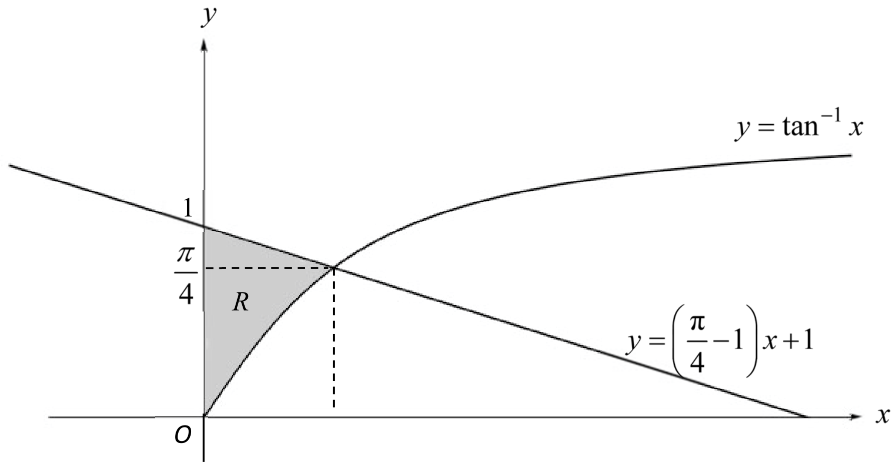
# Applications of Integration [part FM]

**1** **(a)**




**(i)** The shaded region $R$ in the diagram above is bounded by the curve with equation $y = \tan^{-1} x$, the line with equation $y = \left(\frac{\pi}{4} - 1\right)x + 1$ and the $y$-axis.

Show that the exact area of $R$ is $\frac{1}{2}\ln 2 + \frac{4-\pi}{8}$. [5]

**(ii)** Find the volume generated when $R$ is rotated completely about the $y$-axis. [3]

**(b)** A curve has parametric equations $x = t\ln t$, $y = \sin^2 t$, where $1 \le t \le \pi$. Find the numerical value of the area of the region bounded by the curve, the $x$-axis and the $y$-axis. [3]

[2014 JJC/Promo/9]

**2** **(a)** Find $\int x\mathrm{e}^{-x}\,\mathrm{d}x$. [3]

**(b)** The curve $C$ has parametric equations

$$x = \frac{(t+2)^2}{2}, \quad y = \mathrm{e}^{-t}, \quad \text{for } t \ge -2.$$

**(i)** Show that the equation of the normal to the curve at $t = 0$ is $y = 2x - 3$. [4]

**(ii)** Find the exact value of the area of the region bounded by $C$, the line $y = 2x - 3$, the $x$-axis and the $y$-axis. [4]

[2014 RVHS/Promo/11]

3 **(a)** Using an algebraic method, find the exact value of $\displaystyle\int_1^4 \frac{|x-2|}{x}\,\mathrm{d}x$. [3]

**(b)** Sketch and shade the finite region bounded by the curve $y = x^2 + 2$, the lines $y = x$ and $x = 1$, and the $y$-axis. Find the exact volume of the solid formed when the region is rotated $2\pi$ radians about the $y$-axis. [4]

[2014 TJC/Promo/6]

4 **(i)** Use the substitution $x = 2\tan\theta$ to show that

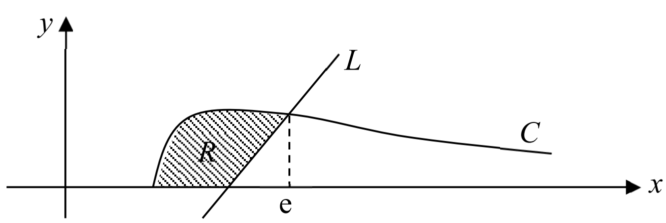
$$\int \frac{1}{(4+x^2)^2}\,\mathrm{d}x = \frac{1}{16}\left(\frac{2x}{x^2+4} + \tan^{-1}\frac{x}{2}\right) + C.$$ [5]

**(ii)** The region $R$ is bounded by the curve $y = \sqrt{\left(\frac{1}{x} - 4\right)}$, the line $y = 2$, the $x$-axis and the $y$-axis. Find the exact volume of the solid formed when $R$ is rotated $2\pi$ radians about the $y$-axis. [4]

[2014 RVHS/Promo/10]

5 **(a)** Find $\displaystyle\int \frac{6+2x}{\sqrt{(1-4x-x^2)}}\,\mathrm{d}x$ [5]

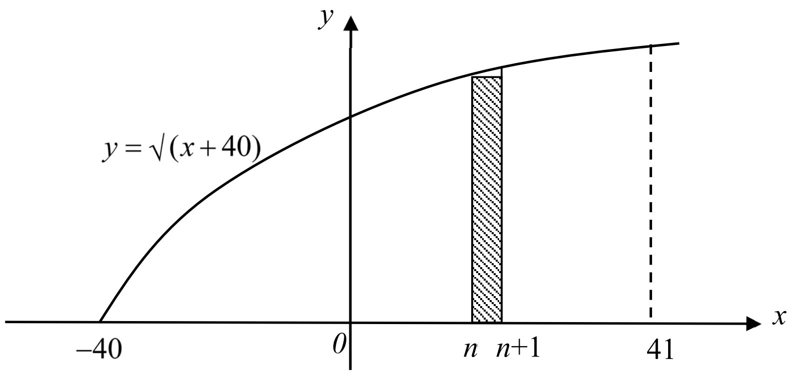
**(b)**




The diagram above shows the region $R$ bounded by the curve $C$ with equation $y = \dfrac{\sqrt{(\ln x)}}{x}$, $x \geq 1$, the $x$-axis and the line $L$ with equation $y = \dfrac{1}{\mathrm{e}(\mathrm{e}-2)}(x-2)$. Find the exact volume of the solid of revolution when $R$ is rotated completely about the $x$-axis. [5]

[2014 AJC/1/3]

**6** The graph of $y = \sqrt{(x+40)}$ is shown in the diagram below.




By considering the shaded rectangle, show that $\sqrt{n+40} < \int_{n}^{n+1} \sqrt{x+40}\ \mathrm{d}x$. [1]

Deduce that $\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{80} < \int_{-40}^{41} \sqrt{x+40}\ \mathrm{d}x$. [2]

Show also that $\sqrt{n+41} > \int_{n}^{n+1} \sqrt{x+40}\ \mathrm{d}x$. [2]

Deduce that $\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{81} > \int_{-40}^{41} \sqrt{x+40}\ \mathrm{d}x$. [1]

Hence, deduce the value $a$, where $a \in \mathbb{Z}$, that satisfies the following inequality,

$$9a < \sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{80} < 9(a+1).$$ [2]

[2014 AJC/1/9]

**7** A curve has equation given by $y = (\ln x)^2 - 1$, where $x > 0$.

**(i)** Sketch the graph, indicating the exact coordinates of the $x$-intercepts and the turning point.

[4]

The region $R$ is bounded by the curve and the $x$-axis.

**(ii)** Find the exact area of $R$. [4]

**(iii)** Find the volume of the solid generated when $R$ is rotated through $2\pi$ radians about the $y$-axis. [4]

[2014 HCI/1/11]

**8** A curve $C$ is defined by the parametric equations $x = \cos t$, $y = \sin 2t$, for $0 \le t \le 2\pi$.

**(i)** Sketch the curve, stating the coordinates of any points of intersection with the axes. [2]

**(ii)** Show that the area enclosed by the curve $C$ is $8\int_0^{\frac{\pi}{2}} \sin^2 t \cos t \, \mathrm{d}t$. Hence, find the exact area enclosed by the curve $C$. [4]

**(iii)** Find the exact value of the volume of revolution formed when the area enclosed by the curve $C$ is rotated completely about the $x$- axis. [3]

[2014 MI/I/8(modified)]

**9** The curve $C$ has equation

$$y = x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} + \lambda$$

where $\lambda > 0$ and $0 \le x \le 3$. The length of $C$ is denoted by $s$. Prove that $s = 2\sqrt{3}$. [4]

The area of the surface generated when $C$ is rotated through one revolution about the $x$-axis is denoted by $S$. Find $S$ in terms of $\lambda$. [5]

[FM/N06/I/12ORpart]

**10** Prove that $\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)\right] = -\frac{1}{x\sqrt{1+x^2}}, x > 0.$

Hence, or otherwise, find the exact length of arc of the curve whose equation is $y = -\ln x$ from the point where $x = \frac{1}{\sqrt{3}}$ to the point where $x = 1$. [SRJC2001/1/4]

**11** The parametric equations of a curve $C$ are given by

$$x = \tan t - t,\ y = \ln(\sec t) \text{ for } 0 \le t < \tfrac{1}{2}\pi.$$

**(i)** On the curve $C$, $O$ and $P$ are points corresponding to $t = 0$ and $t = p$ respectively. Prove that the arc length of $OP$ of $C$ is $\sec p - 1$. [3]

**(ii)** The arc $OP$ of length 1 unit is rotated about the $x$-axis through 4 right angles. Find the area of the curved surface so formed, leaving your answer in exact form. [5]

[TJC FM2001/1/4 (modified)]

**12** A curve is defined parametrically by $x = ae^{-t}\cos t, y = ae^{-t}\sin t$ where $t \geq 0$ and $a$ is a positive constant.

**(i)** Show that $\dfrac{dy}{dx} = \tan(t - \frac{1}{4}\pi)$. [3]

**(ii)** $A$ and $B$ are points on this curve corresponding to $t = 0$ and $t = \pi$. Find the length of the arc $AB$. [4]

**(iii)** The portion of the curve from $t = 0$ and $t = \pi$ is rotated through $2\pi$ radians about the $x$-axis. Prove that the area of the curved surface generated is $\dfrac{2\sqrt{2}}{5}\pi a^2(1 + e^{-2\pi})$ units$^2$. [5]

[FMSAJC 2001/1/5]

**13** A curve has the parametric equations $x = a(\cos t - 1), y = a(\sin t - t)$, where $0 \leq t \leq \frac{1}{2}\pi$ and $a$ is a positive constant.

The arc of the curve between $t = 0$ and $t = \frac{1}{3}\pi$ is rotated through 1 revolution about the $x$-axis. Show that the area of the surface of revolution formed is given by $A = 4\pi a^2 \int_0^{\frac{\pi}{3}} (\sin t - t)\sin\left(\dfrac{t}{2}\right) dt$.

Hence, or otherwise, show that $A = \dfrac{2}{3}\pi a^2\left(2\sqrt{3}\pi - 11\right)$. [8]

[TPJC FM 2001/1/5]

**14(a)** Let $I_n = \int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx$, $n \in \mathbb{Z}^+$.

**(i)** Show that $I_n = \dfrac{2n-1}{2n} I_{n-1}$, $n \in \mathbb{Z}^+$. [3]

**(ii)** Find $I_0$ and $I_5$ in terms of $\pi$. [3]

**(b)** By using the shell method, find the exact volume of the solid of revolution formed by rotating the region bounded by the graphs of $y = x - 1$ and $y = (x-1)^2$ about the $y$-axis. [3]

[2016 HCI/Promo/8]

**15** 3D printing is a term that is commonly used to describe any process in which a 3D object is created from a computer model. To be able to print in 3 dimensions, a computer-aided manufacturing (CAM) software is used to control the motors and the movement of a 3D printer nozzle which is the part that does the actual printing.

A certain CAM software is designed to move the nozzle along the path $C$ described by the parametric equations $x = \sin^3\theta,\ y = \cos^3\theta - 2$ for $0 \le \theta \le \frac{1}{2}\pi$.

Find $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ in terms of $\theta$. [4]

Find the exact length of the path covered by the nozzle, from $\theta = 0$ to $\theta = \frac{1}{2}\pi$. [3]

After completion of the printing for $C$, the printer continues to print on the surface generated by rotating $C$ through $2\pi$ radians about the $x$-axis. Find the exact area of the surface generated. [4]

[2016 HCI/Promo/10]

**16** A newly discovered planet revolving around a distant star is modelled to be in an elliptical trajectory described by the polar equation below, with the star positioned at the origin and $x$-axis as the major axis.

$$r = \frac{1}{1 + 0.3\cos\theta}$$

**(i)** Show that the distance covered by the planet as it moves from the point $A$ where $\theta = 0$ to the point $B$ where $\theta = \frac{1}{2}\pi$ is given by

$$D = \int_0^{\frac{\pi}{2}} \frac{1}{(1 + 0.3\cos\theta)^2}\sqrt{1.09 + 0.6\cos\theta}\ \mathrm{d}\theta\,. \qquad [3]$$

**(ii)** Using trapezium rule with 5 ordinates, approximate the integral in part **(i)** to 4 decimal places. [2]

**(iii)** Using Simpson's rule with 5 ordinates, approximate the integral in part **(i)** to 4 decimal places. [2]

**(iv)** Explain, with the aid of a diagram, whether the approximation in part **(ii)** is an overestimation or underestimation of the integral in part **(i)**. Briefly explain which method produces a better approximation to the integral in part **(i).** [3]

Suppose that the planet and star obey the Kepler's Law of Planetary Motion, which states that the area swept by the line segment joining the star and the planet is proportional to the time the planet takes to cover the distance. That is,

Area swept by line segment = $k$ × Time taken to cover the distance.

**(v)** Assuming that the planet is traversing at an average speed of 0.2 unit per earth month as it traverses through the estimated distance chosen in part **(iv)**, deduce an estimate for $k$ to 4 decimal places. [2]

[2016 SRJC/Promo/11]

| S/N | Answers |
|---|---|
| 1 | (ii) 0.899 (iii) 1.98 |
| 2 | (a) $-e^{-x}(x+1)+C$ (b)(ii) $e^2-\frac{13}{4}$ |
| 3 | (a) 1 (b) $\frac{11}{6}\pi$ |
| 4 | (ii) $\frac{\pi}{64}(2+\pi)$ |
| 5 | (a) $2\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)-2\sqrt{1-4x-x^2}+c$ (b) $\pi\left(1-\frac{7}{3e}+\frac{2}{3e^2}\right)$ |
| 6 | 53 |
| 7 | (ii) $4e^{-1}$ (iii) 12.2 |
| 8 | (ii) $\frac{8}{3}$ (iii) $\frac{16}{15}\pi$ |
| 9 | $\pi\left[2(2\lambda+1)\sqrt{3}-\frac{3}{2}\right]$ |
| 10 | $\ln(2+\sqrt{3})-\ln(1+\sqrt{2})+\sqrt{2}-\frac{2}{3}\sqrt{3}$ |
| 11 | $2\pi[2\ln 2-1]]$ |
| 12 | (ii) $\sqrt{2}a(1-e^{-\pi})$ |
| 14 | (a)(ii) $\frac{63}{512}\pi$ (b) $\frac{\pi}{2}$ |
| 15 | (i) $\frac{1}{3}\operatorname{cosec}^4\theta\sec\theta$ (ii) $\frac{3}{2}$; Area = $\frac{24\pi}{5}$ |
| 16 | (ii) 1.3566 (to 4 dp.) (iii) 1.3522 (to 4 dp.)<br>(iv) approximation using trapezium rule in part (ii) is an overestimation.<br>(v) $k\approx 0.08348$ (4 dp.) |