EUNOIA JUNIOR COLLEGE

JC1 Promotional Examination 2023

General Certificate of Education Advanced Level

Higher 2

FURTHER MATHEMATICS 9649/01

Paper 1

28 September 2023

3 hours

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

An answer booklet will be provided with this question paper. You should follow the instructions on the front cover of the answer booklet. If you need additional answer paper ask the invigilator for a continuation booklet.

Answer **all** questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

1 The curve C has polar equation $r = 2 + \sin 3\theta,\ 0 \le \theta \le 2\pi$.

(a) Sketch C, indicating clearly all key features and symmetries of the curve. [3]

(b) Given that r attains a minimum value at the point with $\theta = \frac{11}{6}\pi$, find the exact cartesian equation of the tangent to C at this point. [3]

2 ~~**(a)** Given that y is a function of x, find expressions for $\frac{\mathrm{d}}{\mathrm{d}x}(xy)$, $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(xy)$ and $\frac{\mathrm{d}^3}{\mathrm{d}x^3}(xy)$ in terms of x and derivatives of y. Hence conjecture an expression for $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(xy)$ in terms of x and derivatives of y, for positive integers n. [2]~~

~~**(b)** Use mathematical induction to prove the correctness of your conjecture. [4]~~

~~**(c)** Hence find an expression for $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x\mathrm{e}^{-x}\right)$ in terms of x. [2]~~

3 ~~Consider the system of equations~~

$$\begin{aligned} x + 5y + z &= 1 \\ \cancel{2x + 11y + az} &\cancel{= 4} \\ x + 2ay + 16z &= -11 \end{aligned}$$

~~where a is a real constant.~~

~~**(a)** Determine the values of a such that the system of equations is consistent. [3]~~

~~**(b)** Given that infinitely many solutions exist, find the solutions of this system. [3]~~

~~**(c)** Hence state the solutions of the system of equations~~

$$\begin{aligned} x + 5y + z &= 2 \\ \cancel{2x + 11y + az} &\cancel{= 4 + a} \\ x + 2ay + 16z &= 5 \end{aligned}$$

~~where a is a constant such that infinitely many solutions exist. [2]~~

4 The curve T has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = x^3 - 3x - 1$.

(a) Show that T has exactly one root α in the interval [1, 2]. [2]

(b) Use one stage of the linear interpolation process to find an approximation, β, to the root α. Explain why this β will be an underestimate. [2]

(c) **(i)** Write down the Newton-Raphson formula in the form $x_{n+1} = \mathrm{g}(x_n)$, where#$x_n$ is the n-th approximation using the formula. [1]

(ii) A student proposes to use either one of the end values from the interval as his initial approximation, $x_1 = 1$ or $x_1 = 2$. Explain why one of these cases will fail and evaluate α correct to 3 decimal places using the other initial approximation. [3]

(d) Given that the linear interpolation process always produces an underestimate over the interval $[u_n, 2]$, where $u_0 = 1$ and u_n is the approximated root after the n-th iteration of the process, write down an expression for this iterative process in the form $u_{n+1} = \mathrm{h}(u_n)$. Hence comment on the efficiency of this process in comparison to the Newton-Raphson process for finding this root of the curve T. [2]

5 ~~Let $\mathbb{R}^3$ be the set of vectors of the form $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, where a, b and c are real numbers.~~

~~**(a)** S is defined as the subset of $\mathbb{R}^3$ for which $2a - b + 4c = 0$. Show that S is a linear space. [2]~~

~~The transformation T, from $\mathbb{R}^3 \to \mathbb{R}^3$, is defined by~~

$$\mathrm{T}: \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a + 3c \\ 6a + 4b + 2c \\ a + b - c \end{pmatrix}.$$

~~**(b)** Show that T is a linear transformation. [2]~~

~~It is given that T can be represented in the form~~

$$\mathrm{T}: \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \mathbf{A} \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \text{ where } \mathbf{A} \text{ is a } 3 \times 3 \text{ matrix with real entries.}$$

~~**(c)** Determine the matrix **A**, and hence find a basis for the null space of **A**. [3]~~

~~**(d)** State the rank of matrix **A**. [1]~~

~~**(e)** Hence, or otherwise, show that the linear space S is the range space of T. [2]~~

~~**(f)** Find the subset of $\mathbb{R}^3$ whose image under T is the point with position vector $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$. [2]~~

6 A sequence of real numbers $\{u_n\}$ is defined by $u_0 = 1$, $u_1 = \alpha$ where α is a constant, and the recurrence relation

$$u_n = 4u_{n-1} + u_{n-2},\ n \geq 0.$$

(a) Solve the recurrence relation to determine an expression for u_n in terms of α. [4]

(b) Hence state the value of α for which the sequence $\{u_n\}$ converges, justifying your answer. [2]

(c) It is now given that $\alpha = 0$. Another sequence $\{s_n\}$ is defined by the expression

$$s_n = \frac{2u_n + u_{n-1}}{2u_n},\ n \geq 2.$$

(i) Show that $s_{n+1} = 1 + \dfrac{1}{4(1+s_n)}$. [2]

(ii) Given that the sequence $\{s_n\}$ converges, find the exact value of the limit. [2]

(iii) State the exact value of s_8, and deduce a rational approximation for the value of $\sqrt{5}$. [2]

7 ~~Points P and Q lie on the x-axis with coordinates $(-c, 0)$ and $(c, 0)$ respectively, with $c > 0$. The conic section curve L is defined as the locus of points R for which the difference between the distances, $|PR - RQ| = 2a$, where $0 < a < c$. It is given that $PR = r$ and $\angle RPQ = \theta$.~~

~~**(a)** For $PR - RQ = 2a$, show that $r(1 - u\cos\theta) = v$, giving the constants u and v as fractions in terms of a and c. [4]~~

~~**(b)** Hence write down an expression for the eccentricity, e, of L in terms of a and c. [1]~~

~~**(c)** Using (x, y) as the coordinates of R and suitable substitutions for $r\cos\theta$, $r\sin\theta$ and r^2, show that the cartesian equation of the curve L is $(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$. [2]~~

~~Given now that $a = \sqrt{3}$ and $c = 2$, we let the line S with equation $y = mx + k$, where $k > 0$, be a tangent to L.~~

~~**(d)** Show that $k^2 = 3m^2 - 1$. [4]~~

~~Let T be another tangent to L with equation $y = -\dfrac{1}{m}x + \sqrt{\mathrm{f}(m)}$, where T is perpendicular to S.~~

~~**(e)** Write down the expression for $\mathrm{f}(m)$. [1]~~

~~**(f)** Using S and T, show that the intersection points between two perpendicular tangents lie on the circle with equation $x^2 + y^2 = 2$. [3]~~

8 A student, Ian, from the Makerspace Club plans to use the curve $\left(x^2+y^2\right)^2=7y^3+2x^2y$ to form the shape of a crest for a club logo.

(a) Show that the polar equation of the curve, $r=\mathrm{f}(\theta)$ can be written in the form $r=5\sin^3\theta+2\sin\theta$. [2]

It is known that the area bounded by the curve, $A=\int_0^{\frac{1}{2}\pi}\left[\mathrm{f}(\theta)\right]^2\,\mathrm{d}\theta$. Another student, Rian, from the same club suggests to get an estimate of A using the Simpson's rule with 3 ordinates.

(b) Evaluate Rian's estimate, leaving the answer in the form of $\dfrac{p}{q}\pi$, where $p,q\in\mathbb{Z}^+$. [3]

A third student, Brian, believes that the estimate is unlikely to be reliable and suggests the following method to evaluate the exact value of A.

(c) Use integration by parts to show that for integers $n\geq 2$,

$$\int_0^{\frac{1}{2}\pi}\sin^n\theta\,\mathrm{d}\theta=\left(\frac{n-1}{n}\right)\int_0^{\frac{1}{2}\pi}\sin^{n-2}\theta\,\mathrm{d}\theta.$$ [4]

(d) By applying the formula in part **(c)**, evaluate the exact value of A. [3]

(e) Determine the percentage error of the estimate in part **(b)** and comment on Brian's claim. [2]

9 ~~The National Population Commission wants to use a population model to study the migration patterns between two cities connected by a bridge, city B and city E. The population, in thousands, of city B and city E at the end of the n^{th} year after 2022 is defined as B_n and E_n respectively. In the model, it is assumed that every year 2% of city B's population migrates to city E, and 10% of city E's population migrates to city B. Migration patterns with other cities are assumed to be negligible.~~

~~The population model can be expressed in the form~~

$$\begin{pmatrix} \mathrm{B}_n \\ \mathrm{E}_n \end{pmatrix} = \mathbf{A} \begin{pmatrix} \mathrm{B}_{n-1} \\ \mathrm{E}_{n-1} \end{pmatrix}, \text{ for } n \geq 1.$$

~~Data from the end of 2022 is used to determine the values of B_0 and E_0 for the model.~~

~~**(a)** Write down the matrix **A**, and use the matrix to show that $\mathrm{B}_n + \mathrm{E}_n$ is a constant for all integer values of $n \geq 0$.~~ ~~[3]~~

~~**(b)** Determine the eigenvalues and corresponding eigenvectors of matrix **A**.~~ ~~[4]~~

~~**(c)** The Commission ran a simulation where at the end of 2022, city B has a population of 50000 while city E has a population of 80000. By expressing **A** in the diagonalised form $\mathbf{QDQ}^{-1}$, find an expression for the vector $\begin{pmatrix} \mathrm{B}_n \\ \mathrm{E}_n \end{pmatrix}$ in terms of n.~~ ~~[4]~~

~~**(d)** One population planner hypothesizes that city E will be abandoned eventually, since a larger proportion of people leave city E for city B every year. Explain whether the model supports this position.~~ ~~[2]~~

~~The Commission ran a new simulation which follows the same migration patterns each year as the original model, but introduces a government relocation programme in 2025 to move 5000 people from city B to city E. They modelled that in the long run, the projected population sizes would be almost the same as the original simulation without relocation.~~

~~**(e)** Determine the year in which the difference between the population of city B under the new simulation and the original simulation first falls below 100.~~ ~~[2]~~