^4$,

$$f(r-1) = (r-1)^4 = r^4 - 4r^3 + 6r^2 - 4r + 1$$

$$\therefore \quad f(r) - f(r-1) = 4r^3 - 6r^2 + 4r - 1 \text{ --- } (*)$$

where $a = 4$ and $b = -6$. (Shown)

$$\therefore \quad \sum_{r=1}^{n}\left(4r^3 - 6r^2 + 4r - 1\right) = \sum_{r=1}^{n}\left[f(r) - f(r-1)\right]$$

$$\begin{aligned}
\Rightarrow \sum_{r=1}^{n}\left(4r^3 - 6r^2 + 4r - 1\right) = [\quad & f(1) - f(0) \\
& + f(2) - f(1) \\
& + f(3) - f(2) \\
& + \ldots \\
& + f(n-1) - f(n-2) \\
& + f(n) - f(n-1) \quad] \\
= {} & f(n) - f(0) \\
= {} & n^4
\end{aligned}$$

$$\Rightarrow 4\sum_{r=1}^{n} r^3 - 6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 = n^4$$

$$\Rightarrow 4\sum_{r=1}^{n} r^3 - 6\left(\frac{n}{6}\right)(n+1)(2n+1) + 4\left(\frac{n}{2}\right)(1+n) - n = n^4$$

$$\begin{aligned}
\Rightarrow \sum_{r=1}^{n} r^3 &= \frac{1}{4}\left[n^4 + n(n+1)(2n+1) - 2n(n+1) + n\right] \\
&= \frac{1}{4}\left[n^4 + 2n^3 + 3n^2 + n - 2n^2 - 2n + n\right] \\
&= \frac{1}{4}\left[n^4 + 2n^3 + n^2\right] \\
&= \frac{1}{4}n^2\left(n^2 + 2n + 1\right) \\
&= \frac{1}{4}n^2\left(n^2 + 1\right)^2
\end{aligned}$$

12 Two planes p_1 and p_2 have equations $ax-3y-z=b$ and $4x+y+bz=2a$ respectively. They intersect at the line *l* which contains the point $A(1,0,-1)$.

(i) Find the values of *a* and *b*. [2]

(ii) Without the use of a graphic calculator, find a vector equation of the line *l*. [2]

Given that the point $N(-4,-6,12)$ is the foot of perpendicular from point $B(1,c,d)$ to the line *l*, show that $6c-13d=-217$. [3]

Another plane p_3 is parallel to the plane p_2 and contains *B*. Given that the distance between planes p_3 and p_2 is $\dfrac{5}{\sqrt{21}}$. Find the values of *c* and *d*. [5]

Hence, write down 2 possible equations of plane p_3. [2]

[Solution]

(i) Two planes p_1 and p_2 contains the point $A(1,0,-1)$:

$a(1)-3(0)-(-1)=b \Rightarrow a-b=-1$----(1)

$4(1)+(0)+b(-1)=2a \Rightarrow 2a+b=4$----(2)

Solving (1) and (2): $a=1; b=2$

(ii)

Direction vector of the line *l* = $\begin{pmatrix}1\\-3\\-1\end{pmatrix}\times\begin{pmatrix}4\\1\\2\end{pmatrix}=\begin{pmatrix}-5\\-6\\13\end{pmatrix}$

Vector equation of the line *l* : $\mathbf{r}=\begin{pmatrix}1\\0\\-1\end{pmatrix}+\lambda\begin{pmatrix}5\\6\\-13\end{pmatrix}$ for $\lambda\in\mathbf{R}$

(iii) Given $N(-4,-6,12)$ is the foot of perpendicular from point $B(1,c,d)$ to the line *l*,

$\Rightarrow \overline{BN} \perp \begin{pmatrix}5\\6\\-13\end{pmatrix}$

$$\Rightarrow \left[\begin{pmatrix}-4\\-6\\12\end{pmatrix}-\overrightarrow{OB}\right]\bullet\begin{pmatrix}5\\6\\-13\end{pmatrix}=0$$

$$\Rightarrow \overrightarrow{OB}\bullet\begin{pmatrix}5\\6\\-13\end{pmatrix}=\begin{pmatrix}-4\\-6\\12\end{pmatrix}\bullet\begin{pmatrix}5\\6\\-13\end{pmatrix}=-20-36-156$$

$$\Rightarrow \begin{pmatrix}1\\c\\d\end{pmatrix}\bullet\begin{pmatrix}5\\6\\-13\end{pmatrix}=-212$$

$$\Rightarrow 5+6c-13d=-212$$

$\therefore 6c-13d=-217$ (shown)

Plane p_3, parallel to plane p_2 and contains B, is of distance $\frac{5}{\sqrt{21}}$ units from plane p_2:

$$\Rightarrow \left|\frac{\overrightarrow{BN}\cdot\begin{pmatrix}4\\1\\2\end{pmatrix}}{\sqrt{16+1+4}}\right|=\frac{5}{\sqrt{21}}$$

$$\Rightarrow \left|\left(\begin{pmatrix}-4\\-6\\12\end{pmatrix}-\begin{pmatrix}1\\c\\d\end{pmatrix}\right)\cdot\begin{pmatrix}4\\1\\2\end{pmatrix}\right|=5$$

$$\Rightarrow (-16-6+24)-(4+c+2d)=\pm 5$$

$\Rightarrow c+2d=-7$ or $c+2d=3$

Consider

$6c-13d=-217$ ----(1)

$c+2d=-7$ ----------(2)

Solving (1) and (2): $c=-21; d=7$

Also

$6c-13d=-217$ ----(3)

$c+2d=3$ -----------(4)

Solving (3) and (4): $c = -15.8; d = 9.4$

Equations of plane p_3 are

$$\mathbf{r} \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -21 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = 4 - 21 + 14 = -3$$

and

$$\mathbf{r} \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -15.8 \\ 9.4 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = 4 - 15.8 + 18.8 = 7$$