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$$\begin{pmatrix}1&1&0&k-3\\4&8&k&1\\5&9&8&k\end{pmatrix}\xrightarrow[-5R_1+R_3]{-4R_1+R_2}\begin{pmatrix}1&1&0&k-3\\0&4&k&-4k+13\\0&4&8&-4k+15\end{pmatrix}\xrightarrow{-R_2+R_3}\begin{pmatrix}1&1&0&k-3\\0&4&k&-4k+13\\0&0&8-k&2\end{pmatrix}$$

For all real values of k, there are exactly 3 pivots in the REF. Therefore, the only possible value for $\dim(S_k) = 4 - 3 = 1$.

For $x_4 = 0$ in the solution, we require $k = 8$.

A basis for S_k when $k = 8$ is $\left\{\begin{pmatrix}2\\-2\\1\\0\end{pmatrix}\right\}$.

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<tr><td>2</td><td>

(a) $\det \mathbf{M} = 0 \Rightarrow 15 - a(-a) + b(-3a) = 0$

$$15 + a^2 = 3ab$$

$$b = \frac{5}{a} + \frac{a}{3}$$

(b) $\begin{pmatrix}1&a&b\\0&3&1\\a&0&5\end{pmatrix}\begin{pmatrix}2\\1\\-2\end{pmatrix}=\begin{pmatrix}2\\1\\-2\end{pmatrix} \Rightarrow \begin{array}{l}2+a-2b=2\\3-2=1\\2a-10=-2\end{array}$

$$\Rightarrow a = 4,\ b = 2$$

(c) Let the characteristic equation of $\mathbf{M}$ be $\mathrm{f}(\lambda) = 0$.

$\mathrm{f}(\lambda) = \det(\lambda\mathbf{I} - \mathbf{M}) = (\lambda-1)(\lambda-3)(\lambda-5) - 3 = 0$

$\mathrm{f}(2) = (1)(-1)(-3) - 3 = 0$. Since $\lambda = 2$ satisfies the characteristic equation, 2 is an eigenvalue.

$\lambda^3 - 9\lambda^2 + 23\lambda - 18 = 0$

$(\lambda-2)(\lambda^2 - 7\lambda + 9) = 0$ (by comparing coefficients)

$\lambda^2 - 7\lambda + 9 = 0 \Rightarrow \lambda = \dfrac{7 \pm \sqrt{49-4(9)}}{2} = \dfrac{7 \pm \sqrt{13}}{2}$

(d) Solve $\begin{pmatrix}1&-\sqrt{3}&0\\0&-1&-1\\-\sqrt{3}&0&-3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\begin{pmatrix}1&-\sqrt{3}&0\\0&-1&-1\\-\sqrt{3}&0&-3\end{pmatrix}\xrightarrow{R_3\to R_3+\sqrt{3}R_1}\begin{pmatrix}1&-\sqrt{3}&0\\0&-1&-1\\0&-3&-3\end{pmatrix}\xrightarrow[\substack{R_1\to R_1-\sqrt{3}R_2\\R_2\to -R_2}]{R_3\to R_3-3R_2}\begin{pmatrix}1&0&\sqrt{3}\\0&1&1\\0&0&0\end{pmatrix}$$

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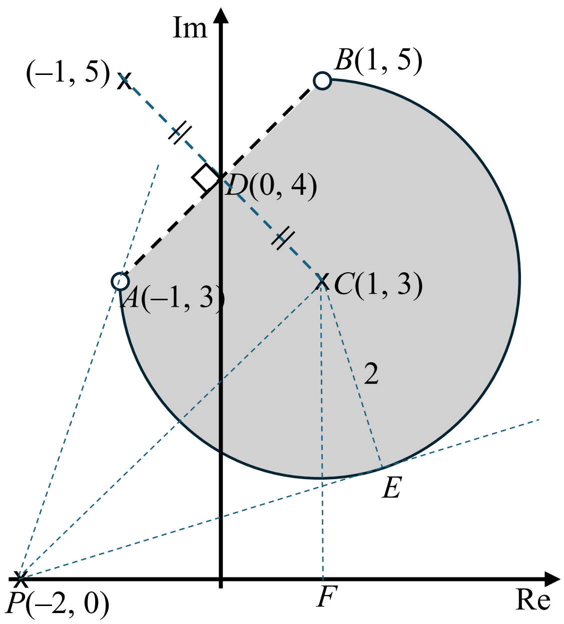
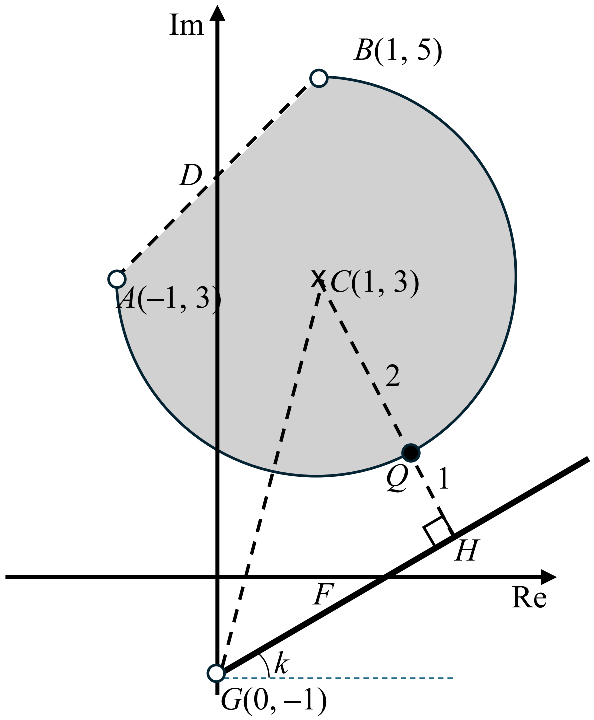
	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} -\sqrt{3} \\ -1 \\ 1 \end{pmatrix}$, $z \in \mathbb{R}$. An eigenvector corresponding to 2 is $\begin{pmatrix} \sqrt{3} \\ 1 \\ -1 \end{pmatrix}$. Alternatively, use cross-product: $\begin{pmatrix} 1 \\ -\sqrt{3} \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 1 \\ -1 \end{pmatrix}$
3	 (a) $\lvert 3+\mathrm{i}(z-1)\rvert \le 2 \Rightarrow \lvert \mathrm{i}(z-(1+3\mathrm{i})\rvert \le 2 \Rightarrow \lvert (z-(1+3\mathrm{i})\rvert \le 2$ $\lvert z-(1+3\mathrm{i})\rvert < \lvert z-(-1+5\mathrm{i})\rvert$ (b) $\tan\angle APF = \frac{3}{1} \Rightarrow \angle APF = \tan^{-1} 3$ $\tan\angle CPF = \frac{3}{3} \Rightarrow \angle CPF = \frac{\pi}{4}$. $\sin\angle CPE = \frac{2}{\sqrt{3^2+3^2}} = \frac{\sqrt{2}}{3}$. $\frac{\pi}{4} - \sin^{-1}\frac{\sqrt{2}}{3} \le \arg(z+2) < \tan^{-1} 3$ $0.295 \le \arg(z+2) < 1.25$ (3 s.f.)



(c) $\tan \angle DGC = \dfrac{1}{4}$

$\sin \angle CGH = \dfrac{2+1}{\sqrt{1^2+4^2}} = \dfrac{3}{\sqrt{17}}$

$k = \dfrac{\pi}{2} - \tan^{-1}\dfrac{1}{4} - \sin^{-1}\dfrac{3}{\sqrt{17}}$

$\approx 0.510991 = 0.511$ (3 s.f.)

4

(a) $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = \dfrac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta$, $\quad \dfrac{\mathrm{d}y}{\mathrm{d}\theta} = \dfrac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta$.

$$\int \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int \sqrt{\left[1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}x}\,\mathrm{d}x$$

$$= \int \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

$$= \int \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta\right)^2}\,\mathrm{d}\theta$$

$$= \int \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\cos^2\theta - 2r\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta\cos\theta + r^2\sin^2\theta + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\sin^2\theta + 2r\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta\cos\theta + r^2\cos^2\theta}\,\mathrm{d}\theta$$

$$= \int \sqrt{\left[r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\right]\left(\sin^2\theta + \cos^2\theta\right)}\,\mathrm{d}\theta$$

$$= \int \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

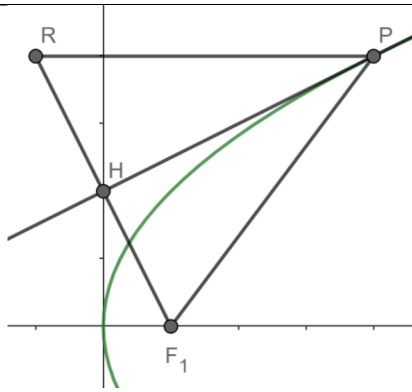
(b)(i) Find intersections:

$3 = 2(1+\cos\theta) \Rightarrow \cos\theta = \dfrac{1}{2} \Rightarrow \theta = \pm\dfrac{\pi}{3}$

Exact area

	$= 2\times\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi}\left[3^2 - 4(1+\cos\theta)^2\right]\mathrm{d}\theta$ $= \int_{\frac{\pi}{3}}^{\pi} 9 - 4(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$ $= \int_{\frac{\pi}{3}}^{\pi} 5 - 8\cos\theta - 4\left(\frac{1+\cos 2\theta}{2}\right)\mathrm{d}\theta$ $= \int_{\frac{\pi}{3}}^{\pi} 3 - 8\cos\theta - 2\cos 2\theta\,\mathrm{d}\theta$ $= \left[3\theta - 8\sin\theta - \sin 2\theta\right]_{\frac{\pi}{3}}^{\pi}$ $= 3\pi - \left(\pi - 4\sqrt{3} - \frac{\sqrt{3}}{2}\right)$ $= 2\pi + \frac{9\sqrt{3}}{2}$ (b)(ii) Perimeter $= 2\left[\overbrace{3\left(\frac{2\pi}{3}\right)}^{\text{circle arc}} + \overbrace{\int_{\frac{\pi}{3}}^{\pi}\sqrt{4(1+\cos\theta)^2 + 4\sin^2\theta}\ \mathrm{d}\theta}^{\text{cardioid arc}}\right]$ $= 4\pi + 4\int_{\frac{\pi}{3}}^{\pi}\sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta}\ \mathrm{d}\theta$ $= 4\pi + 4\int_{\frac{\pi}{3}}^{\pi}\sqrt{r}\ \mathrm{d}\theta$ (shown), $k = 4\pi,\ p = 4,\ \alpha = \frac{\pi}{3},\ \beta = \pi$ $= 4\pi + 4\sqrt{2}\int_{\frac{\pi}{3}}^{\pi}\sqrt{1+\cos\theta}\ \mathrm{d}\theta$ $= 4\pi + 4\sqrt{2}\int_{\frac{\pi}{3}}^{\pi}\sqrt{1+2\cos^2\frac{\theta}{2} - 1}\ \mathrm{d}\theta$ $= 4\pi + 8\int_{\frac{\pi}{3}}^{\pi}\cos\frac{\theta}{2}\ \mathrm{d}\theta$ $= 4\pi + 16\left[\sin\frac{\theta}{2}\right]_{\frac{\pi}{3}}^{\pi}$ $= 4\pi + 16\left(1 - \frac{1}{2}\right)$ $= 4\pi + 8$

5



(a) $y^2 = 4px$

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4p \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2p}{y}$$

Equation of L: $y - 2pt = \frac{2p}{2pt}(x - pt^2)$

$$y - 2pt = \frac{1}{t}\left(x - pt^2\right)$$

$$y = \frac{1}{t}x + pt$$

H has coordinates $(0,\ pt)$. F_1 has coordinates $(p, 0)$.

Gradient of line $F_1H = \frac{0 - pt}{p - 0} = -t$

Since $(\text{Gradient of } F_1H)(\text{Gradient of } L) = (-t)\left(\frac{1}{t}\right) = -1$, the lines are perpendicular.

(b) Since $RH = F_1H$, R has coordinates $(-p,\ 2pt)$.

The directrix equation is $x = -p$. Therefore, R lies on the directrix.

(c) The points $(-a,\ b)$ and $(a,\ -b)$ lie on the directrix, since they are points of reflection of F in the y- and x-axis respectively.

Gradient of directrix $= \frac{-b-b}{a+a} = -\frac{b}{a}$.

Equation of directrix: $y + b = -\frac{b}{a}(x - a) \quad \Rightarrow \quad bx + ay = 0$ (shown)

(d) Let the feet of perpendicular from D and E to the directrix be M and N respectively.
Since distance from point on parabola to focus = distance from point on parabola to directrix,

$FD = MD$ and $FE = NE$

$$\sqrt{a^2 + (b-1)^2} = \frac{a}{\sqrt{b^2 + a^2}} \qquad \sqrt{(a-2)^2 + b^2} = \frac{2b}{\sqrt{b^2 + a^2}}$$

$$a^2 + (b-1)^2 = \frac{a^2}{b^2 + a^2} \text{ --- (1)} \qquad (a-2)^2 + b^2 = \frac{4b^2}{b^2 + a^2} \text{ --- (2)}$$

	$4\times(1)+(2): \quad 4a^2+4(b^2-2b+1)+a^2-4a+4+b^2=\dfrac{4a^2+4b^2}{b^2+a^2}$ $$5a^2-4a+5b^2-8b+4=0$$ Completing the squares: $5\left[a^2-\frac{4}{5}a+\left(\frac{2}{5}\right)^2\right]-\frac{4}{5}+5\left[b^2-\frac{8}{5}b+\left(\frac{4}{5}\right)^2\right]-\frac{16}{5}+4=0$ $$\left(a-\frac{2}{5}\right)^2+\left(b-\frac{4}{5}\right)^2=0$$ The only possible solution is $a=\frac{2}{5}$ and $b=\frac{4}{5}$.
6	(a) Using the same people will eliminate any differences in grip strengths between different people and so will only compare the grip strengths of the dominant and non-dominant hands of the same person. (b) A 95% confidence interval for population mean grip strength $=2.79\pm 2.20099\times\frac{3.92}{\sqrt{12}}$ $=2.79\pm 2.49065$ $=(0.29935, 5.28065)$ (c) The population of differences in grip strengths must be normally distributed. (d) Let μ_d be the **population mean** difference in grip strengths (dominant – non-dominant) $H_0: \mu_d = 2$ (physiotherapist's claim) $H_1: \mu_d \neq 2$ Since the value 2 lies inside the confidence interval (0.29935, 5.28065), we do not reject H_0. Hence there is insufficient evidence at 5% significance level to refute the physiotherapist's claim. (e) If samples of the **same size are drawn repeatedly** many times and a **95% confidence interval is computed for each sample** in a similar manner, then about **95% of these intervals would contain the population mean difference in grip strength..**
7	The p.d.f. of V is $f(v)=\begin{cases}\frac{1}{20} & 60\le v\le 80\\ 0 & \text{otherwise}\end{cases}$ (a) $P(V\le v)=\frac{1}{20}(v-60)$ (using area of rectangle)

	$$\mathrm{F}(v)=\begin{cases}0 & v<60\\ \dfrac{v-60}{20} & 60\le v\le 80\\ 1 & v>80\end{cases}$$ (b) $T=\dfrac{240}{V}$ For $60\le v\le 80$, $3\le t\le 4$, $$\mathrm{G}(t)=\mathrm{P}(T\le t)=\mathrm{P}\left(\frac{240}{V}\le t\right)=\mathrm{P}\left(V\ge\frac{240}{t}\right)=1-\mathrm{P}\left(V<\frac{240}{t}\right)=1-\frac{\frac{240}{t}-60}{20}=4-\frac{12}{t}$$ $$\therefore \mathrm{G}(t)=\begin{cases}0, & t<3\\ 4-\dfrac{12}{t}, & 3\le t\le 4\\ 1, & t>4\end{cases}$$ (c) Let t hrs be the journey time from house to capital city. Then $\mathrm{P}(T\le t)\ge 0.80\Rightarrow 4-\dfrac{12}{t}\ge 0.80\Rightarrow t\ge 3\tfrac{3}{4}=3$ hrs 45 mins $\therefore$ the latest time to leave the house is 6.15 am.
8	(a) Flaws occur independently and at a constant mean rate. (b) $N=$ no. of flaws in 200 m length of material $\sim \mathrm{P_0}(3.2)$ $\mathrm{P}(N\ge 5)=1-\mathrm{P}(N\le 4)=0.219$ (c) $\mathrm{F}(x)=\mathrm{P}(X\le x)$ = P(at least 1 flaws in x metres) $=\mathrm{P}(Y\ge 0)$, where $Y\sim \mathrm{P_0}\left(\dfrac{0.8}{50}x\right)=\mathrm{P_0}(0.016x)$ $=1-\mathrm{P}(Y=0)$ $=1-\mathrm{e}^{-0.016x}$ for $x\ge 0$. $$\therefore \mathrm{f}(x)=\begin{cases}0.016\mathrm{e}^{-0.016x} & x>0\\ 0 & x\le 0\end{cases}$$ X follows an exponential distribution with mean $=\dfrac{1}{0.016}=62.5$ (d) $\mathrm{P}(X\ge 50)=1-\mathrm{F}(50)=\mathrm{e}^{-0.8}=0.449$

9

(a) The underlying population for the estimated time is normally distributed.

(b) The hypotheses to be tested are:

H_0 : Estimates are normally distributed.

H_1 : Estimates are not normally distributed.

Level of significance 5%

Using the unbiased estimates in (a), we test the fit of the data to the normal distribution $N(61.1, 7.564^2)$.

Estimates x (seconds)	Observed frequency O_i	Expected frequency E_i	$\frac{(O_i - E_i)^2}{E_i}$
$x < 51.5$	4	6.1314	0.7409
$51.5 \le x < 55.5$	10	7.6412	0.7281
$55.5 \le x < 59.5$	9	11.2016	0.4327
$59.5 \le x < 63.5$	17	12.4951	1.6242
$63.5 \le x < 67.5$	15	10.6060	1.8204
$x \ge 67.5$	5	11.9247	4.0212

Degree of freedom: $\nu = 6 - 1 - 2 = 3$ (constraints: total frequency, 2 estimated paramters)

Test Statistic: $\sum \frac{(O_i - E_i)^2}{E_i} \sim \chi_3^2$

Rejection region: $\chi^2 \ge 7.815$

Conclusion: Since p-value = 0.0248 > 0.05 **OR** Since χ^2value = $9.3676 > 7.815$,

$\therefore$ H_0 is rejected. Hence, there is sufficient evidence at 5% level of significance that the estimates are not normally distributed.

(c) H_0 : $M = 0$, where M = median of t

H_1 : $M < 0$

Level of significance: 5%

Under H_0, Test Statistic: $S = S_+ \sim B(18, 0.5)$

Let $S_+ =$ no. of positive signs out $18 = 10$

$p - \text{value} = P(S \le 10) = 0.760 > 0.05$ $\therefore$ H_0 is not rejected.

Hence, there is insufficient evidence at 5% level of significance to conclude that the estimates of one minute tends to be shorter after lunch than before lunch.

(c) Wilcoxon matched pair sign test makes use of the magnitudes of the differences rather than just their signs, thus making it more powerful than sign test.
Not appropriate to use Wilcoxon test here because of too many tied ranks.

10	(a) For a two-sample *t*-test, the test statistic when testing for equality of the population means is $$T=\frac{\bar{x}_1-\bar{x}_2}{s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \text{ where } s_p^2=\frac{\sum\left(x_1-\bar{x}_1\right)^2+\sum\left(x_2-\bar{x}_2\right)^2}{n_1+n_2-2}$$ Given that $n_1=n_2=n$, $\sum\left(x_1-\bar{x}_1\right)^2=\left(n_1-1\right)s_1^2=\left(n-1\right)s_1^2$ $\sum\left(x_2-\bar{x}_2\right)^2=\left(n_2-1\right)s_2^2=\left(n-1\right)s_2^2$ $\therefore\ s_p^2=\frac{\left(n_1-1\right)s_1^2+\left(n_2-1\right)s_2^2}{2n-2}=\frac{1}{2}\left(s_1^2+s_2^2\right)$ Hence, $T=\frac{\bar{x}_1-\bar{x}_2}{\sqrt{\frac{1}{2}\left(s_1^2+s_2^2\right)\frac{2}{n}}}=\frac{\left(\bar{x}_1-\bar{x}_2\right)\sqrt{n}}{\sqrt{s_1^2+s_2^2}}$ Degree of freedom $\nu=n_1+n_2-2=2n-2=2\left(n-1\right)$. Let μ be the population mean mass of a melon. Test $\mathrm{H}_0:\ \mu=600$ $\mathrm{H}_1:\ \mu\neq600$ Level of significance: 5% Under H_0, test statistic $T=\frac{\bar{x}-600}{\frac{s}{\sqrt{n}}}\sim t_{11}$ Using GC, *p*-value = 0.0566 > 0.05 , we do not reject H_0 Hence there is insufficient evidence at 5% level of significance to conclude that the sample from *A* does not come from a population with mean 600 g. **OR** Critical/Rejection region = $\{t:t\leq-2.201 \text{ or } t\geq2.201\}$ For supplier A, t value $=\frac{593.3-600}{\sqrt{\frac{118.8}{12}}}=-2.129>-2.201$ Hence there is insufficient evidence at 5% level of significance to conclude that the sample from *A* does not come from a population with mean 600 g. (c) Let X_A and X_B be the mass of a melon from supplier A and B respectively with population mean mass μ_A and μ_B respectively . $\mathrm{H}_0:\ \mu_A-\mu_B=0$ $\mathrm{H}_1:\ \mu_A-\mu_B\neq0$ Level of significance: 1% Assumptions: 1. The mass of melons from supplier B follows a normal distribution.

	2. There is a common population variance of the masses of melons from suppliers *A* and *B*. 3. The samples are drawn independently Perform a 2-sample *t*-test, Under H_0 , test statistic is $T = \frac{(\bar{x}_A - \bar{x}_B)\sqrt{n}}{\sqrt{s_A^2 + s_B^2}}$ (from result in first part) Degree of freedom $v = 2(12-1) = 22$ Critical/Rejection region is $t \le -2.819$ or $t \ge 2.819$. $t - \text{value} = \frac{(593.3 - 607.2)\sqrt{12}}{\sqrt{118.8 + 153.8}} = -2.916 < -2.819$ **Or** using GC, *p*-value = 0.00800 < 0.01 Hence H_0 is rejected and we may conclude that there is sufficient evidence at 1% significance level that there is a difference between the mean weight of melons from the two suppliers