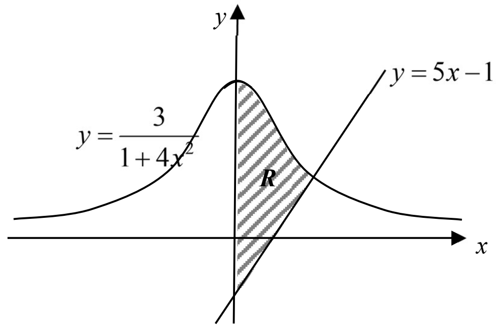
Additional Practice Questions (Chapter 6B Applications of Integration)

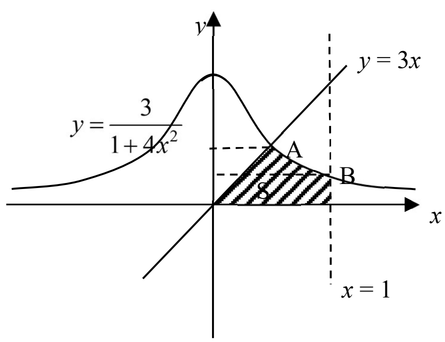
1. **[2008/Promo/AJC/Q7]**

a) The diagram below shows (not to scale) the region R which is bounded by the curve $y=\dfrac{3}{1+4x^2}$, the line $y=5x-1$ and the y-axis. Find the exact area of R. [4]



b) Another region S is bounded by the curve $y=\dfrac{3}{1+4x^2}$, the lines $y=3x$, $x = 1$ and the x-axis. Find the volume generated when S is rotated through 2π about the y-axis, giving your answer to 3 significant figures. [5]

Solution:

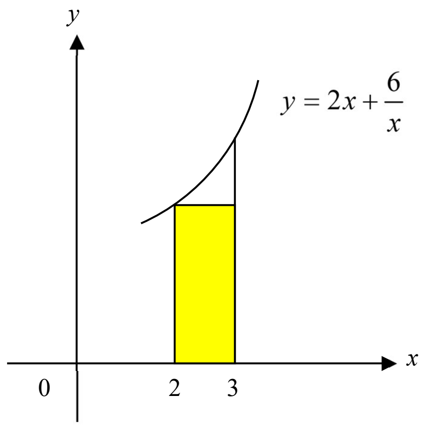
a.	$\text{Area}=\int_0^{0.5}\left(\dfrac{3}{1+4x^2}-(5x-1)\right)dx$ $=\left[\dfrac{3}{2}\tan^{-1}(2x)-\dfrac{5x^2}{2}+x\right]_0^{\frac{1}{2}}$ $=\dfrac{3}{2}\tan^{-1}(1)-\dfrac{5}{2}\left(\dfrac{1}{4}\right)+\dfrac{1}{2}=\dfrac{3}{2}\left(\dfrac{\pi}{4}\right)-\dfrac{1}{8}=\dfrac{1}{8}(3\pi-1)$
b.	 Point A = (0.5, 1.5) from GC Point B, y = 3/5 V_1 = volume of cylinder = $\pi(1)^2\left(\dfrac{3}{5}\right)=\dfrac{3\pi}{5}$ $V_2=\int_{\frac{3}{5}}^{\frac{3}{2}}\pi x^2\,dy=\pi\int_{\frac{3}{5}}^{\frac{3}{2}}\dfrac{1}{4}\left(\dfrac{3}{y}-1\right)dy=1.45210$ V_3 = volume of cone = $\dfrac{1}{3}\pi\left(\dfrac{1}{2}\right)^2\left(\dfrac{3}{2}\right)=\dfrac{\pi}{8}$ Required volume = $V_1+V_2-V_3=2.94$

Concept/Point of View

For finding volume generated for part (b), it would be very tedious to use $\int_p^q \mathrm{f}(y)-\mathrm{g}(y)\,\mathrm{d}y$ approach. Instead, we can use consider "big overall volume minus unwanted volume" approach.

We first determine V_1 (vol of cylinder) and V_2 and then subtract V_3 (vol of cone) from the sum of V_1 and V_2!

2. **[2008/Promo/ACJC/Q11]**



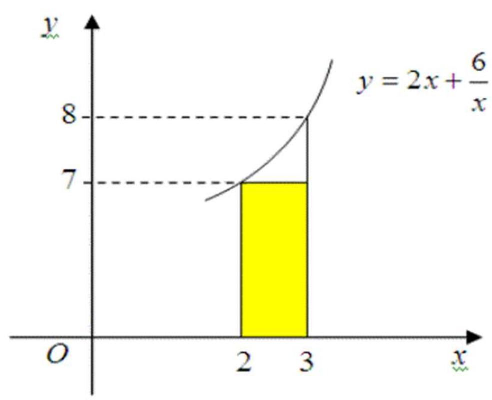
The diagram shows a sketch of part of the graph of $y = 2x + \frac{6}{x}$. By considering the shaded rectangle and the area of the region between the graph and the x-axis for $2 \le x \le 3$, show that

$$\int_2^3 \left(2x + \frac{6}{x}\right) dx > 7. \qquad [1]$$

Show also that $\int_2^3 \left(2x + \frac{6}{x}\right) dx < 8$. [1]

Hence deduce that $\frac{1}{p} < \ln 1.5 < \frac{1}{q}$, where p and q are positive integers to be determined. [4]

Solution:



$$\int_2^3 \left(2x + \frac{6}{x}\right) dx > \text{area of shaded rectangle} = (7)(1) = 7 \text{ (shown)}$$

$$\int_2^3 \left(2x + \frac{6}{x}\right) dx < (8)(1) = 8 \text{ (shown)}$$

$$\int_2^3 \left(2x + \frac{6}{x}\right) dx = \left[x^2 + 6\ln x\right]_2^3 = (9 + 6\ln 3) - (4 + 6\ln 2) = 5 + 6\ln\frac{3}{2}$$

$$\therefore\ 7 < 5 + 6\ln\frac{3}{2} < 8 \Rightarrow \frac{1}{3} < \ln(1.5) < \frac{1}{2} \qquad \therefore\ p = 3,\ q = 2$$

3. **[2008/Promo/HCI/Q13]**

i) Use the substitution $u = 2x+1$ to find $\int x\sqrt{2x+1}\ \mathrm{d}x$. [3]

The region R is bounded by the curve $y = x\sqrt{2x+1}$, the y-axis and line $y = \frac{1}{\sqrt{2}}$.

ii) Find the exact area of region R using your result in part (i). [3]

iii) Find the volume of the solid generated when R is rotated through four right angles about the x–axis. [3]

Solution:

i

$$u = 2x+1 \Rightarrow \frac{du}{dx} = 2$$

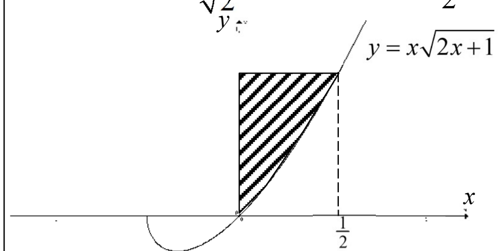
$$\int x\sqrt{2x+1}\ \mathrm{d}x = \int \frac{u-1}{2}\sqrt{u}\ \frac{1}{2}\mathrm{d}u = \frac{1}{4}\int u^{3/2} - u^{1/2}\ \mathrm{d}u$$

$$= \frac{1}{4}\frac{u^{5/2}}{5/2} - \frac{1}{4}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C$$

ii

Let $x\sqrt{2x+1} = \frac{1}{\sqrt{2}}$. From GC, $x = \frac{1}{2}$



Area of R

$$= \frac{1}{2}\frac{1}{\sqrt{2}} - \int_0^{1/2} x\sqrt{2x+1}\ \mathrm{d}x$$

$$= \frac{\sqrt{2}}{4} - \left[\frac{(2x+1)^{5/2}}{10} - \frac{(2x+1)^{3/2}}{6}\right]_0^{1/2}$$

$$= \frac{\sqrt{2}}{4} - \left[\frac{2\sqrt{2}}{5} - \frac{\sqrt{2}}{3} - \left(\frac{1}{10} - \frac{1}{6}\right)\right]$$

$$= \frac{11\sqrt{2}}{60} - \frac{1}{15} \qquad = \frac{1}{60}(11\sqrt{2} - 4) \text{ units}^2$$

iii

$$\text{Volume of solid } = \pi\left(\frac{1}{\sqrt{2}}\right)^2\left(\frac{1}{2}\right) - \pi\int_0^{1/2} x^2(2x+1)\ \mathrm{d}x$$

$$= \frac{\pi}{4} - \pi\left[\frac{x^4}{2} + \frac{x^3}{3}\right]_0^{1/2}$$

$$= \frac{17\pi}{96} = 0.556 \text{ units}^3$$

4. **[NJC/2009Promo/Q11]**

A curve C is defined by the parametric equations

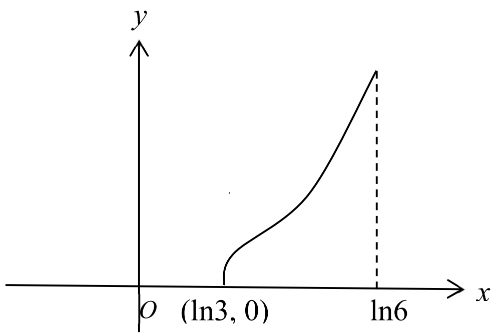
, where

(i) Sketch C for showing the exact coordinates of the point of intersection of C with the x-axis. [2]

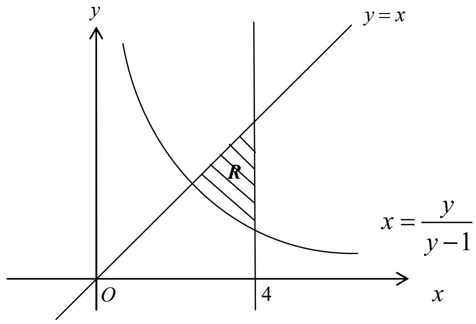
(ii) The finite region S is bounded by C, the line $x = \ln 6$ and the x-axis.

Show that the area of S can be expressed as $\int_p^q \frac{\sqrt{t^2-9}}{t}\,\mathrm{d}t$, where p and q are constants to be determined. By using the substitution $t = 3\sec\theta$, find the exact value of this area. [7]

Solution:

4(i)		
(ii)	$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{t}$ Area of S $= \int_{\ln 3}^{\ln 6} y\,\mathrm{d}x$ $= \int_3^6 \sqrt{t^2-9}\left(\frac{1}{t}\right)\mathrm{d}t$ $= \int_3^6 \frac{\sqrt{t^2-9}}{t}\,\mathrm{d}t$ (shown) Given $t = 3\sec\theta \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}\theta} = 3\sec\theta\tan\theta$ Area of S $= \int_0^{\frac{\pi}{3}} \sqrt{9\sec^2\theta - 9}\left(\frac{1}{3\sec\theta}\right)\left(3\sec\theta\tan\theta\right)\mathrm{d}\theta$ $= 3\int_0^{\frac{\pi}{3}} \tan^2\theta\,\mathrm{d}\theta$ $= 3\int_0^{\frac{\pi}{3}} \sec^2\theta - 1\,\mathrm{d}\theta$ $= 3\left[\tan\theta - \theta\right]_0^{\frac{\pi}{3}}$ $= 3\left(\sqrt{3} - \frac{\pi}{3}\right)$ $= 3\sqrt{3} - \pi$	

5. **[2009/Promo/NJC/Q10(b)]**



The diagram above shows part of the graph of a curve C given by $x = \dfrac{y}{y-1}$.

The region R is bounded by C and the lines $y = x$ and $x = 4$.

Write down the equation of the curve obtained when C is translated by 4 units in the negative x-direction. Hence, or otherwise, write down also an expression for the volume V of the solid formed when R is rotated 2π radians about the line $x = 4$ and find its numerical value, giving your answer correct to 3 decimal places. [5]

Solution:

$$x + 4 = \frac{y}{y-1} \Rightarrow x = \frac{y}{y-1} - 4$$

$$\text{Volume} = \pi\int_{\frac{4}{3}}^{2}\left(\frac{y}{y-1} - 4\right)^2 \mathrm{d}y + \frac{\pi}{3}(2)^2(2)$$

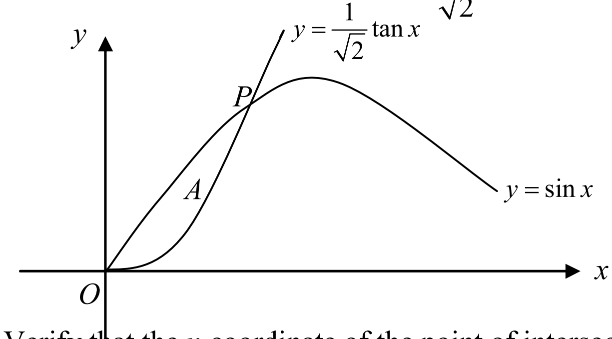
$$= \pi(1.40833) + \frac{8\pi}{3} \quad \text{(by GC)}$$

$$= 12.80197$$

$$\approx 12.802 \text{ units}^3 \quad \text{(to 3 dec. pl.)}$$

6. **[RVHS/2009/Promo/Q12]**

The region A is bounded by the curves $y=\frac{1}{\sqrt{2}}\tan x$ and $y=\sin x$ (see diagram).



(i) Verify that the x-coordinate of the point of intersection of the 2 curves, P, is $\frac{\pi}{4}$. [1]

(ii) Find the exact area of the region A, giving your answer in the form $a+b\ln 2$ where a and b are exact constants to be determined. [4]

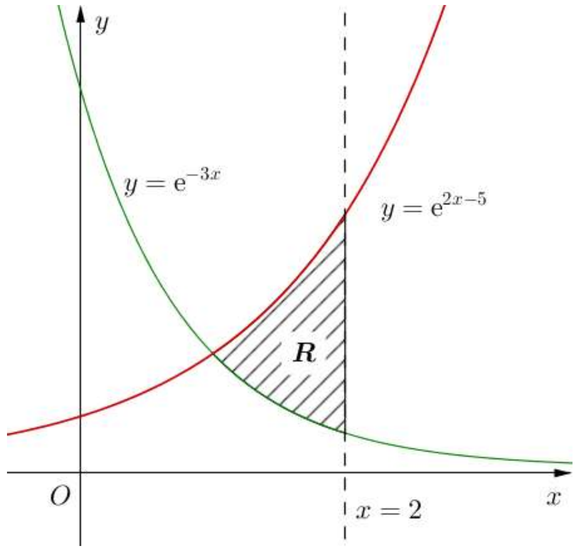
(iii) Find the exact volume of the solid of revolution formed when the region A is rotated through 360° about the x-axis. [4]

(iv) The region A is rotated through 360° about the y-axis instead. Find the volume of the solid of revolution in this case, giving your answer correct to 4 decimal places. [2]

Solution:

(i)	Since $\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}\tan\frac{\pi}{4}=\frac{1}{\sqrt{2}}$, x-coordinate of P is $\frac{\pi}{4}$.
(ii)	Area of A $=\int_0^{\frac{\pi}{4}}\sin x\ \mathrm{d}x-\int_0^{\frac{\pi}{4}}\frac{1}{\sqrt{2}}\tan x\ \mathrm{d}x$ $=-[\cos x]_0^{\frac{\pi}{4}}-\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos x}\ \mathrm{d}x$ $=1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}[\ln\lvert\cos x\rvert]_0^{\frac{\pi}{4}}$ $=1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\ln(\frac{1}{\sqrt{2}})\quad=1-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{4}\ln 2.$
(iii)	$V_x=\pi\int_0^{\frac{\pi}{4}}\sin^2 x\ \mathrm{d}x-\pi\int_0^{\frac{\pi}{4}}\frac{1}{2}\tan^2 x\ \mathrm{d}x$ $=\pi\int_0^{\frac{\pi}{4}}\frac{1-\cos 2x}{2}\ \mathrm{d}x-\frac{\pi}{2}\int_0^{\frac{\pi}{4}}\tan^2 x\ \mathrm{d}x$ $=\frac{\pi}{2}[x-\frac{1}{2}\sin 2x]_0^{\frac{\pi}{4}}-\frac{\pi}{2}\int_0^{\frac{\pi}{4}}\sec^2 x-1\ \mathrm{d}x$ $=\frac{\pi}{2}[\frac{\pi}{4}-\frac{1}{2}]-\frac{\pi}{2}[\tan x-x]_0^{\frac{\pi}{4}}$ $=\frac{\pi^2}{4}-\frac{3\pi}{4}.$
(iv)	$V_y=\pi\int_0^{\frac{1}{\sqrt{2}}}(\tan^{-1}\sqrt{2}y)^2\,dy-\pi\int_0^{\frac{1}{\sqrt{2}}}(\sin^{-1}y)^2\,dy$ $=0.1280.$

7. 2015 NJC Promo/ 6



The diagram shows the curves with equations $y = e^{2x-5}$ and $y = e^{-3x}$.
The shaded region R is bounded by the curves and the line $x = 2$.

(i) Find the x-coordinate of the point of intersection of the curves. [1]

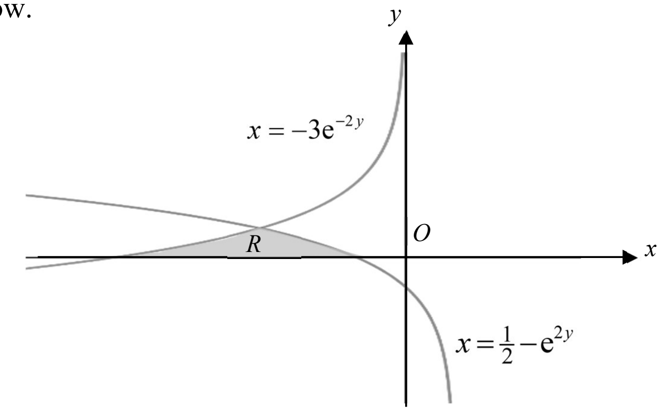
(ii) Find the exact area of R, giving your answer in the form $\frac{a}{e}+\frac{b}{e^3}+\frac{c}{e^6}$, where a, b and c are constants to be determined. [4]

(iii) Find the numerical value of the volume of revolution formed when R is completely rotated about the y-axis. [4]

(i)	$e^{-3x} = e^{2x-5}$ $-3x = 2x-5$ $x = 1$
(ii)	$\text{Area} = \int_1^2 e^{2x-5} - e^{-3x}\ dx$ $= \left[\frac{e^{2x-5}}{2}+\frac{e^{-3x}}{3}\right]_1^2$ $= \frac{e^{-1}}{2}+\frac{e^{-6}}{3}-\frac{e^{-3}}{2}-\frac{e^{-3}}{3}$ $= \frac{1}{2e}-\frac{5}{6e^3}+\frac{1}{3e^6} \qquad \therefore a=\frac{1}{2}, b=-\frac{5}{6}, c=\frac{1}{3}$
(iii)	$\text{volume} = \pi(2)^2\left(e^{-1}-e^{-6}\right)-\pi\int_{e^{-6}}^{e^{-3}}\left(-\frac{1}{3}\ln y\right)^2\ dy$ $-\pi\int_{e^{-3}}^{e^{-1}}\left(\frac{\ln y+5}{2}\right)^2\ dy$ $\approx 4.59176 - 0.25218 - 2.81111$ $= 1.52847 = 1.53$ (3 s.f.)

8. DHS/2019/Promo/Q6

The shaded region R is bounded by the curves $x=-3\mathrm{e}^{-2y}$, $x=\frac{1}{2}-\mathrm{e}^{2y}$ and the x-axis as shown in the diagram below.



(i) Find the exact area of the region R. [5]

(ii) Find the volume of the solid of revolution formed when R is rotated through 4 right angles about the y-axis, giving your answer correct to 2 decimal places. [2]

1(i)	$\frac{1}{2}-\mathrm{e}^{2y}=-3\mathrm{e}^{-2y}$ $\mathrm{e}^{2y}-\frac{1}{2}-3\mathrm{e}^{-2y}=0$ $2\mathrm{e}^{4y}-\mathrm{e}^{2y}-6=0$ $\left(\mathrm{e}^{2y}-2\right)\left(2\mathrm{e}^{2y}+3\right)=0$ $\mathrm{e}^{2y}=2$ or $\mathrm{e}^{2y}=-\frac{3}{2}$(rej. since $\mathrm{e}^{2y}>0$) $2y=\ln 2$ $y=\frac{\ln 2}{2}$ Exact area of region $R=-\int_0^{\frac{\ln 2}{2}}-3\mathrm{e}^{-2y}-\left(\frac{1}{2}-\mathrm{e}^{2y}\right)\mathrm{d}y$ $=-\left[-3\left(\frac{\mathrm{e}^{-2y}}{-2}\right)-\frac{1}{2}y+\frac{\mathrm{e}^{2y}}{2}\right]_0^{\frac{\ln 2}{2}}$ $=-\left[\left(\frac{3}{2}\left(\frac{1}{2}\right)-\frac{\ln 2}{4}+1\right)-\left(\frac{3}{2}+\frac{1}{2}\right)\right]=\frac{\ln 2+1}{4}$ unit^2
(ii)	Volume of solid of revolution $=\pi\int_0^{\frac{\ln 2}{2}}\left(-3\mathrm{e}^{-2y}\right)^2-\left(\frac{1}{2}-\mathrm{e}^{2y}\right)^2\ \mathrm{d}y$ $=4.24$ unit^3 (2 d.p.)

9. **NJC/2019/Promo/Q5**

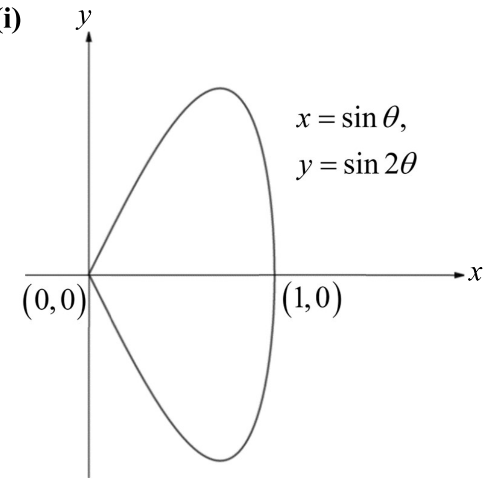
A curve C has parametric equations

$$x = \sin\theta, \qquad y = \sin 2\theta, \qquad \text{for } 0 \le \theta \le \pi.$$

(i) Sketch C, labelling the coordinates of any axial intercepts. [2]

(ii) Find the exact area of the region bounded by C. [5]

(i)



(ii)

When $x = 0$,
$\sin\theta = 0$
$\theta = 1.$

When $x = 1$,
$\sin\theta = 1$
$\theta = \frac{\pi}{2}.$

$$\frac{dx}{d\theta} = \cos\theta.$$

<u>Method 1</u>: Using the form $f'(x)\left[f(x)\right]^n$.

Area of region bounded by C

$$= 2\int_0^{\frac{\pi}{2}} \sin 2\theta \cos\theta \ d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 2\sin\theta\cos^2\theta \ d\theta = -4\int_0^{\frac{\pi}{2}} (-\sin\theta)\cos^2\theta \ d\theta = -4\left[\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}} = -4\left(0 - \frac{1}{3}\right) = \frac{4}{3} \text{ units}^2$$

<u>Method 2:</u> Using the factor formula.

Area of region bounded by C

$$= 2\int_0^{\frac{\pi}{2}} \sin 2\theta \cos\theta \ d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin 3\theta + \sin\theta \ d\theta = \left[-\frac{\cos 3\theta}{3} - \cos\theta\right]_0^{\frac{\pi}{2}} = 0 - \left(-\frac{1}{3} - 1\right) = \frac{4}{3} \text{ units}^2$$

10. TMJC/2019/Promo/Q3

The curve C has equation $y = \dfrac{\sqrt{x}}{(16-x^2)^{\frac{1}{4}}}$, $0 \le x < 4$.

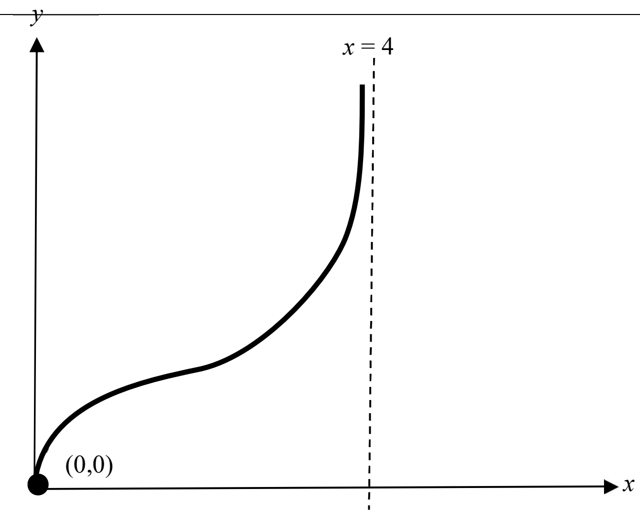
(i) Sketch C. [2]

(ii) Find the exact volume of revolution when the region bounded by C, the line $x = 2\sqrt{3}$ and the x-axis, is rotated 2π radians about the x-axis. [4]

(iii) A horizontal line l intersects C at $x = 2\sqrt{3}$. Find the exact volume of revolution when the region bounded by C, the horizontal line l and the line $x = 0$, is rotated 2π radians about the x-axis. [2]

(i)

$y = \dfrac{\sqrt{x}}{(16-x^2)^{\frac{1}{4}}}$



Volume generated

$$= \pi \int_0^{2\sqrt{3}} \left[\frac{\sqrt{x}}{\left(16-x^2\right)^{\frac{1}{4}}} \right]^2 \mathrm{d}x$$

$$= -\frac{\pi}{2} \int_0^{2\sqrt{3}} \left[\frac{-2x}{\sqrt{16-x^2}} \right] \mathrm{d}x = -\frac{\pi}{2} \left[\frac{\sqrt{16-x^2}}{\frac{1}{2}} \right]_0^{2\sqrt{3}} = -\pi \left[\sqrt{16-\left(2\sqrt{3}\right)^2} - \sqrt{16} \right] = -\pi\left(2-4\right) = 2\pi$$

(iii)

Volume generated

$$= \pi \left[\frac{\sqrt{2\sqrt{3}}}{\left(16-\left(2\sqrt{3}\right)^2\right)^{\frac{1}{4}}} \right]^2 \left(2\sqrt{3}\right) - 2\pi$$

$$= \pi \left(\frac{2\sqrt{3}}{2} \right) \left(2\sqrt{3}\right) - 2\pi$$

$$= 4\pi$$