2012 H2 MH Prelim P1 Solutions

Qn	Solution
1	$\dfrac{4x}{x-3} \geq 1$ $\Rightarrow \dfrac{4x}{x-3} - \dfrac{x-3}{x-3} \geq 0$ $\Rightarrow \dfrac{3x+3}{x-3} \geq 0$ $\Rightarrow \dfrac{x+1}{x-3} \geq 0$ $\Rightarrow x \leq -1$ or $x > 3$ Replace x by $\lvert x \rvert$, $\Rightarrow \lvert x \rvert \leq -1$ (N.A.) or $\lvert x \rvert > 3$ $\Rightarrow x > 3$ or $x < -3$

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2	(i) Let P_n be the statement "$u_n = n^2 2^{-n}$ for $n \in \mathbb{Z}^+$" LHS of P_1 = $u_1 = u_0 - 2^{-1}\left[(1)^2 - 4(1) + 2\right] = 2^{-1} = \dfrac{1}{2}$ RHS of $P_1 = (1)^2\, 2^{-1} = \dfrac{1}{2}$ = LHS of P_1 $\therefore P_1$ is true. Assume that P_k is true for some $k \in \mathbb{Z}^+$, i.e. $u_k = k^2 2^{-k}$ We want to prove P_{k+1}, i.e. $u_{k+1} = (k+1)^2\, 2^{-(k+1)}$ LHS of $P_{k+1} = u_{k+1} = u_k - 2^{-(k+1)}\left[(k+1)^2 - 4(k+1) + 2\right]$ $= k^2 2^{-k} - 2^{-(k+1)}\left[(k+1)^2 - 4(k+1) + 2\right]$ $= 2^{-(k+1)}\left[k^2 2 - (k+1)^2 + 4(k+1) - 2\right]$ $= 2^{-(k+1)}\left[2k^2 - k^2 - 2k - 1 + 4k + 4 - 2\right]$ $= 2^{-(k+1)}\left[k^2 + 2k + 1\right]$ $= 2^{-(k+1)}(k+1)^2$ = RHS of P_{k+1} $\therefore P_{k+1}$ is true. Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.

	(ii) $\sum_{n=1}^{N}\left[-2^{-n}\left(n^2-4n+2\right)\right]=\sum_{n=1}^{N}\left(u_n-u_{n-1}\right)$ $=u_1-u_0$ $+u_2-u_1$ $+u_3-u_2$ $\vdots$ $+u_N-u_{N-1}$ $=u_N=N^2 2^{-N}$
	(iii) $S_\infty=0$
3 (a)	**(i)** $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^2-1}=\frac{x}{\sqrt{x^2-1}}$
	(ii) $\int x\cos^{-1}\left(\frac{1}{x}\right)\mathrm{d}x=\frac{x^2}{2}\cos^{-1}\left(\frac{1}{x}\right)-\frac{1}{2}\int\frac{x}{\sqrt{x^2-1}}\,\mathrm{d}x$ $=\frac{x^2}{2}\cos^{-1}\left(\frac{1}{x}\right)-\frac{1}{2}\sqrt{x^2-1}+c$

(b)	Let $u=\dfrac{1}{x} \;\Rightarrow\; x=\dfrac{1}{u} \;\Rightarrow\; \dfrac{dx}{du}=-\dfrac{1}{u^2}$ when $x = 3$, $u=\dfrac{1}{3}$; when $x = 6$, $u=\dfrac{1}{6}$ $\displaystyle\int_3^6 \frac{1}{x\sqrt{x^2-9}}\,dx = \int_{\frac{1}{3}}^{\frac{1}{6}} \frac{u}{\sqrt{\frac{1}{u^2}-9}}\left(-\frac{1}{u^2}du\right)$ $\displaystyle= \int_{\frac{1}{3}}^{\frac{1}{6}} \frac{u}{\sqrt{\frac{1-9u^2}{u^2}}}\left(-\frac{1}{u^2}du\right)$ $\displaystyle= -\int_{\frac{1}{3}}^{\frac{1}{6}} \frac{1}{\sqrt{1-9u^2}}\,du$ $\displaystyle= -\left[\frac{1}{3}\sin^{-1}3u\right]_{\frac{1}{3}}^{\frac{1}{6}}$ $\displaystyle= -\frac{1}{3}\left[\frac{\pi}{6}-\frac{\pi}{2}\right]$ $\displaystyle=\frac{\pi}{9}$

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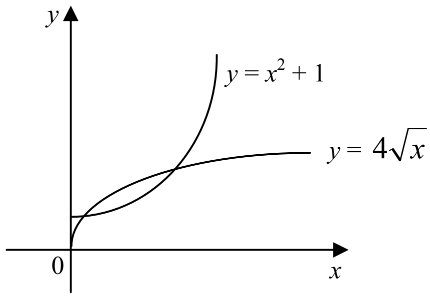
4	(i)  Points of intersection: $4\sqrt{x} = x^2 + 1$ $\Rightarrow$ (0.062997, 1.0040) and (2.2301, 5.9734) Area = $\int_{0.062997}^{2.2301} 4\sqrt{x} - (x^2 + 1)\ \mathrm{d}x$ = 2.9747 ≈ 2.97
	(ii) Let $x = c$ such that $\int_{0.062997}^{c} 4\sqrt{x} - (x^2 + 1)\ \mathrm{d}x = \frac{1}{2}(2.9747)$ $\left[\frac{8}{3}x^{\frac{3}{2}} - \frac{x^3}{3} - x\right]_{0.062997}^{c} = \frac{1}{2}(2.9747)$ $\frac{8}{3}c^{\frac{3}{2}} - \frac{c^3}{3} - c = 1.4664$ $\Rightarrow c = 1.07$
	(iii) Volume generated about y-axis $= \pi\int_{1.0040}^{5.9734} (y-1) - \frac{y^4}{256}\ \mathrm{d}y$ = 20.2

Qn	Solution
5(i)	Given $y=\frac{1}{2}\ln\left(1+\tan x\right)$, $e^{2y}=1+\tan x$ Differentiate throughout w.r.t x. $e^{2y}\left(2\frac{dy}{dx}\right)=\sec^2 x$ $2e^{2y}\frac{dy}{dx}=\sec^2 x$ (shown) Differentiate throughout w.r.t x. $2e^{2y}\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(4e^{2y}\frac{dy}{dx}\right)=2\sec x\left(\sec x\tan x\right)$ $e^{2y}\frac{d^2y}{dx^2}+2e^{2y}\left(\frac{dy}{dx}\right)^2=\sec^2 x\tan x$ When $x=0$, $y=0$, $\frac{dy}{dx}=\frac{1}{2}$, $\frac{d^2y}{dx^2}=-\frac{1}{2}$ By Maclaurin's series, $f(x)=f(0)+x\,f'(0)+\frac{x^2}{2!}f''(0)+...$ $f(x)=0+x\left(\frac{1}{2}\right)+\frac{x^2}{2!}\left(-\frac{1}{2}\right)+...$ $=\frac{1}{2}x-\frac{1}{4}x^2+...$
(ii)	$\frac{x}{a+bx}=x\left(a+bx\right)^{-1}$ $=\frac{x}{a}\left(1+\frac{b}{a}x\right)^{-1}$ $=\frac{x}{a}\left(1-\frac{b}{a}x+...\right)$ $=\frac{x}{a}-\frac{b}{a^2}x^2+...$ Given $\frac{x}{a}-\frac{b}{a^2}x^2=\frac{1}{2}x-\frac{1}{4}x^2$, Comparing coefficient of x, $\frac{1}{a}=\frac{1}{2}\Rightarrow a=2$ Comparing coefficient of x^2, $\frac{b}{a^2}=\frac{1}{4}\Rightarrow b=\frac{a^2}{4}=1$ $\therefore \underline{\underline{a=2,b=1}}$

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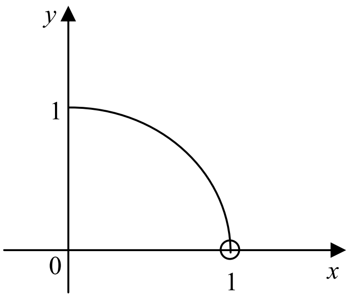
6	(i) 
	(ii) $x=\cos\left(\mathrm{e}^t\right)$, $y=\sin\left(\mathrm{e}^t\right)$, where $t\le\ln\frac{\pi}{2}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=-\mathrm{e}^t\sin\left(\mathrm{e}^t\right)$ $\quad\frac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{e}^t\cos\left(\mathrm{e}^t\right)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\cot\left(\mathrm{e}^t\right)$ Gradient of normal $=\tan\left(\mathrm{e}^t\right)$ Equation of normal: $y-\sin\left(\mathrm{e}^t\right)=\tan\left(\mathrm{e}^t\right)\left[x-\cos\left(\mathrm{e}^t\right)\right]$ $y-\sin\left(\mathrm{e}^t\right)=\tan\left(\mathrm{e}^t\right)x-\sin\left(\mathrm{e}^t\right)$ $\therefore y=\tan\left(\mathrm{e}^t\right)x$ Since y-intercept is 0, the normal passes through the origin.
	(iii) Given equation of normal is $y=x$, $\tan\left(\mathrm{e}^t\right)=1$ $\mathrm{e}^t=\frac{\pi}{4}$ $t=\ln\frac{\pi}{4}$ $x=\cos\left(\mathrm{e}^{\ln\frac{\pi}{4}}\right)=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$ $y=\sin\left(\mathrm{e}^{\ln\frac{\pi}{4}}\right)=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-1$ Equation of tangent: $y-\frac{\sqrt{2}}{2}=-\left[x-\frac{\sqrt{2}}{2}\right]$ $y=-x+\sqrt{2}$

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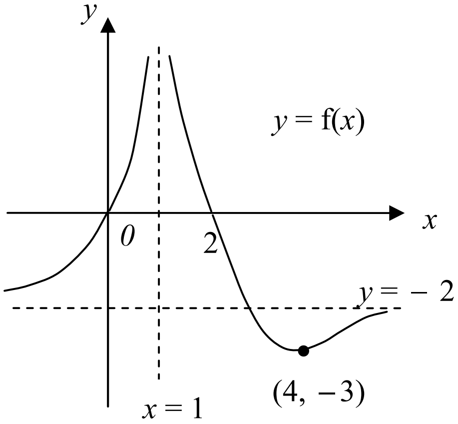
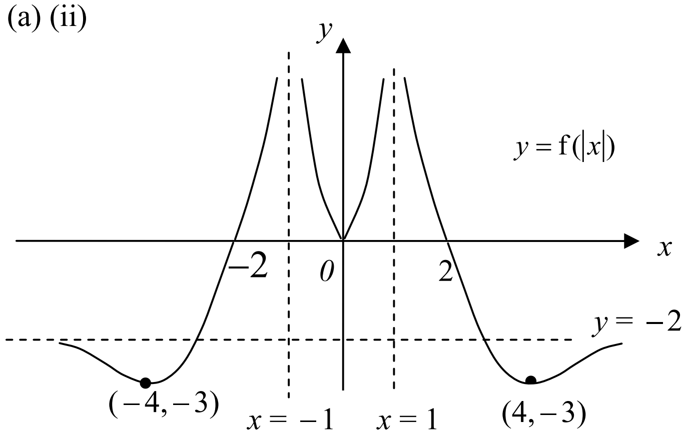
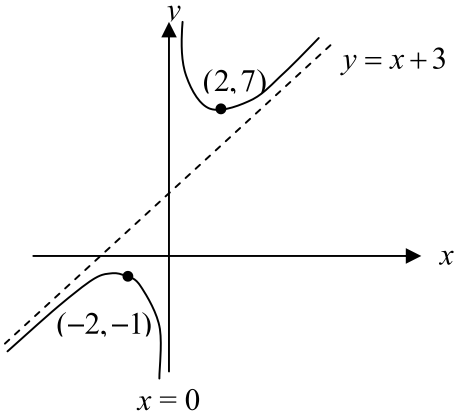
7(a)	Let a cm be the height of the shortest doll. Since the heights of the dolls are in A.P., sum of all their heights $=\frac{7}{2}\left(a+4a\right)=70$ Therefore, $a=\frac{20}{5}=4\text{ cm}$ Height of the tallest doll $=4+(7-1)d=4(4)$ $\therefore d=\frac{12}{6}=2\text{ cm}$
7(b)	Let T_1 be the time interval between 1st and 2nd bounces, T_2 be the time interval between 2nd and 3rd bounces, and so on … Hence T_1, T_2, T_3, …, T_n is a G.P. where $T_1=4,\quad r=0.9$. Given $T_k<0.4$ $\Rightarrow 4\left(0.9\right)^{k-1}<0.4$ $\Rightarrow \left(k-1\right)\ln\left(0.9\right)<\ln\left(0.1\right)$ $\Rightarrow k>22.854$ $\therefore k=23$ Total time from 1st to kth bounce $=S_{22}=\frac{4\left[1-\left(0.9\right)^{22}\right]}{1-0.9}$ ≈ 36.061 $=36$ (nearest sec.)

Qn	Solution
8	(a)(i) 
	(a) (ii) 
(b)	(i) $y=\dfrac{ax^2+3x+b}{x}=ax+3+\dfrac{b}{x}$ $\Rightarrow$ Asymptotes are $y=x+3$ and $x=0$ Given that $y=x+3$ is an oblique asymptote, $a=1$ (ii) When $y=0$, $ax^2+3x+b=0$ C has no x-intercept $\Rightarrow$ Discriminant <0 $\Rightarrow\ 9-4(1)(b)<0$ $\Rightarrow b>\dfrac{9}{4}$ (Shown)

(iii) $y = \dfrac{x^2+3x+4}{x} = x+3+\dfrac{4}{x}$

$\Rightarrow$ Asymptotes are $y = x+3$ and $x = 0$

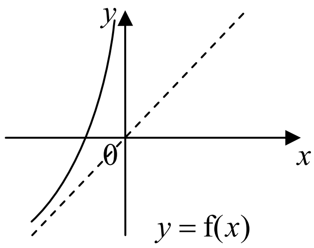


$$\frac{ax^2+3x+b}{x(kx+3)} = 1$$

With $b = 4$ and $a = 1$, $\dfrac{x^2+3x+4}{x} = kx+3$

From the graph, to have two real roots, $k > 1$.

Qn	Solution
9	(i)



Since any horizontal line $y = k$, $k \in \mathbb{R}$ will cut the graph of f exactly once, hence f is one-one. Thus, f^{-1} exists.

(ii) Let $y = x - \dfrac{1}{x}$

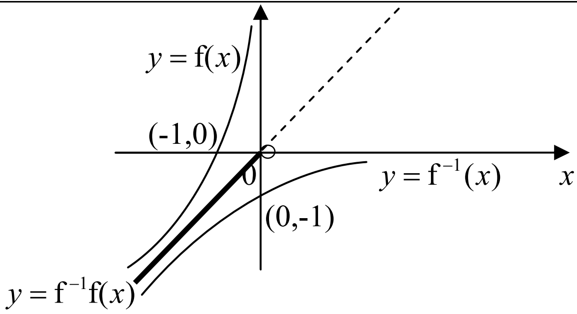
$\therefore x = \dfrac{y}{2} \pm \dfrac{1}{2}\sqrt{y^2+4}$

But $x < 0$, $\therefore x = \dfrac{y}{2} - \dfrac{1}{2}\sqrt{y^2+4}$

Hence, $f^{-1} : x \to \dfrac{x}{2} - \dfrac{1}{2}\sqrt{x^2+4}$, $x \in (-\infty, \infty)$

(iii)

y　　　　　$y = x$



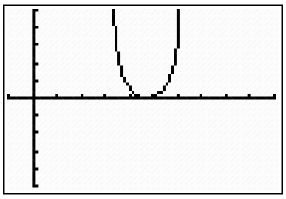
(iv) D_f : $(-\infty,0)$ and R_g : $[-1 , 1]$.

Since $R_g \not\subseteq D_f$, therefore fg does not exist.

(v) $fh(x) = f(\sin x)$

$$= \sin x - \frac{1}{\sin x}$$

Hence, fh : $x \rightarrow \sin x - \dfrac{1}{\sin x}, \ \pi < x < 2\pi$



Range of fh = $[0,\infty)$

Qn	Solution
10	**(a)(i)** $\overrightarrow{AB}\cdot\overrightarrow{OP}=(\mathbf{b}-\mathbf{a})\bullet\mathbf{p}$ $=\mathbf{b}\bullet\mathbf{p}-\mathbf{a}\bullet\mathbf{p}$ $=\mathbf{a}\bullet\mathbf{p}-\mathbf{a}\bullet\mathbf{p}$ (since $\mathbf{b}\bullet\mathbf{p}=\mathbf{a}\bullet\mathbf{p}$) $=0$ Hence, *AB* is perpendicular to *OP*. **OR** $\mathbf{b}\bullet\mathbf{p}=\mathbf{a}\bullet\mathbf{p}$ $\mathbf{b}\bullet\mathbf{p}-\mathbf{a}\bullet\mathbf{p}=0$ $(\mathbf{b}-\mathbf{a})\bullet\mathbf{p}=0$ $\overrightarrow{AB}\cdot\overrightarrow{OP}=0$ Hence, *AB* is perpendicular to *OP*.
	(ii) Since $\lvert\mathbf{a}\rvert=\lvert\mathbf{b}\rvert$, then *P* must be the midpoint of *AB*. Using ratio theorem, $\overrightarrow{OP}=\frac{1}{2}(\mathbf{a}+\mathbf{b})$ Thus, $\overrightarrow{OR}=2\overrightarrow{OP}$ $=2\left(\frac{1}{2}(\mathbf{a}+\mathbf{b})\right)$ $=\mathbf{a}+\mathbf{b}$
	(iii) $\lvert\mathbf{a}\times\mathbf{b}\rvert$ represents the area of rhombus *OARB* or *OBRA*.
	(b) Equation of l_1 is $\underset{\sim}{r}=\begin{pmatrix}10\\8\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\14\\a-3\end{pmatrix}$, $\lambda\in\mathbb{R}$ Given that l_1 and l_2 are perpendicular, $\begin{pmatrix}1\\14\\a-3\end{pmatrix}\bullet\begin{pmatrix}2\\2\\-5\end{pmatrix}=0\Rightarrow 2+28-5(a-3)=0$ $\therefore a=9$ $l_1:\ \underset{\sim}{r}=\begin{pmatrix}10\\8\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\14\\6\end{pmatrix}$ Given that l_1 and l_2 intersect at a point, $\begin{pmatrix}10+\lambda\\8+14\lambda\\3+6\lambda\end{pmatrix}=\begin{pmatrix}b+2\mu\\-10+2\mu\\7-5\mu\end{pmatrix}$ $\lambda-2\mu-b=-10$ $14\lambda-2\mu=-18$ $6\lambda+5\mu=4$ Using GC, $b=5$

Qn	Solution
11(a)	(i) $\left(z-re^{i\theta}\right)\left(z-re^{-i\theta}\right)=z^2-r\left(e^{-i\theta}+e^{i\theta}\right)z+r^2e^{i\theta}e^{-i\theta}$ $=z^2-r\left[\cos\theta-i\sin\theta+\cos\theta+i\sin\theta\right]z+r^2$ $=z^2-\left(2r\cos\theta\right)z+r^2$ (Shown)
	(ii) $z^4=-81$ $z^4=81e^{i\pi}$ $z^4=81e^{i(\pi+2k\pi)}$ $z=3\,e^{i\left(\frac{\pi+2k\pi}{4}\right)},\quad k=-2,-1,0,1$ $z=3e^{-i\left(\frac{3\pi}{4}\right)},3\,e^{-i\left(\frac{\pi}{4}\right)},3\,e^{i\left(\frac{\pi}{4}\right)},3\,e^{i\left(\frac{3\pi}{4}\right)}$
	(iii) z^4+81 $=\left(z-3e^{i\left(\frac{3\pi}{4}\right)}\right)\left(z-3e^{-i\left(\frac{3\pi}{4}\right)}\right)\left(z-3e^{i\left(\frac{\pi}{4}\right)}\right)\left(z-3e^{-i\left(\frac{\pi}{4}\right)}\right)$ $=\left[z^2-\left(6\cos\frac{3\pi}{4}\right)z+9\right]\left[z^2-\left(6\cos\frac{\pi}{4}\right)z+9\right]$ $=\left[z^2+3\sqrt{2}z+9\right]\left[z^2-3\sqrt{2}z+9\right]$

(b)	$\left\lvert \dfrac{w^*}{(1-\mathrm{i})^2} \right\rvert = \dfrac{\lvert w \rvert}{\lvert 1-\mathrm{i} \rvert^2} = \dfrac{4}{2} = 2$ $\arg\left(\dfrac{w^*}{(1-\mathrm{i})^2}\right) = \arg(w^*) - 2\arg(1-\mathrm{i})$ $= \dfrac{\pi}{6} - 2\left(-\dfrac{\pi}{4}\right) = \dfrac{2\pi}{3}$
	$p = 2\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$ $p^n = 2^n \mathrm{e}^{\frac{2n\pi \mathrm{i}}{3}}$ p^n is real $\Rightarrow$ $\mathrm{Im}(p^n) = 0$ $\Rightarrow 2^n \sin\left(\dfrac{2n\pi}{3}\right) = 0$ $\Rightarrow \sin\left(\dfrac{2n\pi}{3}\right) = 0$ $\Rightarrow \dfrac{2n\pi}{3} = k\pi$ where $k \in \mathbb{Z}$ $\Rightarrow \dfrac{2}{3}n\pi = \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ 5\pi,\ 6\pi,\ \cdots$ $n = \dfrac{3}{2},\ 3,\ \dfrac{9}{2},\ 6,\ \dfrac{15}{2},\ 9,\ \cdots$ Since $n \in \mathbb{Z}^+$, $n = 3,\ 6,\ 9,\ \cdots$