Section A: Pure Mathematics (40 marks)

1 **(a)** The variables *w*, *x* and *y* are connected by the following differential equations:

$$e^{2x}\frac{dw}{dx} = w^2 \qquad \text{and} \qquad \frac{dy}{dx} = w.$$

(i) Verify that $w = \dfrac{2}{e^{-2x} + A}$ is the general solution of $e^{2x}\dfrac{dw}{dx} = w^2$, where A is an arbitrary constant. [2]

(ii) Hence find *y* in terms of *x*. [3]

(b) A tank initially contains 50 grams of salt dissolved in 100 litres of water. Brine that contains 2 grams of salt per litre of brine flows into the tank at a rate of 5 litres per minute. The solution is kept thoroughly mixed and flows out from the tank at a rate of 5 litres per minute.

Given that the amount of salt in the tank at time *t* minutes is given by S, show that

$$\frac{dS}{dt} = \frac{200 - S}{20}.$$

Hence find the time, in minutes, at which the concentration of salt in the tank reaches 1 gram per litre. [7]

2 Solve the equation $z^5 + 32 = 0$, expressing your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [2]

z_1, z_2 and z_3 are three of the roots of $z^5 + 32 = 0$ such that $0 < \arg z_1 < \arg z_2 < \arg z_3 \le \pi$.

(i) Find the smallest positive integer *n* such that $\left(\dfrac{z_1}{{z_2}^*}\right)^n$ is real and positive. [3]

(ii) The points *A* and *B* represent the roots z_1 and z_3 respectively in the Argand diagram. The line segment BA' is obtained by rotating the line segment BA through $\dfrac{\pi}{2}$ clockwise about the point *B*. Find the real part of the complex number represented by point A', giving your answer in exact trigonometric form. [4]

3 Referred to the origin O, the position vectors of points A, B and C are $6\mathbf{i}$, $4\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ and $3\mathbf{i}+2\mathbf{j}+6\mathbf{k}$ respectively. The plane p_1 is given by the equation $x-y+z=1$.

(i) Find the equation of the plane ABC in scalar product form. [3]

(ii) The perpendicular from the point $D(3, -1, 4)$ to the plane p_1 meets the plane ABC at the point S. Find the coordinates of S. [3]

(iii) Find the equation of the plane p_2 such that every point on p_2 is equidistant from points S and D. [3]

4 Betty needs to decorate a wall of length 5 metres for a party. She attaches hooks, starting from the extreme left end of the wall, and numbers each hook "1", "2", "3" and so on. The spacing between the 1st and 2nd hook is 50 cm and each subsequent spacing is 2 cm shorter than the previous spacing. This will continue till she is unable to place the next hook due to insufficient space.

(i) How many hooks can she attach in total? [4]

Betty also cuts a 6 metre roll of ribbon into pieces of varying lengths. The first piece cut off is of length 80 cm and each subsequent piece cut off is 10 % shorter than the previous piece.

(ii) If the length of the remaining roll of ribbon is less than 1 metre after n cuts, find the smallest value of n. [3]

Betty wants to hang the ribbons between every 2 hooks such that the ribbons are of decreasing lengths starting from that between the 1st and 2nd hook. She starts by using the ribbon of length 80 cm.
(Assume negligible length of ribbon is required to hang them on the hooks.)

(iii) By considering the length of the ribbon to be hung between the n^{th} and $(n+1)^{\text{th}}$ hook, write down an inequality to be satisfied by n. [1]

(iv) Hence find the number of hooks which will have ribbons hung on them. [2]

Section B: Statistics (60 marks)

5 As part of SG50 celebrations, a local grassroots organization invited the residents of the neighbourhood to the screening of a film on the 1964 racial riots.

(i) A news reporter wants to interview 15 people at the event venue to find out how the film has impacted them. Describe how a quota sample can be obtained in this context. [1]

Before the screening of the film, residents were asked to register at a counter, providing their personal particulars such as names, contact numbers and their ages. Their age profiles were as follows:

Age group	below 21 years old	21 – 35 years old	36 – 50 years old	above 50 years old
Number of residents	130	195	260	65

The organiser intends to collect feedback on the event based on the various age groups by contacting some of them through telephone after the event.

(ii) Describe in context, how a representative random sample of size 30 can be obtained by the organizer. State a disadvantage of using this method as compared to using quota sampling (besides the need for sampling frame and profile information). [3]

6 Three friends, Andy, Ben and Charlie, attend a graduation lunch with their parents. How many ways can the nine people be seated around a round table if each family must be seated together? [2]

After the lunch, the nine people stand in a single row to take a group photograph. How many ways can they be arranged if Andy and Ben are each standing between their respective parents and Charlie is standing either beside his father or his mother (or both)? [4]

7 A nursery supplies plants to Evergreen Pte. Ltd. The heights, in centimetres, of plants supplied may be taken to follow the distribution $\mathrm{N}\left(\mu, \sigma^2\right)$.

Over the years, Evergreen rejects 6% of the plants supplied as too short and 4% as too tall. It is known that Evergreen only accepts plants of heights between 25 cm and 91 cm.
Show that $\sigma = 20$ correct to the nearest integer. Hence find the median height of the plants accepted by Evergreen. [5]

Evergreen inspects the plants supplied in batches of 10. Find the probability that a total of at most 4 plants are rejected in two randomly chosen batches. [2]

Three children, Anna, Bella and Cinderella, each chooses a plant at random from the nursery. Find the probability that the average height of the three plants is at least twice the average height of the plants chosen by Anna and Bella. [3]

8 It is believed that there is a correlation between the number of fogging sessions and the number of new dengue cases. To test the effectiveness of fogging on controlling the spread of dengue, a particular neighbourhood is subjected to a number of fogging sessions (x) spread out regularly in a month and the corresponding number of new dengue cases (y) is recorded. The results are summarised in the following table:

No. of fogging sessions, x	2	4	6	8	10	12	14
No. of new dengue cases, y	20	14	9	6	7	2	1

(i) Calculate the product moment correlation coefficient between x and y, and explain whether your answer suggests that a linear model is appropriate. [2]

(ii) Draw a scatter diagram for the data. [1]

One of the data points appears to be incorrect.

(iii) Indicate this point by labelling it as ***P***, and explain why the scatter diagram for the remaining points may be consistent with a model of the form $y = A\mathrm{e}^{bx}$. [2]

(iv) Omitting ***P***, use an appropriate regression line to give the best estimate, to the nearest whole number, of the number of fogging sessions in a month when the number of new dengue cases is 7. [2]

9 Three red balls and two blue balls are placed in a bag. All the balls are indistinguishable except for their colours.

Mr Red and Mr Blue take turns to draw a ball from the bag at random with replacement. The first player to draw the ball whose colour matches his name wins the game and the game stops immediately.

If Mr Red draws first, find the probability that

(i) Mr Red wins the game at his third draw. [2]

(ii) Mr Blue wins the game at his n^{th} draw. [2]

(iii) Mr Red wins the game given that the winner wins the game at his third draw. [2]

(iv) Mr Blue wins the game. [2]

10 A farmer claims that the mean weight of the melons grown in his farm is at least 1.2 kg. A random sample of 10 melons is chosen and the weight, x kg, of each melon is recorded. The results are as follows:

1.037	0.914	1.019	1.234	1.110
1.417	1.008	0.846	1.105	1.331

Assuming the weight of a randomly chosen melon follows a normal distribution, carry out an appropriate test of the farmer's claim at the 10% significance level. [4]

A buyer concludes that the sample could not provide sufficient evidence, at α% significance level, that the mean weight differs from 1.2 kg. State the range of values of α for him to draw such a conclusion. [1]

Another random sample of 40 melons is weighed and the results are as follows:

$$\sum x = 45.738 \quad \text{and} \quad \sum x^2 = 72.576$$

(i) By combining the 2 sets of data, find unbiased estimates of the population mean and population variance of the weight of a melon grown by this farmer. [2]

(ii) The farmer decides to change the claim to "the mean weight of the melons grown in his farm is at least m kg". Based on the combined sample of size 50, find the set of values of m for which the farmer's claim is not rejected at the 5% significance level. [3]

11 (a) Good Papa Café sells only two types of muffins, chocolate and banana. The number of chocolate and the number of banana muffins sold in a randomly chosen hour have independent Poisson distributions with means 3 and λ respectively.

Given that in a particular hour, the probability of selling 2 muffins is the same as the probability of selling 4 banana muffins, find the value of λ. [3]

If the café opens for 8 hours a day, use appropriate approximations to find the probability that the daily sale of chocolate muffins exceeds that of banana muffins. [3]

11(b) Steven, who loves soccer, trains his shooting skills by trying to kick balls into the net from distances varying from 15 metres to 35 metres. The probability, p, that he kicks the ball into the net is given by $p = \frac{2}{65}(40 - d)$, where d is the distance between the ball and the goalpost in metres.

Each kick by him is assumed to be independent of previous kicks.

(i) Find the maximum value of *d* (to the nearest integer) for there to be more than 60% chance of Steven kicking the ball into the net. [2]

(ii) Steven kicks 15 balls from a fixed distance of *x* metres from the goalpost. Find, to the nearest integer, the value of *x* such that Steven has 90% chance of kicking at least two balls into the net. [4]

(iii) Steven trains for 60 consecutive days by kicking 100 balls per day at a distance of 24 metres from the goalpost. His target is to kick an average of 50 balls into the net per day. Find the probability that the average number of balls that he kicks into the net per day over this period differs from his target by less than 1. [3]

END OF PAPER