RAFFLES INSTITUTION
H2 Mathematics 9758
2023 Year 6 Term 3 Revision 7b (Tutorial)

Topic: Vectors 3 (Lines and Planes)

Summary for Lines and Planes
[Refer to Revision 7a]

Revision Tutorial Questions

Source of Question: NJC Prelim 9758/2018/02/Q2

1 The planes p_1 and p_2, have equations $2x+3y+6z=0$ and $\mathbf{r}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=6$ respectively.

(i) Find a vector equation of the line of intersection, l, between p_1 and p_2. [2]

The line m passes through the points $A(2,1,1)$ and $B(5,4,2)$.

(ii) Verify that A lies on p_2. [1]

(iii) Find the coordinates of the points on m that are equidistant from planes p_1 and p_2. [5]

Solution

(i)	$p_1:\ 2x+3y+6z=0$ $p_2:\ x+2y+2z=6$ Using GC, $$\mathbf{r}=\begin{pmatrix}-18\\12\\0\end{pmatrix}+\lambda\begin{pmatrix}-6\\2\\1\end{pmatrix},\lambda\in\mathbb{R}$$
(ii)	Since $2+2(1)+2(1)=6$, or $\begin{pmatrix}2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=2+2+2=6$, the point A lies on p_2.
(iii)	Let the point that is equidistant from both planes be C. $\begin{pmatrix}5\\4\\2\end{pmatrix}-\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}3\\3\\1\end{pmatrix}$

	$\overrightarrow{OC}=\begin{pmatrix}2\\1\\1\end{pmatrix}+t\begin{pmatrix}3\\3\\1\end{pmatrix}$ for some $t\in\mathbb{R}$ Distance of C from p_1 = Distance of C from p_2 $$\frac{\left|\left[\begin{pmatrix}2+3t\\1+3t\\1+t\end{pmatrix}-\begin{pmatrix}2\\1\\1\end{pmatrix}\right]\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}\right|}{\sqrt{1^2+2^2+2^2}}=\frac{\left|\left[\begin{pmatrix}2+3t\\1+3t\\1+t\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}\right]\cdot\begin{pmatrix}2\\3\\6\end{pmatrix}\right|}{\sqrt{2^2+3^2+6^2}}$$ $$\frac{\lvert 3t+6t+2t\rvert}{3}=\frac{\lvert 4+6t+3+9t+6+6t\rvert}{7}$$ $$\frac{11\lvert t\rvert}{3}=\frac{\lvert 13+21t\rvert}{7}$$ $$77\lvert t\rvert=\lvert 39+63t\rvert$$ $$77t=-39-63t \quad\text{or}\quad 77t=39+63t$$ $$140t=-39 \quad\text{or}\quad 14t=39$$ $$t=-\frac{39}{140} \quad\text{or}\quad t=\frac{39}{14}$$ $$\overrightarrow{OC}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\left(-\frac{39}{140}\right)\begin{pmatrix}3\\3\\1\end{pmatrix}=\frac{1}{140}\begin{pmatrix}163\\23\\101\end{pmatrix}\text{ or }\overrightarrow{OC}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\left(\frac{39}{14}\right)\begin{pmatrix}3\\3\\1\end{pmatrix}=\frac{1}{14}\begin{pmatrix}145\\131\\53\end{pmatrix}$$ The two points are $\left(\frac{163}{140},\frac{23}{140},\frac{101}{140}\right)$ and $\left(\frac{145}{14},\frac{131}{14},\frac{53}{14}\right)$.

Source of Question: NJC JC2 Mid-Year CT 9758/2018/01/Q6

2 The equation of the plane p is given by $ax-2y-5z=7$, where a is a real constant.

(i) Given that the line l with equation $\mathbf{r}=\begin{pmatrix}3\\5\\6\end{pmatrix}+\lambda\begin{pmatrix}5\\-2\\0\end{pmatrix}$, $\lambda\in\mathbb{R}$ intersects the plane p exactly once, find the possible values of a. [1]

Assume for the remainder of this question that $a = 3$.

(ii) Find the coordinates of the point of intersection between l and p. [3]

(iii) Find the acute angle between the line l and the plane p. [2]

(iv) Find the vector equation of the line of reflection of the line l in the plane p. [5]

Solution

(i)	Since $\begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ -2 \\ -5 \end{pmatrix} \neq 0 \Rightarrow 5a+4 \neq 0 \Rightarrow a \neq -\frac{4}{5}$ a can be any real number except $-\frac{4}{5}$.
(ii)	Let N be the point of intersection between l and p. Since N lies on l, $\overrightarrow{ON} = \begin{pmatrix} 3+5\lambda \\ 5-2\lambda \\ 6 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$. Since N also lies on plane p, $\overrightarrow{ON} \cdot \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix} = 7$ $\begin{pmatrix} 3+5\lambda \\ 5-2\lambda \\ 6 \end{pmatrix} . \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix} = 7$ $9+15\lambda-10+4\lambda-30=7$ $19\lambda = 7-9+10+30=38$ $\lambda = 2$ Hence, $\overrightarrow{ON} = \begin{pmatrix} 3+5(2) \\ 5-2(2) \\ 6 \end{pmatrix} = \begin{pmatrix} 13 \\ 1 \\ 6 \end{pmatrix}$. Therefore, coordinates of point of intersection are (13,1,6).
(iii)	Let θ be the acute angle required. $\theta = \sin^{-1} \dfrac{\left\| \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix} \right\|}{\sqrt{29}\sqrt{38}}$ $\theta = 34.9^\circ$ or 0.609 rad
(iv)	The equation of the line l_1 passing through $A(3, 5, 6)$. and perpendicular to p has equation $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}, \mu \in \mathbb{R}$. Let F be the foot of the perpendicular from the point $A(3, 5, 6)$ onto p.

Since F lies on l_1, $\overrightarrow{OF} = \begin{pmatrix} 3+3\mu \\ 5-2\mu \\ 6-5\mu \end{pmatrix}$ for some $\mu \in \mathbb{R}$.

Since F also lies on p, $\overrightarrow{OF} \cdot \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix} = 7$

$$\begin{pmatrix} 3+3\mu \\ 5-2\mu \\ 6-5\mu \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix} = 7$$

$$9+9\mu-10+4\mu-30+25\mu = 7$$

$$38\mu = 7-9+10+30 = 38$$

$$\mu = 1.$$

So $\overrightarrow{OF} = \begin{pmatrix} 3+3 \\ 5-2 \\ 6-5 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix} \Rightarrow \overrightarrow{NF} = \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 13 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} -7 \\ 2 \\ -5 \end{pmatrix}$

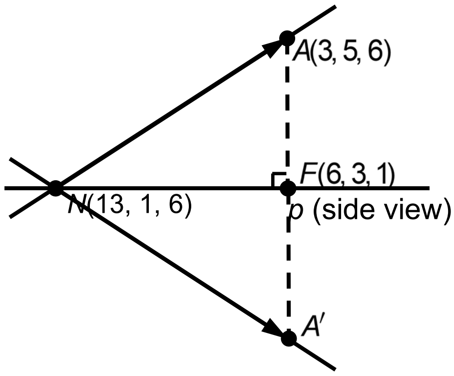
Let A' be the point of reflection of A about p.
By Ratio Theorem,

$$\overrightarrow{OF} = \frac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}$$

$$\Rightarrow \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} + \overrightarrow{OA'}\right]$$

$$\Rightarrow \overrightarrow{OA'} = 2\begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ -4 \end{pmatrix}$$

$$\Rightarrow \overrightarrow{NA'} = \begin{pmatrix} 9 \\ 1 \\ -4 \end{pmatrix} - \begin{pmatrix} 13 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ -10 \end{pmatrix}$$



Therefore, equation of the line of reflection is

$$\mathbf{r} = \begin{pmatrix} 13 \\ 1 \\ 6 \end{pmatrix} + \nu \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}, \ \nu \in \mathbb{R}.$$

Source of Question: YJC Prelim 9758/2018/01/Q9

3 The two lines l_1 and l_2 have equations $x = \frac{z}{-2} - 2,\ y = 0$ and $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ q \\ 2 \end{pmatrix}$ respectively, where $\mu \in \mathbb{R}$ and q is a constant. The point A has coordinates $(4,\ 3,\ k)$ where k is a constant and the two planes p_1 and p_2 have equations $4x = -8 + y - 2z$ and $y + tz = 2t$ respectively, where t is a constant.

(i) Find a vector equation of l_1 and hence show that l_1 lies in p_1. [3]

(ii) Find the coordinates of the foot of perpendicular from A to p_1. Express your answer in terms of k. [3]

(iii) Given that the angle between l_1 and l_2 is 60°, find the possible values of q. [2]

(iv) Given that $q = 1$ and that the point $B(1,\ 2,\ 2)$ is equidistant from l_2 and p_2, find the possible values of t. [6]

Solution

(i)	$l_1:\ x + 2 = \frac{z}{-2}$ Let $\lambda = x + 2 = \frac{z}{-2}$ $x = -2 + \lambda$ $y = 0$ $z = -2\lambda$ Hence, vector equation of l_1 is $$\mathbf{r} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix},\quad \lambda \in \mathbb{R}$$ Note that $\left[\begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}\right] \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ $= -8 + 4\lambda - 4\lambda$ $= -8$ for all values of λ. Hence all points on l_1 lies in p_1. Hence, l_1 lies in p_1.

(ii)

Let the foot of perpendicular from A to p_1 be N.

Equation of the line that passes through A and perpendicular to p_1 is

$$l_{AN} : \mathbf{r} = \begin{pmatrix} 4 \\ 3 \\ k \end{pmatrix} + \alpha \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}, \quad \alpha \in \mathbb{R}.$$

Since N lies on l_{AN}, $\overrightarrow{ON} = \begin{pmatrix} 4+4\alpha \\ 3-\alpha \\ k+2\alpha \end{pmatrix}$, for some $\alpha \in \mathbb{R}$

$$\begin{pmatrix} 4+4\alpha \\ 3-\alpha \\ k+2\alpha \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = -8$$

$$16+16\alpha-3+\alpha+2k+4\alpha=-8$$

$$21\alpha=-21-2k$$

$$\alpha=-1-\frac{2k}{21}$$

$$\therefore \overrightarrow{ON} = \begin{pmatrix} 4-4-\frac{8k}{21} \\ 3+1+\frac{2k}{21} \\ k-2-\frac{4k}{21} \end{pmatrix} = \begin{pmatrix} -\frac{8k}{21} \\ 4+\frac{2k}{21} \\ -2+\frac{17k}{21} \end{pmatrix}$$

Hence, N is the point $\left(-\frac{8}{21}k, 4+\frac{2}{21}k, \frac{17}{21}k-2\right)$.

(iii)

Direction vector of l_2 is $\begin{pmatrix} 0 \\ q \\ 2 \end{pmatrix}$.

$$\cos 60^o = \frac{\left| \begin{pmatrix} 0 \\ q \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right|}{\sqrt{q^2+4}\sqrt{5}}$$

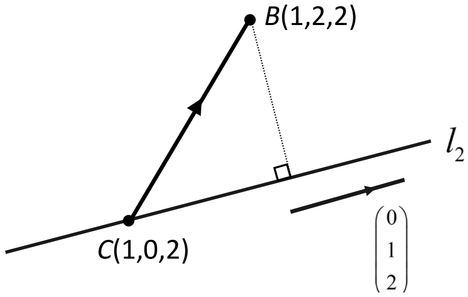
$$\frac{1}{2} = \frac{4}{\sqrt{5q^2+20}}$$

$$\sqrt{5q^2+20}=8$$

$$q^2=\frac{44}{5}$$

$$q=\pm\sqrt{\frac{44}{5}}=\pm 2.97 \text{ (3 s.f.)}$$

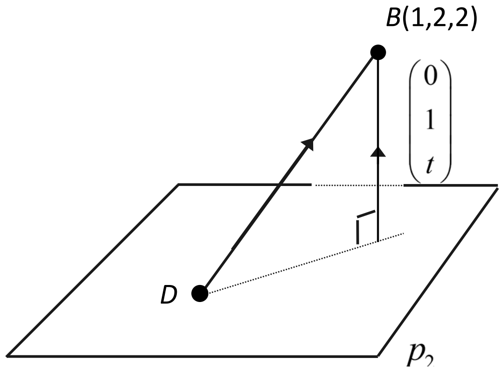
(iv)



$$\overrightarrow{CB}=\begin{pmatrix}0\\2\\0\end{pmatrix}$$

Shortest distance from B to the line l_2:

$$=\frac{\left|\begin{pmatrix}0\\2\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\2\end{pmatrix}\right|}{\sqrt{1+2^2}}=\frac{\left|\begin{pmatrix}4\\0\\0\end{pmatrix}\right|}{\sqrt{1+2^2}}=\frac{4}{\sqrt{5}}$$



Let D be any point on plane p_2.

$$\overrightarrow{OD}=\begin{pmatrix}0\\0\\2\end{pmatrix}\qquad \overrightarrow{DB}=\overrightarrow{OB}-\overrightarrow{OD}=\begin{pmatrix}1\\2\\0\end{pmatrix}$$

Shortest distance from B to plane p_2

$$=\frac{\left|\begin{pmatrix}1\\2\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\t\end{pmatrix}\right|}{\sqrt{1+t^2}}=\frac{2}{\sqrt{1+t^2}}$$

	Since point $B(1,2,2)$ is equidistant from l_2 and p_2, $$\frac{4}{\sqrt{5}}=\frac{2}{\sqrt{1+t^2}}$$ $$\sqrt{1+t^2}=\frac{\sqrt{5}}{2}$$ $$t^2=\frac{1}{4}$$ $$t=\pm\frac{1}{2}$$

Source of Question: SAJC JC2 Mid-Year CT 9758/2018/01/Q8

4 **(a)** Given that $\mathbf{v}=\mathbf{i}+2\mathbf{j}-3\mathbf{k}$, find the direction cosines of **v**. Hence prove that the sum of the squares of the direction cosines is 1. [3]

(b) The planes Π_1 and Π_2 have equations $x+my+3z=6$ and $mx+3y+z+10=0$ respectively. The point P has coordinates (–3, –3, 5) and O is the origin.

(i) Given that point P is on Π_1, find m. Verify that point P is also on Π_2. [2]

(ii) Hence find a vector equation of the line of intersection between the planes Π_1 and Π_2. [2]

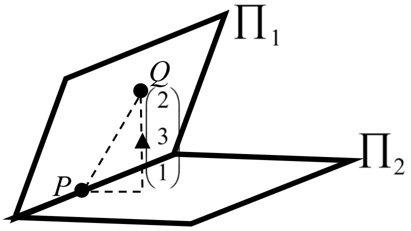
(iii) Point Q with coordinates (–3, 0, 3) is a point on Π_1. Find the shortest distance from point Q to Π_2. [3]

(iv) Suggest in parametric form, the equation of a plane Π_3 that contains points P and Q and is perpendicular to Π_2. [1]

Solution

(a)	Given $\mathbf{v}=\mathbf{i}+2\mathbf{j}-3\mathbf{k}$. Let α, β and γ be the angles formed by **v** with the x-, y- and z-axis respectively. $$\cos\alpha=\frac{\begin{pmatrix}1\\2\\-3\end{pmatrix}.\begin{pmatrix}1\\0\\0\end{pmatrix}}{\sqrt{1+2^2+(-3)^2}}=\frac{1}{\sqrt{14}}$$

	$\cos\beta = \dfrac{2}{\sqrt{14}}$ $\cos\gamma = \dfrac{-3}{\sqrt{14}}$ Sum of squares $\left(\dfrac{1}{\sqrt{14}}\right)^2 + \left(\dfrac{2}{\sqrt{14}}\right)^2 + \left(\dfrac{-3}{\sqrt{14}}\right)^2 = \dfrac{14}{14} = 1$ (shown)
(b) (i)	Since point P is on $\prod_1$, then (–3, –3, 5) must satisfy the equation of the plane. $-3-3m+3(5)=6$ $m=2$ When $m=2$, LHS: $2(-3)+3(-3)+5+10=0=\text{RHS}$ (verified)
(ii)	To find the direction of line of intersection: $\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}2\\3\\1\end{pmatrix}=\begin{pmatrix}-7\\5\\-1\end{pmatrix}$ Hence, the equation of the line required is $\mathbf{r}=\begin{pmatrix}-3\\-3\\5\end{pmatrix}+\lambda\begin{pmatrix}-7\\5\\-1\end{pmatrix},\ \lambda\in\mathbb{R}$
(iii)	$\overrightarrow{PQ}=\begin{pmatrix}-3\\0\\3\end{pmatrix}-\begin{pmatrix}-3\\-3\\5\end{pmatrix}=\begin{pmatrix}0\\3\\-2\end{pmatrix}$ Shortest distance required = $\left\lvert \overrightarrow{PQ}\bullet\dfrac{\begin{pmatrix}2\\3\\1\end{pmatrix}}{\sqrt{14}}\right\rvert = \left\lvert \dfrac{\begin{pmatrix}0\\3\\-2\end{pmatrix}\bullet\begin{pmatrix}2\\3\\1\end{pmatrix}}{\sqrt{14}}\right\rvert = \dfrac{\sqrt{14}}{2}$ units

(iv)	 Π_3 that contains points *P* and *Q* implies Π_3 contains $\overrightarrow{PQ}$. Π_3 is perpendicular to Π_2 implies Π_3 contains $\begin{pmatrix}2\\3\\1\end{pmatrix}$. $\Pi_3 : \mathbf{r} = \begin{pmatrix}-3\\0\\3\end{pmatrix} + g\begin{pmatrix}0\\3\\-2\end{pmatrix} + h\begin{pmatrix}2\\3\\1\end{pmatrix},\ g, h \in \mathbb{R}$.

Source of Question: SRJC JC2 Mid-Year CT 9758/2018/02/Q7

5 The line l_1 and the plane p_1 have equations

$$l_1 : \mathbf{r} = \begin{pmatrix}3\\-2\\5\end{pmatrix} + t\begin{pmatrix}-1\\-2\\3\end{pmatrix},\ t \in \mathbb{R} \quad \text{and} \quad p_1 : \mathbf{r} \bullet \begin{pmatrix}2\\1\\1\end{pmatrix} = 2.$$

It is given that the point *A* has position vector $3\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$.

(i) Find the acute angle between l_1 and p_1. [2]

(ii) Find the coordinates of the foot of perpendicular from the point *A* to p_1. [4]

The plane p_2 and the plane p_3 have equations $-2x + z = 6$ and $\alpha x + 3y + 2z = \beta$ respectively, where $\alpha, \beta \in \mathbb{R}$. Given that the plane p_1 meets the plane p_2 at the line l_2,

(iii) find the equation of l_2. [1]

(iv) Find the values of α and β if p_3 also contains l_2. [2]

(v) The plane p_4 is parallel to the plane p_2 and has equation $-2x + z = k$, where $k \in \mathbb{R}$. Find the possible values of *k* for which the plane p_4 is $\sqrt{5}$ units away from the plane p_2. [2]

Solution

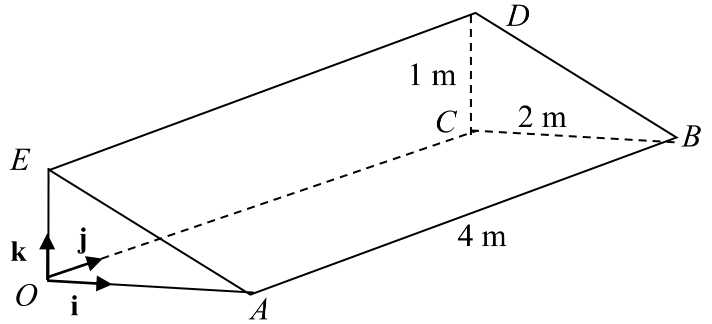
(i)	Let θ be the acute angle between the l_1 and p_1. $\sin\theta = \left\| \dfrac{\begin{pmatrix}-1\\-2\\3\end{pmatrix}\bullet\begin{pmatrix}2\\1\\1\end{pmatrix}}{\sqrt{(-1)^2+(-2)^2+3^2}\sqrt{2^2+1^2+1^2}} \right\| = \left\|\dfrac{-1}{\sqrt{84}}\right\| = \dfrac{1}{\sqrt{84}}$ $\theta = 6.3^\circ$ (1 d.p.)
(ii)	Let the foot of perpendicular from the point *A* to p_1 be *N*. $l_{AN}:\ \mathbf{r} = \begin{pmatrix}3\\-3\\5\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\1\end{pmatrix},\ \lambda\in\mathbb{R}$ Since *N* lies on the line l_{AN}, $\overrightarrow{ON} = \begin{pmatrix}3+2\lambda\\-3+\lambda\\5+\lambda\end{pmatrix}$, for some $\lambda\in\mathbb{R}$. Since *N* lies on the plane p_1, $\begin{pmatrix}3+2\lambda\\-3+\lambda\\5+\lambda\end{pmatrix}\bullet\begin{pmatrix}2\\1\\1\end{pmatrix} = 2$ $6+4\lambda-3+\lambda+5+\lambda = 2$ $\lambda = -1$ This gives $\overrightarrow{ON} = \begin{pmatrix}1\\-4\\4\end{pmatrix}$. Therefore, N is $(1,-4,4)$.
(iii)	Using GC, $x = -3+\frac{1}{2}\mu$ $y = 8-2\mu$ $z = \mu$ Hence, $l_2:\ \mathbf{r} = \begin{pmatrix}-3\\8\\0\end{pmatrix} + \mu\begin{pmatrix}\frac{1}{2}\\-2\\1\end{pmatrix},\ \mu\in\mathbb{R}$ or $l_2:\ \mathbf{r} = \begin{pmatrix}-3\\8\\0\end{pmatrix} + \mu\begin{pmatrix}1\\-4\\2\end{pmatrix},\ \mu\in\mathbb{R}$

(iv)	$p_2: \mathbf{r} \bullet \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 6, \quad p_3: \mathbf{r} \bullet \begin{pmatrix} \alpha \\ 3 \\ 2 \end{pmatrix} = \beta$ If p_1, p_2 and p_3 have a common line of intersection, The normal of p_3 is perpendicular to the direction of l_2. $\begin{pmatrix} \alpha \\ 3 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix} = 0$ $\alpha - 12 + 4 = 0$ $\alpha = 8$ Since any point of l_2 will be on plane p_3, $\begin{pmatrix} -3 \\ 8 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix} = \beta$ $\beta = 0$
(v)	$\left\lvert \dfrac{k-6}{\sqrt{(-2)^2+1^2}} \right\rvert = \sqrt{5}$ $\lvert k-6 \rvert = 5$ $k-6=5$ or $k-6=-5$ $k=11$ or $k=1$

Source of Question: SRJC Prelim 9758/2018/02/Q5

6 A hollow metallic ramp, in the shape of a prism, is constructed for the marching contingent to march onto to reach an elevated platform from the ground during the national day parade.

The diagram below shows the prism with O as the origin of position vectors and the unit vectors **i**, **j** and **k** are parallel to OA, OC and OE respectively.
It is given that $OE = CD = 1$ m, $OA = CB = 2$ m and $OC = AB = ED = 4$ m.



A laser beam in the form of a line l has Cartesian equation $\frac{a+x}{2}=z,\ y=1$, where $a\in\mathbb{R}$, is emitted onto the plane *ABDE.*

(i) Find, in terms of *a,* the coordinates of the point of intersection, *M*, of the laser beam and the plane *ABDE*. [4]

For the following parts of the question assume $a=0$.

(ii) The laser beam is reflected about the plane *ABDE*. By finding the foot of perpendicular from $Q(0,1,0)$ to the plane *ABDE*, find the equation of the reflected beam. [5]

(iii) The path traced out by an ant crawling on the floor *OABC* is given by

$$\mathbf{r}=\begin{pmatrix}2\\0\\0\end{pmatrix}+\beta\begin{pmatrix}-1\\2\\0\end{pmatrix},\beta\in\mathbb{R}.$$

Let *P* be the point on the path, located under the ramp, whereby the ant is equidistant between the planes *ABDE* and *OCDE*. Find the position vector of point *P* exactly. [4]

Solution

(i)	line l : $\mathbf{r}=\begin{pmatrix}-a\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\0\\1\end{pmatrix},a,\lambda\in\mathbb{R}$ normal vector of plane *ABDE*, $\underset{\sim}{n}=\begin{pmatrix}0\\1\\0\end{pmatrix}\times\left[\begin{pmatrix}2\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\1\end{pmatrix}\right]=\begin{pmatrix}-1\\0\\-2\end{pmatrix}=-\begin{pmatrix}1\\0\\2\end{pmatrix}$ plane *ABDE*: $\mathbf{r}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=2$ For point of intersection, $\begin{pmatrix}-a+2\lambda\\1\\\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=2$ $\lambda=\frac{2+a}{4}$ $\overrightarrow{OM}=\begin{pmatrix}-a+\frac{2+a}{2}\\1\\\frac{2+a}{4}\end{pmatrix}=\begin{pmatrix}\frac{2-a}{4}\\1\\\frac{2+a}{4}\end{pmatrix}$

	Coordinates are $\left(\frac{2-a}{2}, 1, \frac{2+a}{4}\right)$.
(ii)	As $a=0$, $\overrightarrow{OM}=\frac{1}{2}\begin{pmatrix}2\\2\\1\end{pmatrix}$. Let F be the foot of perpendicular from Q to the plane *ABDE*. Equation of line QF is $\mathbf{r}=\begin{pmatrix}0\\1\\0\end{pmatrix}+\mu\begin{pmatrix}1\\0\\2\end{pmatrix}, \mu\in\mathbb{R}$. $\overrightarrow{OF}=\begin{pmatrix}\mu\\1\\2\mu\end{pmatrix}$ for some $\mu\in\mathbb{R}$. Since F lies on the plane *ABDE*, $\begin{pmatrix}\mu\\1\\2\mu\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=2$ $\mu=\frac{2}{5}$ $\therefore \overrightarrow{OF}=\frac{1}{5}\begin{pmatrix}2\\5\\4\end{pmatrix}$ Let Q' be the point of reflection of point Q about plane *ABDE*. $\overrightarrow{OQ'}=2\overrightarrow{OF}-\overrightarrow{OQ}=\begin{pmatrix}4/5\\2\\8/5\end{pmatrix}-\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}4/5\\1\\8/5\end{pmatrix}$ $\overrightarrow{MQ'}=\overrightarrow{OQ'}-\overrightarrow{OM}=\begin{pmatrix}4/5\\1\\8/5\end{pmatrix}-\begin{pmatrix}1\\1\\1/2\end{pmatrix}=\begin{pmatrix}-1/5\\0\\11/10\end{pmatrix}=1/10\begin{pmatrix}-2\\0\\11\end{pmatrix}$ Equation of line of reflection of the laser beam: $\mathbf{r}=\begin{pmatrix}4/5\\1\\8/5\end{pmatrix}+\alpha\begin{pmatrix}-2\\0\\11\end{pmatrix}, \alpha\in\mathbb{R}$ **OR** $\mathbf{r}=\frac{1}{2}\begin{pmatrix}2\\2\\1\end{pmatrix}+\alpha\begin{pmatrix}-2\\0\\11\end{pmatrix}, \alpha\in\mathbb{R}$

(iii)	Equation of line AC: $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \beta \in \mathbb{R}$. $\overrightarrow{OP} = \begin{pmatrix} 2-\beta \\ 2\beta \\ 0 \end{pmatrix}$ for some β $\left\| \dfrac{\overrightarrow{OP} \cdot \mathbf{n}_{OCDE}}{\left\|\mathbf{n}_{OCDE}\right\|} \right\| = \left\| \dfrac{\overrightarrow{EP} \cdot \mathbf{n}_{ABDE}}{\left\|\mathbf{n}_{ABDE}\right\|} \right\|$ $\left\| \dfrac{\begin{pmatrix} 2-\beta \\ 2\beta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{\left\|\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\|} \right\| = \left\| \dfrac{\begin{pmatrix} 2-\beta \\ 2\beta \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}{\left\|\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right\|} \right\|$ OR $\left\| \dfrac{\begin{pmatrix} 2-\beta \\ 2\beta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 0}{\left\|\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\|} \right\| = \left\| \dfrac{\begin{pmatrix} 2-\beta \\ 2\beta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - 2}{\left\|\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right\|} \right\|$ $\left\|2-\beta\right\| = \left\| \dfrac{2-\beta-2}{\sqrt{5}} \right\|$ $2-\beta = \pm\dfrac{\beta}{\sqrt{5}}$ OR $5(2-\beta)^2 = \beta^2$ $2\sqrt{5}-\beta\sqrt{5} = \pm\beta$ $\left(2\sqrt{5}-\sqrt{5}\beta-\beta\right)\left(2\sqrt{5}-\sqrt{5}\beta+\beta\right) = 0$ $\therefore \beta = \dfrac{2\sqrt{5}}{\sqrt{5}+1}$ or $\beta = \dfrac{2\sqrt{5}}{\sqrt{5}-1}$ $\beta = \dfrac{2\sqrt{5}\left(\sqrt{5}-1\right)}{4}$ or $\beta = \dfrac{2\sqrt{5}\left(\sqrt{5}+1\right)}{4}$ $\beta = \dfrac{5\pm\sqrt{5}}{2}$ $\overrightarrow{OP} = \dfrac{1}{2}\begin{pmatrix} -1-\sqrt{5} \\ 10+2\sqrt{5} \\ 0 \end{pmatrix}$ (rejected since it gives a point outside of the ramp) Hence $\overrightarrow{OP} = \dfrac{1}{2}\begin{pmatrix} \sqrt{5}-1 \\ 10-2\sqrt{5} \\ 0 \end{pmatrix}$.

Source of Question: TJC JC2 Mid-Year CT 9758/2018/01/Q8

7 The plane Π has equation

$$\mathbf{r}=\begin{pmatrix}-8\\12\\4\end{pmatrix}+\alpha\begin{pmatrix}2\\1\\0\end{pmatrix}+\beta\begin{pmatrix}2\\0\\-1\end{pmatrix},\quad \alpha,\beta\in\mathbb{R}.$$

(i) The point A has coordinates (1, 2, 3). The point B, the reflection of A in Π, has coordinates (–5, c, − 9). By considering the mid-point of AB, or otherwise, find the value of c. [3]

The line l is parallel to Π and has equation

$$\mathbf{r}=\begin{pmatrix}4\\0\\-8\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix},\quad \lambda\in\mathbb{R}.$$

(ii) Find the distance between l and Π. [3]

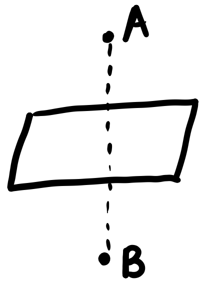
(iii) The variable point P moves along l and the variable point Q moves on Π. The point R lies on the line segment PQ such that $PR = 3RQ$.

Find $\overrightarrow{OR}$ in terms of α, β, and λ. [2]

Hence give a geometrical description of the locus of R. [2]

Solution

<table>
<tr>
<td>(i)</td>
<td>
Position vector of mid-point of $AB=\frac{1}{2}\left[\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}-5\\c\\-9\end{pmatrix}\right]=\begin{pmatrix}-2\\\frac{2+c}{2}\\-3\end{pmatrix}$

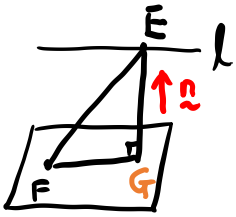
Since mid-point of AB lies on Π :
$\begin{pmatrix}-2\\\frac{2+c}{2}\\-3\end{pmatrix}=\begin{pmatrix}-8\\12\\4\end{pmatrix}+\alpha\begin{pmatrix}2\\1\\0\end{pmatrix}+\beta\begin{pmatrix}2\\0\\-1\end{pmatrix}$
$6 = 2\alpha+2\beta$ --- (1)
$-22 = -c+2\alpha$ --- (2)
$-7 = -\beta$ --- (3)


Solving, $c = 14$

<u>Alternative Solution:</u>
$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-6\\c-2\\-12\end{pmatrix}$
$\begin{pmatrix}-6\\c-2\\-12\end{pmatrix}\bullet\begin{pmatrix}2\\1\\0\end{pmatrix}=0\Rightarrow -12+c-2=0\Rightarrow c=14$
</td>
</tr>
</table>

(ii)

Normal of $\Pi = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$

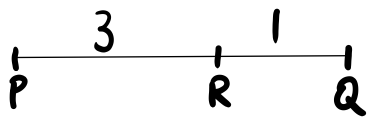


Let E be point (4, 0, –8) on l and F be point (–8, 12, 4) on Π.

Distance between l and $\Pi = \left| \overrightarrow{EF} \bullet \hat{\mathbf{n}} \right| = \left| \frac{1}{3} \begin{pmatrix} -12 \\ 12 \\ 12 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \right| = 4$

(iii)

$\overrightarrow{OP} = \begin{pmatrix} 4+2\lambda \\ \lambda \\ -8 \end{pmatrix}, \quad \overrightarrow{OQ} = \begin{pmatrix} -8+2\alpha+2\beta \\ 12+\alpha \\ 4-\beta \end{pmatrix}$



By Ratio Theorem,

$$\overrightarrow{OR} = \frac{3\overrightarrow{OQ} + \overrightarrow{OP}}{4} = \frac{3}{4} \begin{pmatrix} -8+2\alpha+2\beta \\ 12+\alpha \\ 4-\beta \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 4+2\lambda \\ \lambda \\ -8 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 9 \\ 1 \end{pmatrix} + \frac{3}{4}\alpha \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \frac{3}{4}\beta \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{4}\lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 9 \\ 1 \end{pmatrix} + \left(\frac{3}{4}\alpha + \frac{1}{4}\lambda \right) \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \frac{3}{4}\beta \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

The locus of R is the plane that passes through (–5, 9, 1) and is parallel to $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.

Source of Question: TJC JC2 Mid-Year CT 9758/2018/02/Q5

8 An architectural student created a cuboid model *OABCDEFG* with dimensions 5 cm by 3 cm by 4 cm as shown in Figure 1. Due to poor construction, the cuboid leaned towards the positive *x*-direction to form a parallelepiped *OABCD'E'F'G'* as shown in Figure 2 such that

- the acute angle between the plane *OCG'D'* and the *y-z* plane is 30°,
- the planes *OABC* and *D'E'F'G'* remain parallel to the *x-y* plane, and
- the planes *OAE'D'* and *CBF'G'* remain parallel to the *x-z* plane.

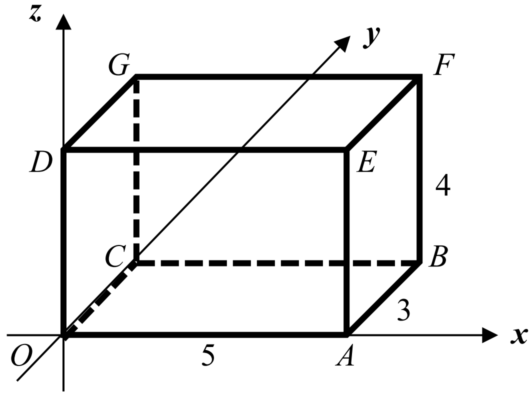


Figure 1

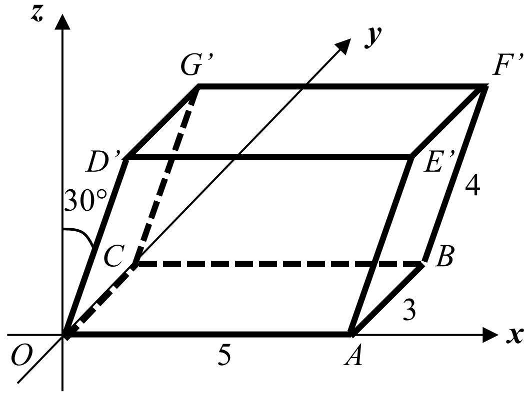


Figure 2

(i) Show that $\overrightarrow{OD'} = 2\mathbf{i} + 2\sqrt{3}\mathbf{k}$. Hence find a vector equation of the plane *ABF'E'* in parametric form. [3]

(ii) Find the acute angle between the plane *ABF'E'* and the plane with equation $\mathbf{r} \bullet \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 3$. [3]

(iii) Find a vector equation of the line of intersection between the plane *ABF'E'* and the *y-z* plane. [3]

(iv) Find the position vector of the point *P* on *OA* which has a shortest distance of $\sqrt{\frac{6}{5}}$ cm from *OF'*. [4]

Solution

(i)	$\overrightarrow{OD'} = \begin{pmatrix} 4\sin 30° \\ 0 \\ 4\cos 30° \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2\sqrt{3} \end{pmatrix}$ (shown) $\overrightarrow{AE'} = \overrightarrow{OD'} = \begin{pmatrix} 2 \\ 0 \\ 2\sqrt{3} \end{pmatrix}$

<table>
<tr><td></td><td>

Vector equation of plane *ABF'E'* is

$$\mathbf{r} = \overrightarrow{OA} + \lambda\,\overrightarrow{AE'} + \mu\,\overrightarrow{AB}$$

$$\mathbf{r} = \begin{pmatrix}5\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\0\\2\sqrt{3}\end{pmatrix} + \mu\begin{pmatrix}0\\3\\0\end{pmatrix}, \quad \lambda,\mu\in\mathbb{R}$$

</td></tr>
<tr><td>

(ii)

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$$\begin{pmatrix}2\\0\\2\sqrt{3}\end{pmatrix}\times\begin{pmatrix}0\\3\\0\end{pmatrix} = \begin{pmatrix}-6\sqrt{3}\\0\\6\end{pmatrix} = 6\begin{pmatrix}-\sqrt{3}\\0\\1\end{pmatrix}$$

Normal to plane *ABF'E'* $= \begin{pmatrix}-\sqrt{3}\\0\\1\end{pmatrix}$

Acute angle $= \cos^{-1}\dfrac{\left|\begin{pmatrix}-\sqrt{3}\\0\\1\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\1\end{pmatrix}\right|}{\sqrt{4}\sqrt{6}} = 56.1^\circ$

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<tr><td>

(iii)

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Cartesian equation of plane *ABF'E'*: $-\sqrt{3}x + z = -5\sqrt{3}$ ----- (1)
Cartesian equation of *y-z* plane: $x = 0$ ----- (2)
Sub (2) into (1): $z = -5\sqrt{3}$

Vector equation of line of intersection: $\mathbf{r} = \begin{pmatrix}0\\0\\-5\sqrt{3}\end{pmatrix} + \alpha\begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \alpha\in\mathbb{R}$

</td></tr>
<tr><td>

(iv)

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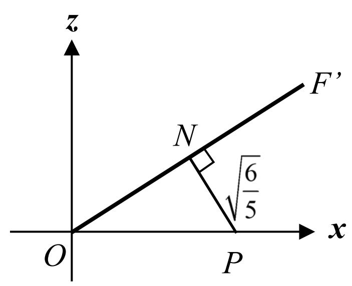
Since *P* lies on *OA*, $\overrightarrow{OP} = k\mathbf{i} = \begin{pmatrix}k\\0\\0\end{pmatrix}, \quad k\in\mathbb{R}$

Shortest distance from *P* to *OF'*

$$= \left|\frac{\overrightarrow{OP}\times\overrightarrow{OF'}}{\left|\overrightarrow{OF'}\right|}\right| = \frac{\left|\begin{pmatrix}k\\0\\0\end{pmatrix}\times\begin{pmatrix}7\\3\\2\sqrt{3}\end{pmatrix}\right|}{\sqrt{49+9+12}}$$

$$= \frac{1}{\sqrt{70}}\left|\begin{pmatrix}0\\-2k\sqrt{3}\\3k\end{pmatrix}\right| = \frac{1}{\sqrt{70}}\sqrt{21k^2} = \sqrt{\frac{3}{10}}k$$



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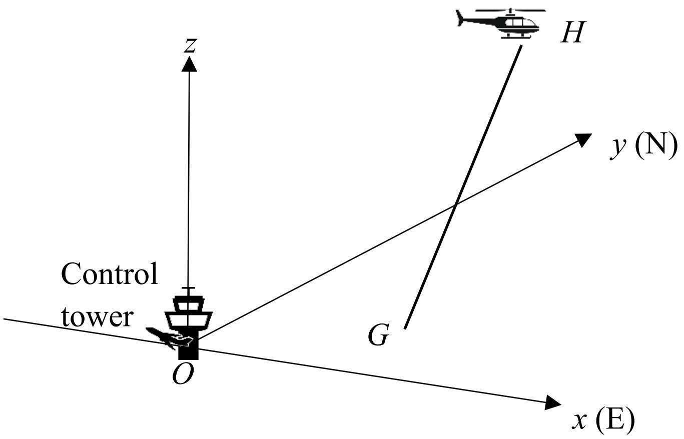
</table>

	$\therefore \sqrt{\frac{3}{10}}k = \sqrt{\frac{6}{5}} \quad \Rightarrow \quad k = 2$ $\overrightarrow{OP} = 2\mathbf{i} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

Source of Question: VJC Prelim 9758/2018/01/Q11
9



The diagram illustrates the initial flight path of a helicopter H taking off from an airport. The origin O is taken to be at the base of the control tower. The x-axis is due east, the y-axis due north, and the z-axis is vertical. The units of distances are measured in kilometres.

The helicopter takes off from the point G on the ground. The position vector $\mathbf{r}$ of the helicopter t minutes after take-off is given by

$$\mathbf{r} = (1 + t)\mathbf{i} + (0.5 + 2t)\mathbf{j} + 2t\mathbf{k}.$$

(i) Write down the coordinates of G and describe the initial flight path. [3]

(ii) Find the acute angle that the helicopter's flight path makes with the horizontal. [3]

(iii) A mountain top is situated at the point M (5, 4.5, 3). Determine how long after take off the helicopter will be nearest to M. [2]

(iv) An eagle sets off from the mountain top to hunt for food. The position of the eagle satisfies the equation

$$\frac{x-2}{3} = \frac{z-2}{1}, \; y = 4.5\,.$$

Determine if the flight path of the helicopter will intersect the path traced out by the eagle, showing your reasoning clearly. [3]

(v) The helicopter enters a cloud at a height of 2 km. Given that the visibility on that day is 3.75 km, determine if the air traffic controller who is situated at 70 m above ground level, in the control tower, will be able to sight the helicopter as it enters the cloud. [2]

Solution

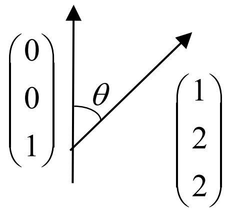
<table>
<tr><td>(i)</td><td>
When $t = 0$, $\overrightarrow{OG} = \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$

The coordinates of G are (1, 0.5, 0).

The flight path is a straight line starting from the point (1, 0.5, 0) moving in the direction of $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.
</td></tr>
<tr><td>(ii)</td><td>
Let θ be the angle the flight path makes with the vertical.

$\cos\theta = \dfrac{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{3} = \dfrac{2}{3}$

$\theta = 48.190°$ $\left(\text{or } 0.84107\right)$

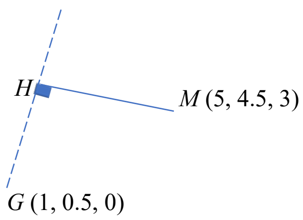


Required angle = $90° - 48.190° = 41.8°$

(or $\dfrac{\pi}{2} - 0.84107 = 0.730$)

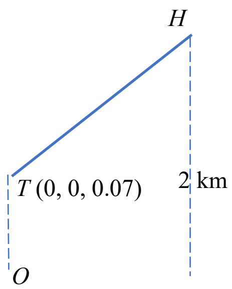
The acute angle the flight path makes with the horizontal is $41.8°$.
</td></tr>
<tr><td>(iii)</td><td>
$\overrightarrow{OH} = \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$\overrightarrow{MH} = \begin{pmatrix} -4 \\ -4 \\ -3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$



$\left(\begin{pmatrix} -4 \\ -4 \\ -3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\right) \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 0$

$-18 + 9t = 0 \Rightarrow t = 2$
</td></tr>
</table>

(iv)	Equation for flight path of eagle is $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 4.5 \\ 2 \end{pmatrix} + s\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, s \in \mathbb{R}$ Suppose the flight paths intersect, $\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4.5 \\ 2 \end{pmatrix} + s\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ $1+t = 2+3s ---(1)$ $0.5+2t = 4.5----(2)$ $2t = 2+s---(3)$ From (2) and (3), $t = 2$ and $s = 2$. Substitute into (1): LHS = -4 and RHS = 1 The flight paths will not intersect.
(v)	When $z = 2$, $t = 1$. $\overrightarrow{OH} = \begin{pmatrix} 2 \\ 2.5 \\ 2 \end{pmatrix}$  Let top of control tower be T. $\overrightarrow{TH} = \begin{pmatrix} 2 \\ 2.5 \\ 1.93 \end{pmatrix}$ $TH = 3.7383$ The controller is able to sight the helicopter.