RAFFLES INSTITUTION
YEAR 5-6 PHYSICS DEPARTMENT

Chapter 7 GRAVITATION

Content

- Gravitational field
- Gravitational force between point masses
- Gravitational field of a point mass
- Gravitational field near to the surface of the Earth
- Gravitational potential
- Circular orbits

Learning Outcomes

Candidates should be able to:

(a) show an understanding of the concept of a gravitational field as an example of field of force and define gravitational field strength at a point as the gravitational force exerted per unit mass placed at that point.

(b) recognize the analogy between certain qualitative and quantitative aspects of gravitational and electric fields (to be taught in Year 6).

(c) recall and use Newton's law of gravitation in the form $F = \frac{Gm_1m_2}{r^2}$.

(d) derive, from Newton's law of gravitation and the definition of gravitational field strength, the equation $g = \frac{GM}{r^2}$ for the gravitational field strength of a point mass.

(e) recall and apply the equation $g = \frac{GM}{r^2}$ for the gravitational field strength of a point mass to new situations or to solve related problems.

(f) show an understanding that near the surface of the Earth g is approximately constant and equal to the acceleration of free fall.

(g) define the gravitational potential at a point as the work done per unit mass in bringing a small test mass from infinity to that point.

(h) solve problems using the equation $\phi = -\frac{GM}{r}$ for the gravitational potential in the field of a point mass.

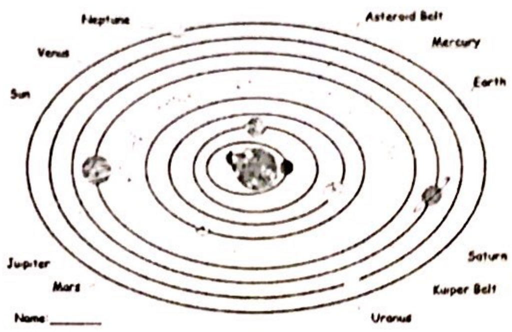
(i) analyse circular orbits in inverse square law fields by relating the gravitational force to the centripetal acceleration it causes.

(j) show an understanding of geostationary orbits and their application.

7.1 Gravitational Force F

Introduction

In 1687, Newton proposed in his *Philosophiæ Naturalis Principia Mathematica* that every mass attracts another mass with a force of gravity. According to him, two seemingly unrelated phenomena – the fall of an apple from an apple tree and the orbital motion of the planets around the Sun – are due to the same reason: gravitational attraction. He came up with the Newton's Law of Gravitation. This law is universally valid and applies to any planets in the solar system and even between distant galaxies.



Newton's Law of Gravitation

Definition

Newton's law of gravitation states that two point masses attract each other with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

The magnitude of the gravitational force F between two particles of masses M and m which are separated by a distance r is given by

Formula

$$\boxed{F = \frac{GMm}{r^2},}$$

where G is the gravitational constant with a value of 6.67×10^{-11} N m^2 kg^{-2}.

G constants can have units

Note:

- Newton's law of gravitation is an example of an inverse square law because the magnitude of the force varies inversely with the square of the separation of the particles. gravitational force, electrostatic force, power (os)
- Gravitational force is attractive in nature and some books indicate this with a negative sign in the formula

$F \propto \frac{1}{r^2}$

Example 7.1 A man of mass 85.0 kg is standing on the surface of the Earth. Given that the Earth has a mass of 5.98×10^{24} kg and a radius of 6.37×10^{6} m, calculate the force that the Earth exerts on the man. Deduce the force that the man exerts on Earth.

Solution g

$$F = \boxed{\frac{GM_E m}{r_E^2}}$$ not F = mg

$$= \frac{(6.67\times10^{-11})\times(5.98\times10^{24})\times(85.0)}{(6.37\times10^{6})^2}$$

$$= 836 \text{ N}$$

By Newton's third law, the force that the man exerts on the Earth is equal in magnitude to the force that the Earth exerts on the man, but in the opposite direction.

Note:
Due the mass of the Earth, we do not expect to see the Earth move:

$$\underbrace{F_{\text{man on Earth}} = -F_{\text{Earth on man}}}_{\text{N3L}} \Rightarrow \underbrace{m_{Earth}a_{Earth} = -m_{man}a_{man}}_{\text{N2L}} \Rightarrow a_{Earth} = -\frac{m_{man}}{m_{Earth}}a_{man} \approx 0$$

Example 7.2 **NJC/H2/Prelims 2010/P1/16 - modified]**
On the surface of the Earth, the gravitational force acting on an object is 45 N. When the object is at a height h above the surface, the gravitational force acting on it is 5 N. If R is the radius of the earth, calculate the approximate value for h in terms of R.

Solution

$$F_{surface} = \frac{GMm}{R^2} = 45$$

$$F_{raised} = \frac{GMm}{(R+h)^2} = 5$$

$$\frac{F_{raised}}{F_{surface}} = \frac{45}{5} = 9 = \frac{(R+h)^2}{R^2}$$

$$\frac{R+h}{R} = 3$$

$$h = 3R - R$$
$$= 2R$$ ✓

$$F_1 = \frac{GMm}{R^2} = 45\text{N}$$

$$F_2 = \frac{GMm}{(R+h)^2} = 5\text{N}$$

$$\frac{F_1}{F_2} = \frac{(R+h)^2}{R^2} = 9$$

$$\therefore \frac{(R+h)}{R} = 3$$

$$h = 2R$$ //

7.2 Gravitational Field Strength *g*

Gravitational Field

Definition

A **gravitational field** is a region of space in which a mass placed in that region experiences a gravitational force.

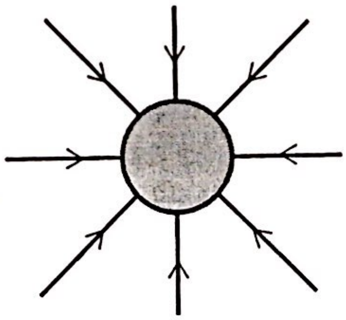
Gravitational Field Lines

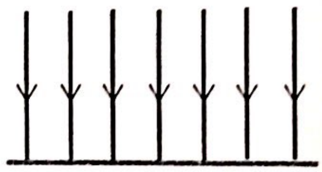
A region of gravitational field can be visualised as consisting of an array of imaginary field lines in which the tangent at a point on a field line acts in the direction of the gravitational force.

- The direction of the field lines indicates the direction of the gravitational field.
- The density of the field lines indicates its strength. A region with a stronger gravitational field strength will have closer or denser field lines, and vice versa.
- For a point mass or a uniform spherical mass, the field lines are directed towards its centre.
- The figure on the left shows the field lines around Earth, which are directed towards its centre. If we zoom in to a region near the surface of the Earth, the field lines seem to be parallel to one another as shown in the figure on the right.
- Hence, near the surface of the Earth, we can consider the gravitational field to be uniform with a magnitude of 9.81 N kg^{-1}.

only attractive and never repulsive :

always into





Definition

The **gravitational field strength** at a point in space is defined as the gravitational force experienced per unit mass at that point.

Formula

$$g = \frac{F}{m}$$

Gravitational Field Strength of a Point Mass

Since the gravitational force between two point masses is given by $F = \frac{GMm}{r^2}$,

Formula

$$g = \frac{GM}{r^2}$$

Note:

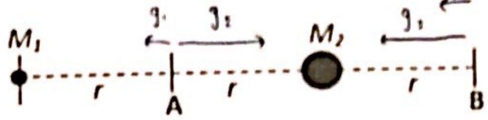
- Gravitational field strength is a vector quantity and it points towards the mass which created it.
- The gravitational field strength at a point is the acceleration of free fall at that point.
- The resultant field strength at a point due to more than one mass can be found by the vector sum of the gravitational field strengths due to all masses.

→ +/- depending

g < acceleration due to ~~field~~ free fall
strength of gravitational field

$ms^{-2} = \frac{N}{kg}$

Example 7.3 Find the resultant gravitational field strength due to the two masses at points **A** and **B**, given that $M_2 > M_1$.



g is a vector, direction must be taken into account

Solution

At point A: $g_1 = G\frac{M_1}{r^2}$ (towards M_1) $\qquad g_2 = G\frac{M_2}{r^2}$ (towards M_2)

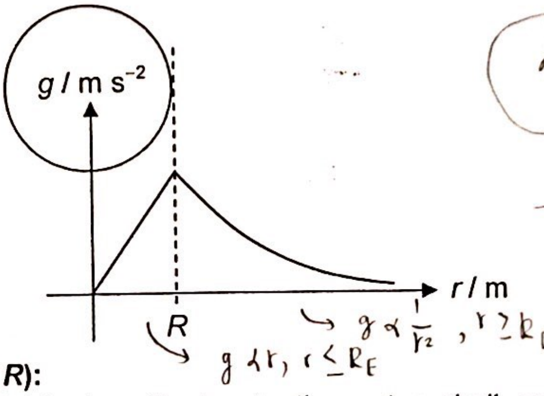
$$g_A = g_2 - g_1 = G\frac{M_2}{r^2} - G\frac{M_1}{r^2} \quad \text{(towards } M_2\text{)}$$

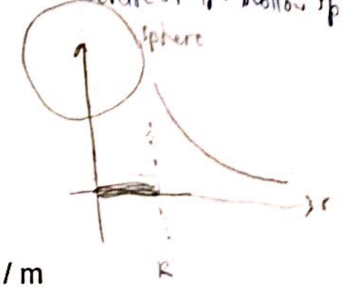
At point B: $g_B = G\frac{M_1}{(3r)^2} + G\frac{M_2}{r^2}$ (towards M_2 and M_1)

Gravitational Field Strength of a Uniform Sphere

For uniform spheres, planets and stars, we can use $F = G\frac{m_1 m_2}{r^2}$ and $g = \frac{GM}{r^2}$ to calculate the gravitational forces and gravitational field strengths. Draw the variation of gravitational field strength with distance from the centre of the hollow sphere

Important Note





$g \propto \frac{1}{r^2}$, $r \geq R_E$

$g \propto r$, $r \leq R_E$

Inside the sphere ($r \leq R$):

- not within syllabus || The gravitational field strength due to the outer shell cancels out and has no contribution to the final value of g (see Appendix A).
- within syllabus || The mass M of the inner sphere is $M = \rho V = \rho\left(\frac{4}{3}\pi r^3\right)$.
- The gravitational field strength due to the inner sphere varies linearly with distance r.

$$g = \frac{GM}{r^2} = \frac{G}{r^2}\left(\frac{4}{3}\rho\pi r^3\right) = \frac{4}{3}G\rho\pi r$$

Outside the sphere ($r \geq R$):

- The gravitational field strength due to a uniform sphere is identical to that of a point mass at the centre of the sphere. Therefore, $g = \frac{GM}{r^2}$.

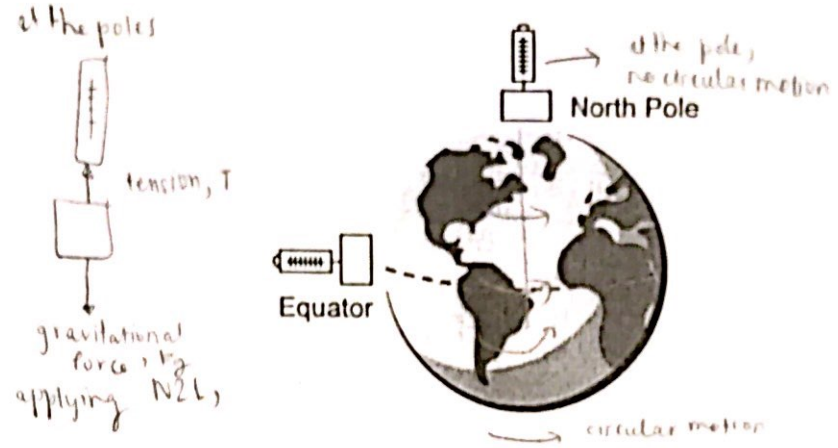
Note:

- The gravitational field inside a hollow sphere is zero (see Appendix A).
- The gravitational field inside a uniform solid sphere varies linearly with distance r.
- The gravitational field outside a uniform solid sphere varies inversely with r^2.

Variation in Gravitational Field Strength due to Earth's Rotation

The figure shows an object of mass m hanging from a spring balance at two places on the Earth – the Equator and the North Pole.

In both cases there are two forces acting on the mass – gravitational force F_g (true weight) and force T due to the spring.



If the Earth is taken to be a uniform sphere, then F_g is the same at both the equatorial and polar regions. However, T will be different because the mass undergoes circular motion at the Equator.

Resultant force F_R at the polar region:

- Applying Newton's second law, $F_R = F_g - T$
- Since there is no circular motion, $F_R = 0$
- Therefore, $T = F_g$
- The spring balance indicates F_g, which is the true weight of the object.

Resultant force F_R at the Equator:

- Applying Newton's second law, $F_R = F_g - T$
- Since the object undergoes circular motion, $F_R = F_C = F_g - T$
- Therefore, $T = F_g - F_C$
- The spring balance indicates a value less than the true weight of the object. This value is known as the **apparent weight.**

Important Note

At the Equator, part of the gravitational force provides for the centripetal force to keep the object moving in a uniform circular motion.

Question: What happens if the Earth spins faster about its axis of rotation?

Example 7.4

(i) Given that the Earth's radius is 6370 km, find the centripetal acceleration of an object placed at the Equator.

(ii) If a 5.0 kg mass is placed on a weighing scale at the Equator, what would the scale read in kg?

Solution

i) acceleration $= \frac{v^2}{r}$

$r = 6370\,000$

$v = 2(6370\,000)(\pi) \div (24 \times 60 \times 60)$

$= 463.24\ ms^{-1}$

acceleration $= 0.0337\ ms^{-2}$ (3s.f.) ✓

ii) $T = F_g - F_c$

$= \frac{GMm}{r^2} - (5.0)(0.0337)$

$= \cancel{6.67\times10^{-11}\times} $

$(5.0)(9.81) - (5.0)(0.0337)$

$= 48.9N$

$m = \frac{T}{g} = \frac{48.9}{9.81} = 4.98\ kg$

how fast must the earth turn to see 0 kg?

$T = F_g - F_c$
$= 0$

$F_g = F_c$

$a = 9.81\ ms^{-2}$

~~9.81 / 0.0337 = 291.2 (3.4s)~~

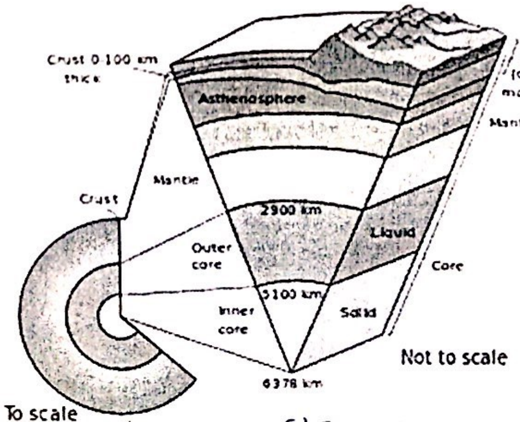
~~24h = 2(6370000)π / r~~

$T = \sqrt{\frac{4\pi^2 r}{g}} = 1.41h$

Other Factors Affecting Gravitational Field Strength

1. The gravitational field strength over the Earth's surface varies as the result of its lack of spherical symmetry. Its polar diameter is about 40 km less than its equatorial diameter.

2. The Earth's density is not uniform. There are local variations in the Earth's gravitational field caused by mountains and trenches, as well as the presence of minerals and oil deposits.



GPE: at least 2 mass

GP: 1 mass

Law of Gravitation

$F_g = mg = \frac{GMm}{r^2}$

don't need to waste time when the height is very small: 1000 m

Gravitational field

$g = \frac{F}{m} = \frac{GM}{r^2}$

field lines

field strength

similar to acceleration

variation in g due to composition, ellipsoidal shape,

variation due to composition, ellipsoidal shape

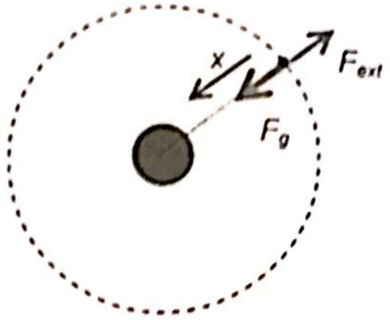
7.3 Gravitational Potential Energy U & Gravitational Potential ϕ

Gravitational Potential Energy

Definition

The **gravitational potential energy** of a mass at a point in a gravitational field is defined as the work done by an external force in bringing the mass from infinity to that point.

Not in syllabus: $U = \int_\infty^r F_{ext}\,dr = \int_\infty^r \frac{GMm}{r^2}dr = \left[-\frac{GMm}{r}\right]_\infty^r = -GMm\left[\frac{1}{r}-\frac{1}{\infty}\right] = -\frac{GMm}{r}$



The gravitational potential energy U of two particles of masses M and m separated by a distance r is given by

Formula

$$\boxed{U = -\frac{GMm}{r}}$$

Note:

- Due to its definition, the negative sign must be indicated.
- For small distances h above the surface of the Earth, $U \approx mgh$ (see Appendix B).

gpe always changes depending on reference point
gpe ≠ mgh unless some conditions are met : set a reference point

Relationship between Force and Potential Energy

Formula

Gravitational force F and gravitational potential energy U are related by the following expression:

differential
½kx² to get Hooke's law →

$$\boxed{F = -\frac{dU}{dr}}$$

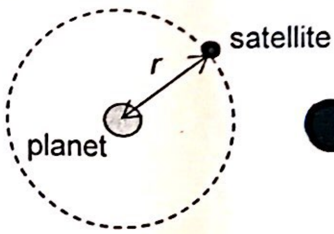
discuss Δ in GPE

The negative sign indicates that the force points in the direction of potential energy.

Example: $F = -\frac{dU}{dr} = -\frac{d}{dr}\left(-\frac{GMm}{r}\right) = GMm\frac{d}{dr}\left(\frac{1}{r}\right) = GMm\left(-\frac{1}{r^2}\right) = -\frac{GMm}{r^2}$

Example 7.5

A satellite of mass m is moving in a uniform circular motion of radius r around a planet of mass M.



What is the total energy of the satellite?

Solution

Since satellite is moving in a uniform circular motion, it experiences a centripetal force. The centripetal force is provided by the gravitational force due to the planet.

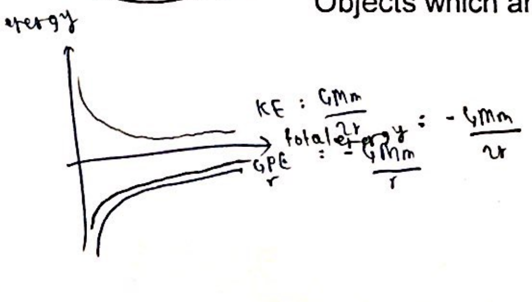
$$F_g = F_c \Rightarrow \frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow v_{orbital} = \sqrt{\frac{GM}{r}}.$$

Important Note

$$\text{Total energy} = \text{KE} + \text{GPE} = \frac{1}{2}mv^2 + \left(-\frac{GMm}{r}\right) = \frac{1}{2}m\left(\frac{GM}{r}\right) + \left(-\frac{GMm}{r}\right) = -\frac{GMm}{2r}$$

Objects which are bounded have a negative total energy!

↳ because of definition of GPE: infinity is reference point



Gravitational Potential

Definition

The **gravitational potential** at a point in a gravitational field is defined as the work done per unit mass by an external force in bringing a small test mass from infinity to that point.

Formula

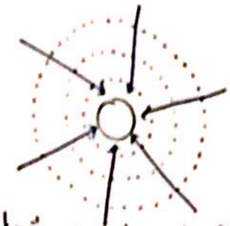
$$\phi = \frac{U}{m}$$

Formula

At a distance r from a point mass M, $U = -\frac{GMm}{r}$, therefore

$$\phi = -\frac{GM}{r}$$

where the mass m of the small test mass cancels out.



[illegible] points at a fixed distance from the mass form an equipotential surface

Important Note

Why is gravitational potential (or potential energy) negative in value? ← (memorise this)

- Gravitational potential (or potential energy) at infinity is zero.
- Since gravitational force is attractive in nature, to bring a mass from infinity to a point in the gravitational field, the direction of the external force is opposite to the direction of displacement of the mass. (We can imagine the external force supporting the mass as it is being lowered towards the Earth.)
- This results in negative work done by the external force.
- Hence, based on its definition, gravitational potential (or potential energy) has a negative value.

Note:

- Gravitational potential is a scalar quantity.
- The negative sign is <u>not</u> an indication of direction and it <u>cannot</u> be omitted.
- The gravitational potential at a point due to two or more masses can be found by adding of the potentials at that point due to each of the mass.

Relationship between Field Strength and Potential

Gravitational field strength is numerically equal to the potential gradient.

Formula

$$g = -\frac{d\phi}{dr}$$

The negative sign indicates that the field strength points in the direction of decreasing potential.

Example 7.6 Given that the mass of the Earth is 5.98 x 10^{24} kg and its radius is 6.37 x 10^{6} m, calculate the change in the gravitational potential as an object is moved from the surface of the Earth to a point 1200 m above the surface.

Solution

$= \phi_f - \phi_i$

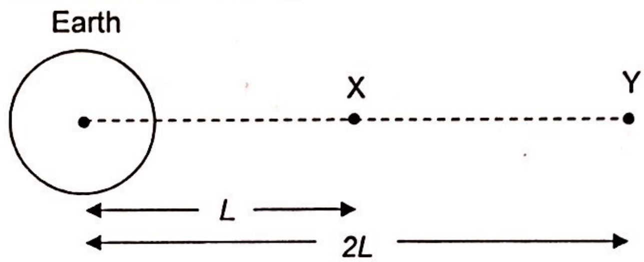
$$\Delta\phi = \left(-\frac{Gm}{r}\right) - \left(-\frac{Gm}{r_E}\right)$$

$$= Gm\left(\frac{1}{r_E} - \frac{1}{r}\right)$$

$$= 6.67 \times 10^{-11} \times 5.98 \times 10^{24}\left(\frac{1}{6.37 \times 10^{6}} - \frac{1}{6.37 \times 10^{6} + 1200}\right)$$

$$= 1.18 \times 10^{4} \text{ J kg}^{-1}$$

Example 7.7 **[SAJC/H2/Prelims 2010/P1/13 - modified]**
The figure shows two points X and Y at distances *L* and *2L* from the centre of the Ear[illegible]
The gravitational potential at X is –8 kJ kg^{-1}.



What is the work done when a 2 kg mass is taken from X to Y?

Solution

~~$\phi_f = \phi_i + \Delta\phi$~~

~~$= \frac{-Gm}{2L}$~~

$\phi_x = -\frac{Gm}{L} = -8$ J kg^{-1}

$\phi_y = \frac{-Gm}{2L} = -4$ J kg^{-1}

work done = gain in GPE = ΔU

$= m\Delta\phi = m(\phi_y - \phi_x)$

$= 2[(-2)-(-8)]$

$= 8$ kJ

7.4 Escape Velocity

Escape Velocity

Have you ever wondered if it is possible to throw an object into the air at such a high velocity that it will never fall back to Earth? This is actually plausible because the acceleration of free fall does not stay at 9.81 m s^{-2} but decreases as the object moves away from the Earth. If this is possible, what is the value of this "escape velocity"? The solution to this problem is simple if we consider the energy of the object.

How much energy does an object of mass m need in order to escape from the Earth's gravitational influence and reach infinity? At infinity, its GPE is defined as zero. If the object has sufficient energy to *just* reach infinity, its KE at infinity is zero. So the total energy at infinity is given by $E_T = E_P + E_K = 0$.

From the principle of conservation of energy, the total energy of an object should remain unchanged throughout its motion. This means that any object with a total energy zero will be able to *just* reach infinity and *stop* there.

Consider an object (e.g. a rocket) of mass m being projected with a velocity v from the surface of the Earth. Its initial GPE and KE are:

$$U = -\frac{GMm}{R} \text{ and } \mathrm{KE} = \frac{1}{2}mv^2$$

Therefore, minimum energy to reach infinity is such that

$$\text{Total energy} = -\frac{GMm}{R} + \frac{1}{2}mv^2 = 0$$

$\sqrt{2\left(\frac{GM}{R^2}\right)R} : \sqrt{2gR}$

(assuming conservation of energy)

Formula

$$\boxed{v = \sqrt{\frac{2GM}{R}} = \sqrt{2gR}}$$

Note:
- The escape velocity is independent of the mass of the object.
- The escape velocity for Earth is 11.2 km s^{-1}.

Example 7.8 A spacecraft of mass 5000 kg propels itself from the surface of the Earth of mass 5.98×10^{24} kg and radius 6.37×10^6 m. If Calculate the minimum escape speed.

Solution

$$v \geq \sqrt{\frac{2GM}{R}}$$
$$= \sqrt{\frac{2\times(6.67\times10^{-11})\times(5.98\times10^{24})}{6.37\times10^{6}}}$$
$$= 1.12\times10^{4}\ \text{m s}^{-1}$$

OR

$$v \geq \sqrt{2gR}$$
$$= \sqrt{2\times9.81\times(6.37\times10^{6})}$$
$$= 1.12\times10^{4}\ \text{m s}^{-1}$$

Example 7.9 **[RI/H2/Prelims 2010/P2/3 - part]**

Given that the mass of Earth is 5.98×10^{24} kg and its radius is 6370 km, determine the minimum kinetic energy required to project a spacecraft of mass 2550 kg from the surface of Earth so that it completely escapes from the gravitational attraction of the Earth. Ignore air resistance.

Solution

By the principle of conservation of energy,

total energy on surface = total energy at infinity

At infinity, GPE is 0 (by definition) and KE is 0 (since spacecraft is launched with the minimum kinetic energy).

Hence,

$$E_{surface} = E_{infinity}$$

$$\cancel{E_{sur}}\ E_{kmin} + \left(-\frac{G m_E m_s}{R_E}\right) = 0$$

$$E_{kmin} = \frac{G m_E m_s}{R_E}$$

$$= \frac{(6.67 \times 10^{-11})(5.98 \times 10^{24})(2550)}{6370 \times 10^{3}}$$

$$= 1.60 \times 10^{11}\ \text{J}$$

7.5 Planetary & Satellite Motions

Recap

For any object of mass m to move in uniform circular motion of radius r, there must be a resultant force F_c acting on the object which is directed towards the centre of the circle. This is known as the centripetal force, which can be expressed as

$$F_c = \frac{mv^2}{r} \text{ or } mr\omega^2$$

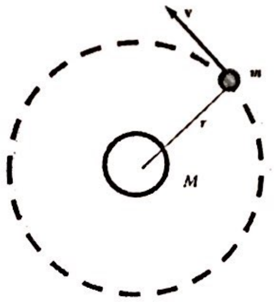
where v is the linear velocity and ω is the angular velocity.

The time taken for the object to make one complete circle is known as the period T, which is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi r}{v}$$

Kepler's Third Law

Consider a satellite of mass m moving with linear speed v in a circular orbit of radius r about a planet of mass M.



The only force acting on the satellite is the gravitational force. Hence, this gravitational force provides the centripetal force necessary for the circular motion of the satellite. Satellites generally do not have any engine propelling it.

Applying Newton's second law of motion, (derive this)

$$F_g = F_c$$

$$G\frac{Mm}{r^2} = mr\omega^2$$

$$G\frac{Mm}{r^2} = mr\left(\frac{2\pi}{T}\right)^2$$

$$T^2 = \frac{4\pi^2}{GM}r^3$$

Formula

$$\boxed{T^2 \propto r^3}$$

where T is the orbital period and r is the orbital radius. This relationship is also known as Kepler's third law.

Example 7.10 **[MI/H2/Prelims 2010/P1/14 - modified]**

Mercury is 5.79×10^{10} m from the Sun and it takes 0.241 Earth years for Mercury to make one revolution around the Sun. If Neptune is 450×10^{10} m from the Sun, calculate the period of Neptune's around the Sun.

Solution

from $T^2 = \frac{4\pi^2}{GM} r^3$,

$$\left(\frac{T_N}{T_M}\right)^2 = \left(\frac{r_N}{r_M}\right)^3$$

$$\left(\frac{T_N}{0.241\ yr}\right)^2 = \left(\frac{450 \times 10^{10}}{5.79 \times 10^{3}}\right)^3$$

T_N = 165 years

radius = r_{earth} + altitude

Example 7.11 **(a)** An object orbits at an altitude of 300 km above the Earth. What is its linear velocity?

(b) Briefly explain why the orbital plane of any satellite includes the centre of the Earth.

Take Earth as a uniform sphere of radius 6370 km and mass 5.98×10^{24} kg.

Solution

(a) orbital speed: use Newton's 2nd law

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$v = \sqrt{\frac{GM}{r}}$$

$$= \sqrt{\frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{6370000 + 300000}}$$

$= 7.73 \times 10^3\ ms^{-1}$

(b) The gravitational force due to the Earth provides for the centripetal force on the satellite. Since the centripetal force is directed towards the centre of the Earth, the orbital plane must contain the centre of the Earth.

special orbit

Geostationary Orbits

Important Note

A **geostationary satellite** is one that remains at a fixed position relative to any point on the Earth's surface.

A geostationary satellite must satisfy the following conditions:

1. Its orbital period is the same as that of the Earth about its axis of rotation (24 hrs).
2. Its direction of rotation is the same as that of the Earth about its axis of rotation (eastward).
3. It lies above the Equator (or in the same plane as the Equator).

The radius of a geostationary orbit is

$$r = \left(\frac{T^2GM}{4\pi^2}\right)^{\frac{1}{3}} = \left(\frac{(24\times3600)^2\times6.672\times10^{-11}\times5.96\times10^{24}}{4\pi^2}\right)^{\frac{1}{3}} = 42200\text{ km}$$

Therefore, the height of the orbit above the Earth's surface = 42200 - 6400 = 35800 km.

Advantages of using a geostationary satellite

1. Continuous surveillance of the region under it.
2. Easy for the ground station to communicate with it.
3. Due to the high altitude, the satellite can transmit and receive signals over a large area.

can it lie at any other latitudes?

Disadvantages

Its distance from the Earth's surface is large compared to the Low-Earth Orbit (LEO) satellites, which typically operate at an altitude of a few hundred kilometres. This leads to

1. a significant loss of signal strengths,
2. poorer resolution in imaging satellites,
3. time lag in telecommunication.

Example 7.12 Which of the following quantities are *not necessarily* the same for satellites that are in geostationary orbits around the Earth?

A. angular velocity
B. centripetal acceleration
C. kinetic energy
D. period of orbit
E. radius of orbit
F. mass
G. momentum

<u>Solution</u>:

C, F, G. (any quantity that involves mass)

Example 7.13 **[N99/III/2]**

A satellite of mass 2400 kg is placed in a geostationary orbit at a distance of 4.23×10^7 m from the centre of the Earth.

(a) Calculate

1. the angular velocity of the satellite,
2. the speed of the satellite,
3. the acceleration of the satellite,
4. the force of attraction between the Earth and the satellite,
5. the mass of the Earth.

(b) Explain why a geostationary satellite

1. must be placed vertically above the equator,
2. must move from west to east.

(c) Why are such satellites often used for telecommunications?

Solutions

(a) 1. $\omega = \dfrac{2\pi}{T}$

$= \dfrac{2\pi}{24 \times 60 \times 60}$

$= 7.27 \times 10^{-5}$ rad s^{-1}

2. $v = r\omega$

$= 4.23 \times 10^7 \times 7.27 \times 10^{-5}$

$= 3.08 \times 10^3$ m s^{-1}

3. $a = \dfrac{v^2}{r}$

$= \dfrac{(3.08 \times 10^3)^2}{4.23 \times 10^7}$

$= 0.224$ m s^{-2}

4. $F = \dfrac{GMm}{r^2}$

$= ma$

$= 2400 \times 0.224$

$= 538$ N

5. $538 = \dfrac{GMm}{r^2}$

$M = \dfrac{538 \times (4.23 \times 10^7)^2}{6.11 \times 10^{-11} \times 2400}$

$= 6.56 \times 10^{24}$ kg

(b) 1. Since the centripetal force on the satellite is directed towards the centre of the Earth, any circular orbit must have its centre at the centre of the Earth. If the orbit is not on the equator, the satellite must sometimes be over the northern hemisphere and sometimes over the southern hemisphere, and so cannot be geostationary.

2. For a satellite to stay above a fixed point, it must have the same direction of rotation as the Earth about its axis of rotation, ie from west to east.

(c) The earthbound transmitters and receivers can be aimed in a fixed direction, with no disruption to the signals.

Summary

- Gravitational force: $F = \frac{GMm}{r^2}$
- Gravitational field strength: $g = \frac{GM}{r^2}$
- Gravitational potential energy: $U = -\frac{GMm}{r}$
- Gravitational potential: $\phi = -\frac{GM}{r}$
- Relationship between F and g: $g = \frac{F}{m}$
- Relationship between U and ϕ: $\phi = \frac{U}{m}$
- Relationship between F and U: $F = -\frac{dU}{dr}$
- Relationship between g and ϕ: $g = -\frac{d\varphi}{dr}$ = negative potential gradient.
- For a body to escape from the Earth's gravitational influence:

$$-\frac{GMm}{R_{Earth}} + \frac{1}{2}mv^2 = 0 \Rightarrow \qquad v_{escape} = \sqrt{\frac{2GM}{R_{Earth}}} = \sqrt{2gR_{Earth}}$$

- For a body to orbit around the Earth at radius r:

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow \qquad v_{orbital} = \sqrt{\frac{GM}{r}}$$

$$\frac{GMm}{r^2} = mr\left(\frac{2\pi}{T}\right)^2 \Rightarrow \qquad T^2 = \frac{4\pi^2}{GM}r^3$$

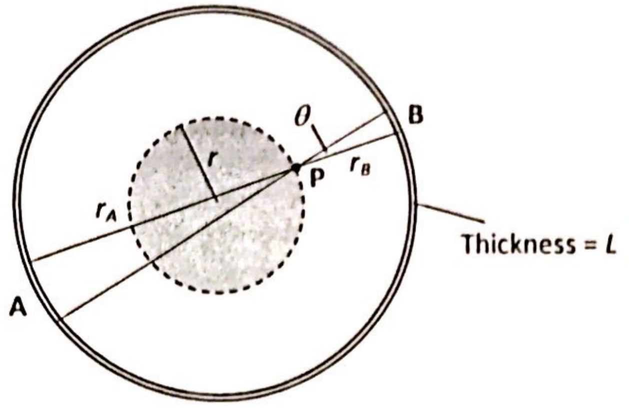
- These conditions will ensure that a geostationary satellite is always above the same point on the Earth's surface, ie its relative position with respect to the Earth remains unchanged:
  1. Same 24-hour orbital period as the Earth.
  2. Same Eastward rotation as the Earth.
  3. Orbits above the equator.

$G = 6.67 \times 10^{-11}$

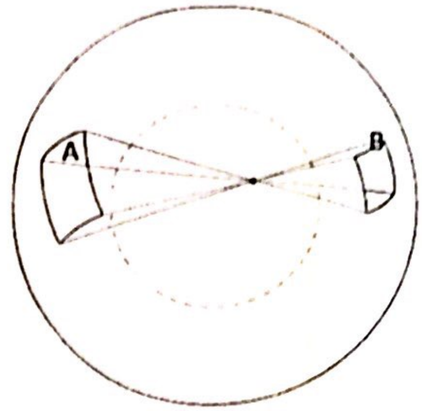
Gravitational Field Inside a Homogenous Spherical Shell of Mass

Appendix A

Cross-sectional view



3D view to show the areas of A and B



Consider a point P located within a solid sphere a distance r from the centre of the sphere. We will prove that the "effective mass" consists of only the mass enclosed by the shaded sphere within dotted sphere of radius r. The approach is to prove that the rest of the mass (the unshaded portion) cancels out each other's gravitational effect such that their net force on any mass placed at P is zero.

First, consider a thin spherical shell of thickness L. P experiences the gravitational field from every portion of the shell. Two pieces of mass A and B are located on the opposite sides of P as shown. If the angle they subtend at P is θ, then their areas (considering a square of sides = arc length, $r\theta$) are given by $(r_A\theta)^2$ and $(r_B\theta)^2$ respectively. So their masses will be $(r_A\theta)^2 L\rho$ and $(r_B\theta)^2 L\rho$ respectively, where ρ is the density of the sphere. Hence, their gravitational field strength at P will be $G\frac{(r_A\theta)^2 L\rho}{r_A^2}$ and $G\frac{(r_B\theta)^2 L\rho}{r_B^2}$ respectively.

In both cases the r's cancel away and we get $G\theta^2 L\rho$ for both. So the gravitational field strengths of A and B are equal in magnitude at P. It is also obvious that they are opposite in direction since A and B are on opposite sides of P. So the resultant field at P due to A and B is zero. Apply this argument to every direction around P, we find that the resultant field due to the whole spherical shell is zero.

A solid sphere can be imagined to be made up of layers of spherical shells. If every shell with radius > r produces zero resultant field at P then clearly their combined effect is still zero. So the effective mass is only those within the dotted shaded sphere.

Derivation of $\Delta U = mgh$

Appendix B

When an object of mass m is lifted through height h from point 1 to point 2 near the surface of the Earth, we usually use the formula mgh to determine the gain in gravitational potential energy U. This is a good approximation if g is constant between point 1 and point 2.

Proof

Take point 1 to be the ground level and point 2 a distance h above it, such that $r_1 = R_E$ and $r_2 = R_E + h$. Then

$$\begin{aligned}
\Delta U &= \left(-\frac{GMm}{r_2}\right) - \left(-\frac{GMm}{r_1}\right) \\
&= \left(-\frac{GMm}{R_E + h_2}\right) - \left(-\frac{GMm}{R_E}\right) \\
&= -\frac{GMm}{R_E}\left(\frac{R_E}{R_E + h} - 1\right) \\
&= -\frac{GMm}{R_E}\left(\frac{R_E - R_E - h}{R_E + h}\right) \\
&= \frac{GMm}{R_E}\left(\frac{h}{R_E + h}\right) \\
&\approx \frac{GMm}{R_E}\left(\frac{h}{R_E}\right) && \text{since } h \ll R_E \\
&= m\left(\frac{GM}{R_E^2}\right)h && \text{since } g = \frac{GM}{R_E^2} \\
&= mgh
\end{aligned}$$