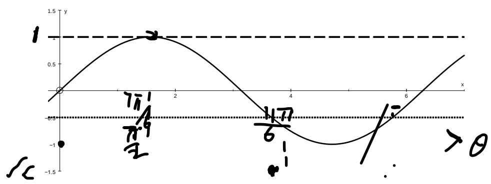
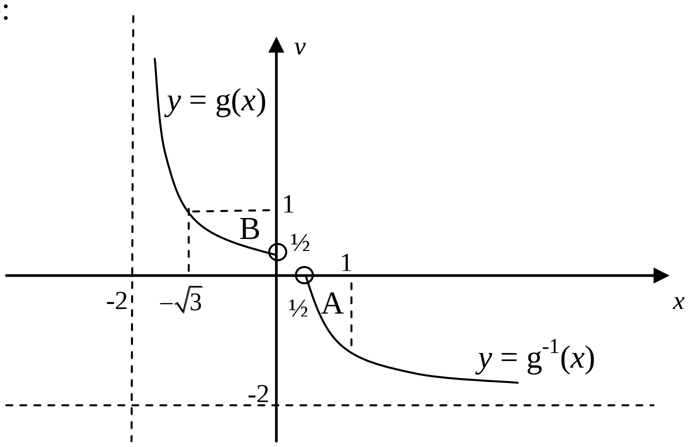
AJC Preliminary Examination 2012
H2 Mathematics Paper 1 (9740/01) Solution

Qn	Soln
1	$S_n = \frac{100}{2}\left[2a+99d\right]=10000 \quad \Rightarrow 2a+99d=200$ a, $a+d$, $a+4d$ are consecutive terms in GP: $\frac{a+d}{a}=\frac{a+4d}{a+d}$ $\Rightarrow (a+d)^2 = a(a+4d)$ $\Rightarrow d^2 = 2ad \quad \Rightarrow \quad d = 2a \quad$ since $d \neq 0$. Sub $d = 2a$ into $2a + 99d = 200$, get $d = 2$ and $a = 1$.
2	$Ax^2+By^2+Cy=8 \qquad (2,1) \Rightarrow 4A + B + C = 8$ -----(1) Diff (*) wrt x : $\quad 2Ax+2By\frac{dy}{dx}+C\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{-2Ax}{2By+c}$ Tangent at (2,1) // y-axis : 2B + C = 0 ------(2) Diff again wrt x : $2A+2B\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]+C\frac{d^2y}{dx^2}=0$ When y = 0 , $\frac{dy}{dx}=\sqrt{\frac{3}{2}}$ and $\frac{d^2y}{dx^2}=\frac{9}{4} \Rightarrow 2A+2B\left(\frac{3}{2}\right)+C\left(\frac{9}{4}\right)=0$ (3) Solve the 3 eqns : get A = 3, B = 4 and C = -8
3	$\Rightarrow \frac{6x-4+(2x-1)(x-3)}{x-3} \le 0 \quad \Rightarrow \quad \frac{2x^2-x-1}{x-3}\le 0 \Rightarrow \frac{(2x+1)(x-1)}{x-3}\le 0$ $\Rightarrow (x-3)(2x+1)(x-1)\le 0$ $\Rightarrow x \le -\frac{1}{2}$ or $1 \le x < 3$  $\frac{6-4\operatorname{cosec}\theta}{1-3\operatorname{cosec}\theta}\le 1-2\sin\theta \Rightarrow \frac{6\sin\theta-4}{\sin\theta-3}\le 1-2\sin\theta$ Replace x by $\sin\theta$, $\quad \sin\theta \le -\frac{1}{2}$, $\quad 1\le \sin\theta < 3$ $\Rightarrow \frac{7\pi}{6}\le\theta\le\frac{11\pi}{6}$ or $\theta=\frac{\pi}{2}$
4 (i)	Reversing the transformations: a. Stretch parallel to y-axis by factor ½ gives $y=\frac{1}{2\sqrt{4-x^2}}$ b. Translate 1 unit to the right gives $y=\frac{1}{2\sqrt{4-(x-1)^2}}$ c. Reflection in y-axis gives $y=\frac{1}{2\sqrt{4-(-x-1)^2}}=\frac{1}{2\sqrt{4-(x+1)^2}}$ = f(x)
4 (ii)	The graphs of y = g(x) and $y = g^{-1}(x)$: 

4 (iii)

area of the region bounded by $y = g^{-1}(x)$, the x-axis and the line $x = 1$ = region A = region B

= Rectangle - $\int_{-\sqrt{3}}^{0} y \, dx$

$$= (1)\left(\sqrt{3}\right) - \int_{-\sqrt{3}}^{0} \frac{1}{\sqrt{4-x^2}} \, dx = \sqrt{3} - \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-\sqrt{3}}^{0} = \sqrt{3} - \left[0 - \left(-\frac{\pi}{3}\right)\right] = \sqrt{3} - \frac{\pi}{3}.$$

5

$$S_n = \frac{2}{1\times2\times3} + \frac{2}{2\times3\times4} + \frac{2}{3\times4\times5} + \dots + \frac{2}{n(n+1)(n+2)}$$

$$S_1 = \frac{1}{3} = \frac{1}{2} - \frac{1}{2\times3}, S_2 = \frac{5}{12} = \frac{1}{2} - \frac{1}{3\times4}, S_3 = \frac{9}{20} = \frac{1}{2} - \frac{1}{4\times5}.$$

(ii) $S_n = \frac{1}{2} - \frac{1}{(n+1)(n+2)}$ by observation.

(iii) Let P_n be the statement "$S_n = \frac{1}{2} - \frac{1}{(n+1)(n+2)}$" for $n \in Z^+$

P_1 is true from (i)

Assume that P_k is true for some $k \in Z^+$ ie. $S_k = \frac{1}{2} - \frac{1}{(k+1)(k+2)}$

We need to show that P_{k+1} is true, ie to prove that $S_{k+1} = \frac{1}{2} - \frac{1}{(k+2)(k+3)}$

LHS = S_{k+1} = S_k + (k+1)th term

$$= \frac{1}{2} - \frac{1}{(k+1)(k+2)} + \frac{2}{(k+1)(k+2)(k+3)}$$

$$= \frac{1}{2} - \frac{k+3-2}{(k+1)(k+2)(k+3)}$$

$$= \frac{1}{2} - \frac{1}{(k+2)(k+3)} = \text{RHS}$$

Therefore P_{k+1} is true.

Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, $\therefore$ by MI, P_n is true for $n \in Z^+$

6

(i) By pythagoras' theorem: $l = \sqrt{4+r^2}$ and $R^2 = r^2 + (2-R)^2 \Rightarrow r^2 = 4R-4$

A = $\pi r l$ $\Rightarrow$ A = $\pi\sqrt{4R-4}\sqrt{4R}$

$$\therefore A = 4\pi\sqrt{R^2-R}$$

(ii) $\frac{dA}{dt} = \frac{dA}{dR} \times \frac{dR}{dV} \times \frac{dV}{dt}$

$$\boxed{V = \frac{4}{3}\pi R^3 \Rightarrow \frac{dV}{dR} = 4\pi R^2}$$

$$\frac{dA}{dt} = \frac{2\pi(2R-1)}{\sqrt{R^2-R}} \times \frac{1}{4\pi R^2} \times 8$$

$$\frac{dA}{dt} = \frac{2\pi(4-1)}{\sqrt{4-2}} \times \frac{1}{4\pi(4)} \times 8 = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

7

(i) Since the sequence converges to L,

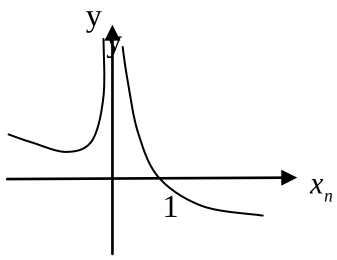
ie as $n \to \infty, x_n \to L$ and $x_{n+1} \to L$ $\quad L = \frac{1}{3}\left(2L + \frac{1}{L^2}\right) \Rightarrow 3L = 2L + \frac{1}{L^2} \Rightarrow L^3 = 1 \Rightarrow L = 1$

(ii) Consider $x_{n+1} - x_n = \frac{1}{3}\left(2x_n + \frac{1}{x_n^2}\right) - x_n$

Method 1: $x_{n+1} - x_n = \frac{1}{3}\left(2x_n + \frac{1}{x_n^2}\right) - x_n = \frac{1}{3x_n^2}\left(2x_n^3 + 1 - 3x_n^3\right) = \frac{1}{3x_n^2}\left(1 - x_n^3\right)$

Since $x_n > L = 1,\ \ 1 - x_n{}^3 < 0 \ \ \Rightarrow x_{n+1} - x_n < 0 \Rightarrow x_{n+1} < x_n$.

Method 2: Use GC, sketch $y = \frac{1}{3}\left(2x + \frac{1}{x^2}\right) - x$



From the graph, for $x_n > L = 1$,

$y < 0 \Rightarrow x_{n+1} - x_n < 0 \Rightarrow x_{n+1} < x_n$.

(iii) The sequence is such that $0 < x_0 < 1$, and from (i) $n \to \infty, x_n \to 1$.

From (ii) , $x_1 > 1, x_2 > 1, x_3 > 1, ...$ and $1 < x_n .... < x_4 < x_3 < x_2 < x_1$

the sequence will decrease and converge to the limit 1 from the right for $n \ge 1$.

Since L =1, $d_{n+1} = x_{n+1} - L = x_{n+1} - 1$

$$d_{n+1} = x_{n+1} - 1 = \frac{1}{3}\left(2x_n + \frac{1}{x_n^2}\right) - 1 = \frac{1}{3}\left(2\left(1 + d_n\right) + \frac{1}{\left(1 + d_n\right)^2}\right) - 1$$

$$= \frac{1}{3}\left(2 + 2d_n + \left(1 + d_n\right)^{-2} - 3\right) = \frac{1}{3}\left(-1 + 2d_n + 1 + (-2)d_n + \frac{(-2)(-3)}{2!}d_n^2 + ...\right) \approx d_n^2$$

Range of validity is $|d_n| < 1 \Rightarrow -1 < d_n < 1$.

8a

$(y+5)^2 = x - 3$ -----(1) $\qquad (y+5)^2 = x - 3 \Rightarrow y = -5 \pm \sqrt{x-3}$

$y = x - 10$ -----(2)

Points of intersections are 4, -6) and (7, -3)

Volume generated

$= \pi\int_3^4 \left(-5 - \sqrt{x-3}\right)^2 \text{ dx } + \pi\int_4^7 (x-10)^2 \text{ dx } - \pi\int_3^7 \left(-5 + \sqrt{x-3}\right)^2 \text{ dx } = 127.2345 \approx 127$ (3 s.f.)

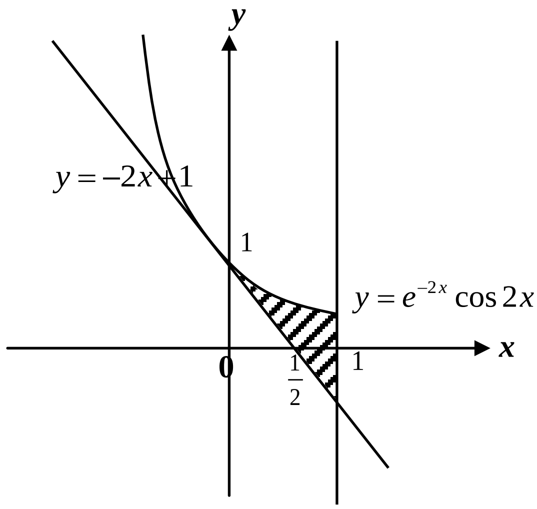
8b

$$\int e^{-2x} \cos x \text{ dx} = -\frac{1}{2} e^{-2x} \cos x - \frac{1}{2}\int e^{-2x} \sin x \text{ dx}$$

$$\int e^{-2x} \cos x \text{ dx} = -\frac{1}{2} e^{-2x} \cos x + \frac{1}{4} e^{-2x} \sin x - \frac{1}{4}\int e^{-2x} \cos x \text{ dx}$$

$$\frac{5}{4}\int e^{-2x} \cos x \text{ dx} = -\frac{1}{2} e^{-2x} \cos x + \frac{1}{4} e^{-2x} \sin x + C'$$

$$\int e^{-2x} \cos x \text{ dx} = -\frac{2}{5} e^{-2x} \cos x + \frac{1}{5} e^{-2x} \sin x + C$$



At $x = 0,\ t = 0$. At $x = 1,\ t = \frac{\pi}{2}$

$$\frac{\text{dy}}{\text{dx}} = \frac{\text{dy}}{\text{dt}}\left(\frac{\text{dt}}{\text{dx}}\right) = \frac{-2e^{-2t}}{\cos t} = -2$$

Equation of tangent at x = 0: $y = -2x + 1$

Exact area bounded

$= \int_0^1 y \text{ dx}$ *(Since the area of both triangles are the same)

$$= \int_0^{\frac{\pi}{2}} e^{-2t} \cos t \text{ dt} = \left[-\frac{2}{5} e^{-2t} \cos t + \frac{1}{5} e^{-2t} \sin t\right]_0^{\frac{\pi}{2}} = \frac{1}{5}\left(e^{-\pi} + 2\right)$$

9

(i) $y=\dfrac{2x^2-a}{x+k} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(x+k)(4x)-(2x^2-a)}{(x+k)^2}=\dfrac{2x^2+4kx+a}{(x+k)^2}$

For the curve to have at least 1 tangent parallel to the x-axis, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ must have real roots,

i.e. $2x^2+4kx+a=0$ has real roots

$(4k)^2-4(2)(a)\ge 0 \Rightarrow 16k^2-8a\ge 0 \Rightarrow 2k^2\ge a$

Since $2k^2\ne a$, $\therefore k^2>\dfrac{a}{2} \Rightarrow k>\sqrt{\dfrac{a}{2}}$ or $k<-\sqrt{\dfrac{a}{2}}$ (rejected $\because k>0$)

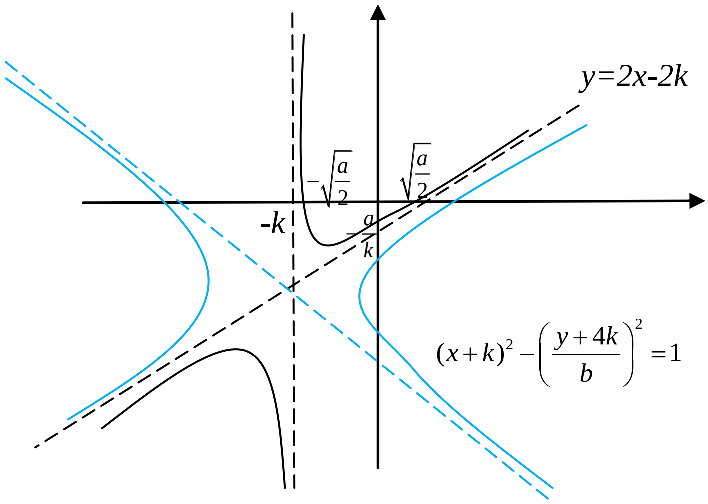
(ii) $y=\dfrac{2x^2-a}{x+k}=2x-2k+\dfrac{2k^2-a}{x+k}$

When $2k^2=a$, $y=2x-2k$

Thus, the graph is a straight line.

(iii)

From diagram, $0<b\le 2$



10 (i)

(i) A(0,1,0) lies on p_2: 8(0)+a(1) + (0) = 4 hence a = 4.

Director vector of L : $\begin{pmatrix}2\\1\\0\end{pmatrix}\times\begin{pmatrix}8\\4\\1\end{pmatrix}=\begin{pmatrix}1\\-2\\0\end{pmatrix}$ $\quad\therefore\ L: \underset{\sim}{r}=\begin{pmatrix}0\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\0\end{pmatrix},\ \lambda\in\mathbb{R}$

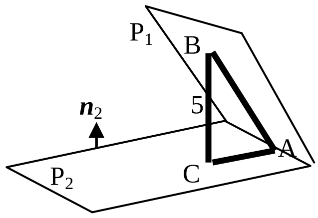
10 (ii)

(ii) $\overrightarrow{AB}\perp L$ and $\overrightarrow{AB}\perp \underset{\sim}{n}_1$

$\Rightarrow \overrightarrow{AB}\parallel\begin{pmatrix}1\\-2\\0\end{pmatrix}\times\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\5\end{pmatrix}=5\begin{pmatrix}0\\0\\1\end{pmatrix}$. Hence $\overrightarrow{AB}\parallel\begin{pmatrix}0\\0\\1\end{pmatrix}$

2nd part: <u>Method 1</u>

Let $\overrightarrow{AB}=k\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\k\end{pmatrix} \Rightarrow \overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\begin{pmatrix}0\\1\\0\end{pmatrix}+\begin{pmatrix}0\\0\\k\end{pmatrix}=\begin{pmatrix}0\\1\\k\end{pmatrix}$



$\dfrac{\left|\overrightarrow{OB}\cdot\begin{pmatrix}8\\4\\1\end{pmatrix}-4\right|}{\sqrt{8^2+4^2+1^2}}=5 \Rightarrow \begin{pmatrix}0\\1\\k\end{pmatrix}\cdot\begin{pmatrix}8\\4\\1\end{pmatrix}-4=\pm 45 \Rightarrow k=\pm 45.$ $\quad\overrightarrow{OB}=\begin{pmatrix}0\\1\\45\end{pmatrix}$ or $\begin{pmatrix}0\\1\\-45\end{pmatrix}$

<u>Method 2</u>

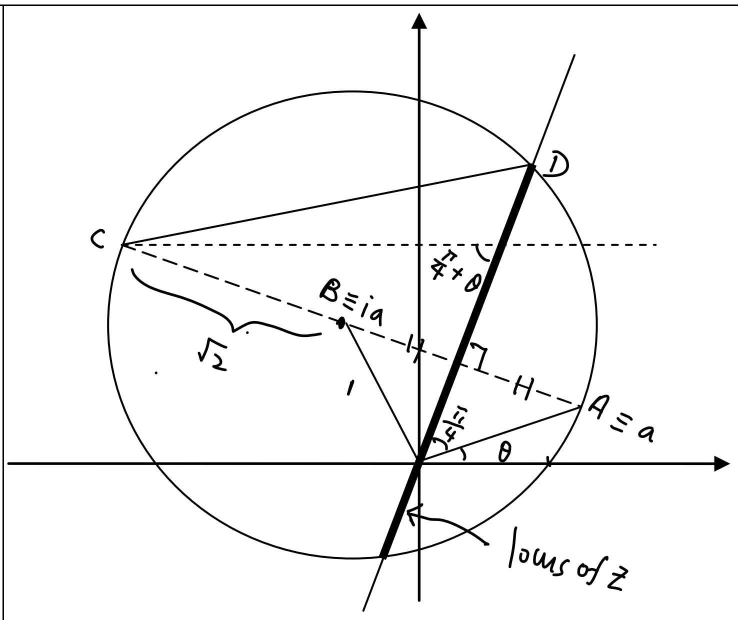
BC = 5 = length of projection of $\overrightarrow{AB}$ onto $\underset{\sim}{n}_2$

$=\left|\overrightarrow{AB}\cdot\hat{\underset{\sim}{n}}_2\right|=\dfrac{1}{\sqrt{8^2+4^2+1^2}}\left|\begin{pmatrix}0\\0\\k\end{pmatrix}\cdot\begin{pmatrix}8\\4\\1\end{pmatrix}\right|=\left|\dfrac{k}{9}\right|.$

Hence $\dfrac{k}{9}=\pm 5 \Rightarrow k=\pm 45$.

	$\overleftrightarrow{AB}=\begin{pmatrix}0\\0\\45\end{pmatrix}$ or $\overleftrightarrow{AB}=\begin{pmatrix}0\\0\\-45\end{pmatrix}$ $\overleftrightarrow{OB}=\overleftrightarrow{OA}+\overleftrightarrow{AB}=\begin{pmatrix}0\\1\\0\end{pmatrix}+\overleftrightarrow{AB}\Rightarrow .\ \overleftrightarrow{OB}=\begin{pmatrix}0\\1\\45\end{pmatrix}$ or $\begin{pmatrix}0\\1\\-45\end{pmatrix}$
10 (iii)	Method 1 : Acute angle between line AB and p_2 = acute angle between p_1 and p_2 = $\left\|\hat{n}_1 \cdot \hat{n}_2\right\|$ $= \cos^{-1}\left\|\hat{n}_1 \cdot \hat{n}_2\right\| = \cos^{-1}\left\|\frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\\0\end{pmatrix}\cdot\frac{1}{\sqrt{64+16+1}}\begin{pmatrix}8\\4\\1\end{pmatrix}\right\| = \cos^{-1}\left\|\frac{20}{9\sqrt{5}}\right\| = 6.4^o$ Method 2 : acute angle between line AB and p_2 $= \sin^{-1}\left\|\frac{\overleftrightarrow{AB}}{\left\|\overleftrightarrow{AB}\right\|}\cdot \hat{n}_2\right\| = \sin^{-1}\left\|\frac{1}{45}\begin{pmatrix}0\\0\\45\end{pmatrix}\cdot\frac{1}{\sqrt{64+16+1}}\begin{pmatrix}8\\4\\1\end{pmatrix}\right\| = \sin^{-1}\left\|\frac{45}{45\times 9}\right\| = 6.4^o$
10 (iv)	$p_3 : 2x + y + \beta z = 6$. $n_3 \cdot \begin{pmatrix}1\\-2\\0\end{pmatrix} = \begin{pmatrix}2\\1\\\beta\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\0\end{pmatrix} = 0$ for all values of β. Hence p_3 // L ----(1) $2(0) + 1+\beta(0) = 1 \neq 6 \Rightarrow$ A(0,1,0) does not lie on p_3 -----(2) Hence line L does not intersect p_3. Therefore p_1, p_2 and p_3 do not meet at a common point. When $\beta = 0$, $p_3 : 2x + y = 6$, $p_1 : 2x + y = 1$, p_2, $8x + 4y + z = 4$ Geometrically, p_1 and p_3 are parallel with p_2 intersecting both p_1 and p_3.

11



Angle that locus of Z makes with the real axis = $\frac{\pi}{4}+\theta$.

$c = 2\mathrm{i}a - a = \mathrm{i}a + (\mathrm{i}a - a) \quad \Rightarrow \overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{AB} = \overrightarrow{OB} + \overrightarrow{BC}$

Geometrical relationship: *AC* is the diameter of the circle with centre *B*.
[Or *A*, *B*, *C* are collinear; Or *B* is the midpt of *A* and *C*]

(i) $|z + a - 2\mathrm{i}a| = |z - c|$ = Distance between *Z* and *C*. Least $|z + a - 2\mathrm{i}a| = \sqrt{2} + \frac{1}{2}\left(\sqrt{2}\right) = \frac{3\sqrt{2}}{2}$

(ii) $\angle ABD = \cos^{-1}\left(\dfrac{\sqrt{2}/2}{\sqrt{2}}\right) = \frac{\pi}{3} \quad \Rightarrow \angle ACD = \frac{\pi}{6}$

Acute angle *CA* makes with the real axis = $\frac{\pi}{2} - \left(\frac{\pi}{4} + \theta\right) = \frac{\pi}{4} - \theta$

Largest $\arg(z + a - 2\mathrm{i}a)$ = $\frac{\pi}{6} - \left(\frac{\pi}{4} - \theta\right) = \theta - \frac{\pi}{12}$