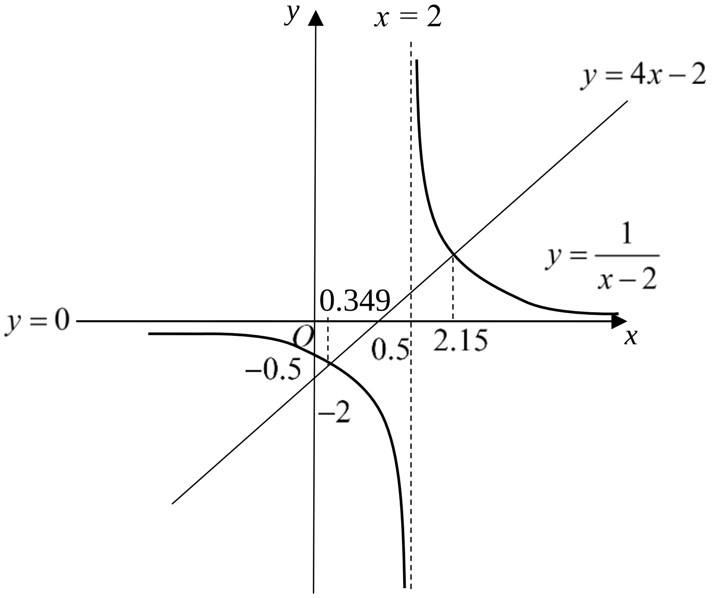
Jurong Pioneer Junior College
H2 Mathematics **JC1-2022**
Year-End Exam Solution

Q1

(i)



(ii)

$0.349 \le x < 2$ or $x \ge 2.15$

Q2

(a)
(i)

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(9+3\mathrm{e}^{3x}\right)=\frac{9\mathrm{e}^{3x}}{9+3\mathrm{e}^{3x}}=\frac{3\mathrm{e}^{3x}}{3+\mathrm{e}^{3x}}$$

(ii)

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^4 x^2=4\sin^3 x^2\cos x^2\left(2x\right)=8x\sin^3 x^2\cos x^2$$

(iii)

$$\frac{\mathrm{d}}{\mathrm{d}x}\sec 3x\sin^{-1}2x$$

$$=\sec 3x\frac{1}{\sqrt{1-(2x)^2}}2+\sin^{-1}2x\left(3\sec 3x\tan 3x\right)$$

$$=\sec 3x\left(\frac{2}{\sqrt{1-4x^2}}+3\tan 3x\sin^{-1}2x\right)$$

(b)

$$y^3-4y+x^2-9x+10=0$$

$$3y^2\frac{\mathrm{d}y}{\mathrm{d}x}-4\frac{\mathrm{d}y}{\mathrm{d}x}+2x-9=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{9-2x}{3y^2-4}$$

Q3

$u=1+t^3 \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}t}=3t^2$ When $t=0$, $u=1$. When $t=2$, $u=9$

$$\int_0^2\frac{t^5}{\left(1+t^3\right)^3}\,\mathrm{d}t=\int_0^2\frac{t^3\left(t^2\right)}{\left(1+t^3\right)^3}\,\mathrm{d}t$$

$$=\frac{1}{3}\int_1^9\frac{u-1}{u^3}\,\mathrm{d}u$$

$$=\frac{1}{3}\int_1^9\frac{u}{u^3}-\frac{1}{u^3}\,\mathrm{d}u$$

$$=\frac{1}{3}\int_1^9 u^{-2}-u^{-3}\,\mathrm{d}u$$

$$=\frac{1}{3}\left[\frac{u^{-1}}{-1}-\frac{u^{-2}}{-2}\right]_1^9$$

$$=\frac{1}{3}\left[-\frac{1}{u}+\frac{1}{2u^2}\right]_1^9=\frac{1}{3}\left[\left(-\frac{1}{9}+\frac{1}{2(81)}\right)-\left(-1+\frac{1}{2}\right)\right]=\frac{32}{243}$$

Q4

(i)

$y=\dfrac{x^2+kx+4}{x+1}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(2x+k)(x+1)-(x^2+kx+4)}{(x+1)^2}$

$=\dfrac{x^2+2x+(k-4)}{(x+1)^2}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=0 \quad \Rightarrow \quad x^2+2x+(k-4)=0$

For two stationary points: $2^2-4(k-4)>0$

$k<5$

<u>Alternative</u>:

$y=\dfrac{x^2+kx+4}{x+1}=x+(k-1)+\dfrac{5-k}{x+1}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=1-\dfrac{(5-k)}{(x+1)^2}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=0 \quad \Rightarrow \quad (x+1)^2=5-k$

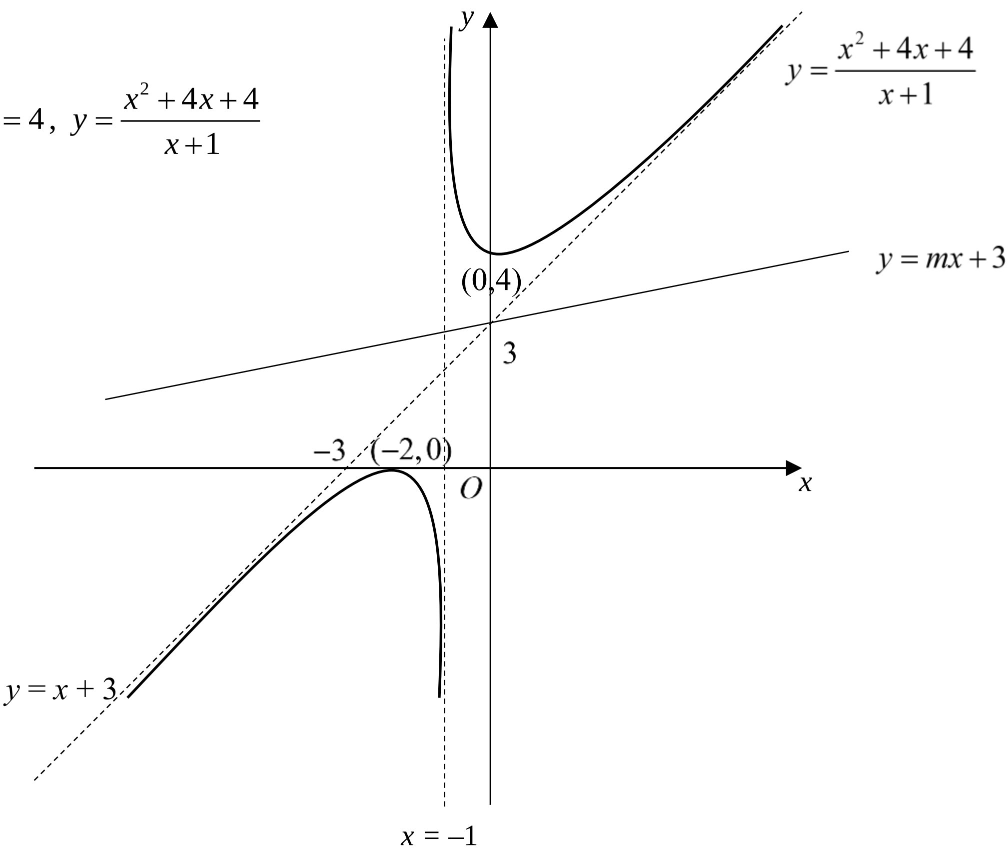
$x+1=\pm\sqrt{5-k}$

For two stationary points: $5-k>0$

$k<5$

(ii)

When $k=4$, $y=\dfrac{x^2+4x+4}{x+1}$



(iii)

$x^2+kx+4-(mx+3)(x+1)=0$

$x^2 + kx + 4 = (mx+3)(x+1)$

$\dfrac{x^2+kx+4}{x+1} = mx+3$, where $k = 4$.

For graphs of $y = \dfrac{x^2+kx+4}{x+1}$ and $y = mx+3$ to have no intersection and hence $x^2+kx+4-(mx+3)(x+1)=0$ to have no real root, $0 < m \le 1$.

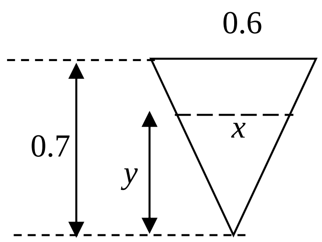
Q5

(i)

Using similar triangles,

$\dfrac{x}{y} = \dfrac{0.6}{0.7}$

$\therefore x = \dfrac{6}{7}y$



Let the volume of water in the tank be V.

$V = \dfrac{1}{2}xy(4) = 2y\left(\dfrac{6}{7}y\right) = \dfrac{12}{7}y^2$

$\dfrac{\mathrm{d}V}{\mathrm{d}y} = \dfrac{24}{7}y$

Since $\dfrac{\mathrm{d}V}{\mathrm{d}t} = \left(\dfrac{\mathrm{d}V}{\mathrm{d}y}\right)\left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)$,

$\therefore 0.0025 = \dfrac{24}{7}(0.4)\left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)$

$\dfrac{\mathrm{d}y}{\mathrm{d}t} = 0.0018$ m/s (4 d.p)

(ii)

Time taken = $\left[\dfrac{1}{2}(0.6)(0.7)(4) - \dfrac{12}{7}(0.4)^2\right] \div 0.0025 \approx 226\text{s}$ (nearest second)

Or

Time taken = $\left[\dfrac{12}{7}(0.7)^2 - \dfrac{12}{7}(0.4)^2\right] \div 0.0025 \approx 226\text{s}$ (nearest second)

Q6

(i)

Using GC, from graphs, the intersection points of C and l are $(2,4)$ and $(5,1)$.

Alternative

$-x+6=x^2-8x+16$

$x^2-7x+10=0$ (or use GC polynomial rootfinder)

$(x-5)(x-2)=0$

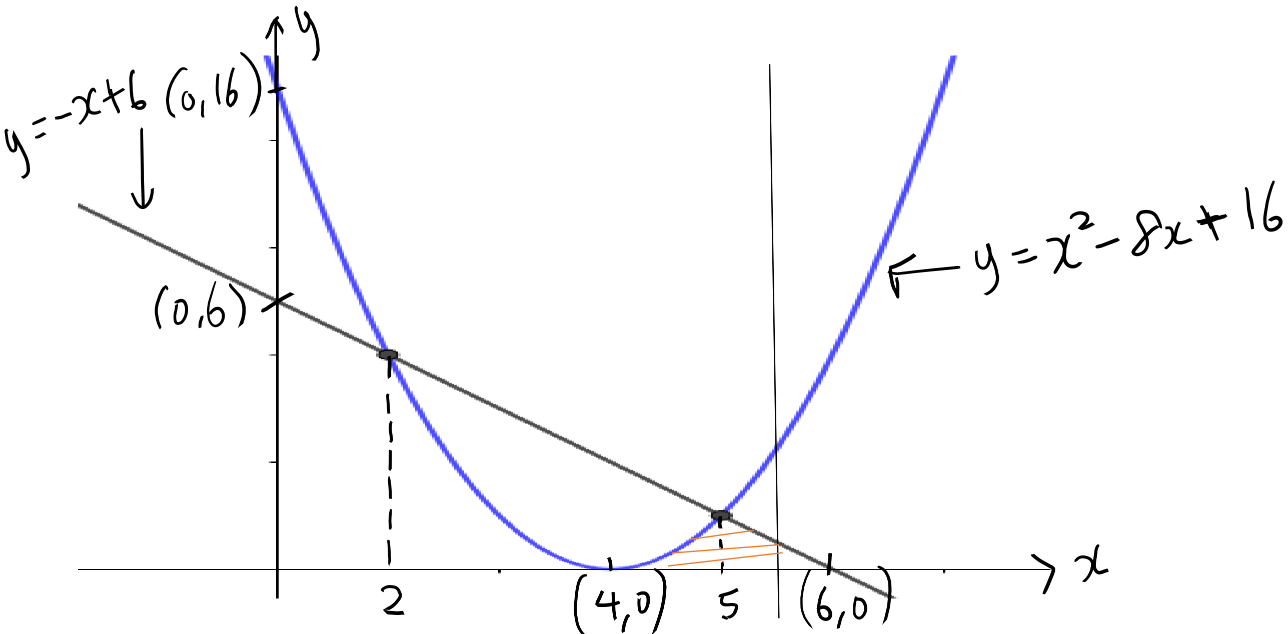
$\therefore\ x=5$ or $x=2$

When $x=2$, $y=-2+6=4$

When $x=5$, $y=-5+6=1$

The intersection points of C and l are $(2,4)$ and $(5,1)$.

(ii)



Area of the region R

$$=\int_5^a -x+6\,\mathrm{d}x+\int_4^5 (x-4)^2\,\mathrm{d}x \quad \text{or} \quad =\int_5^a -x+6\,\mathrm{d}x+\int_4^5 x^2-8x+16\,\mathrm{d}x$$

$$=\left[-\frac{x^2}{2}+6x\right]_5^a+\left[\frac{(x-4)^3}{3}\right]_4^5 \quad \text{or} \quad =\left[-\frac{x^2}{2}+6x\right]_5^a+\left[\frac{x^3}{3}-\frac{8x^2}{2}+16x\right]_4^5$$

$$=-\frac{a^2}{2}+6a+\frac{5^2}{2}-6(5)+\frac{(5-4)^3}{3}$$

$$=-\frac{a^2}{2}+6a-\frac{103}{6}\text{ units}^2$$

for $\int_5^a -x+6\,\mathrm{d}x$, can also use area of trapezium

$\frac{1}{2}(a-5)(1+6-a)$

Q7

(a)

$$\int \frac{10e^x}{5-2e^x}\,dx = \frac{10}{-2}\int \frac{-2e^x}{5-2e^x}\,dx = -5\int \frac{-2e^x}{5-2e^x}\,dx = -5\ln\left|5-2e^x\right| + c$$

(b)

$$\int \frac{x}{\sqrt{1+8x^2}}\,dx = \frac{1}{16}\int 16x\left(1+8x^2\right)^{-\frac{1}{2}}\,dx = \frac{1}{16}\frac{\left(1+8x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{1}{8}\left(1+8x^2\right)^{\frac{1}{2}} + c$$

(c)

$$\int x(\ln x)^2\,dx$$

$$= (\ln x)^2\left(\frac{x^2}{2}\right) - \int 2(\ln x)\left(\frac{1}{x}\right)\left(\frac{x^2}{2}\right)\,dx$$

$$= (\ln x)^2\left(\frac{x^2}{2}\right) - \int x(\ln x)\,dx$$

$$= (\ln x)^2\left(\frac{x^2}{2}\right) - \left[(\ln x)\left(\frac{x^2}{2}\right) - \int\left(\frac{1}{x}\right)\left(\frac{x^2}{2}\right)\,dx\right]$$

$$= (\ln x)^2\left(\frac{x^2}{2}\right) - (\ln x)\left(\frac{x^2}{2}\right) + \frac{1}{2}\int x\,dx$$

$$= (\ln x)^2\left(\frac{x^2}{2}\right) - (\ln x)\left(\frac{x^2}{2}\right) + \frac{x^2}{4} + c$$

Let $u = (\ln x)^2$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = 2(\ln x)\left(\frac{1}{x}\right)$ $\quad v = \frac{x^2}{2}$

Let $u = \ln x$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x}$ $\quad v = \frac{x^2}{2}$

Q8

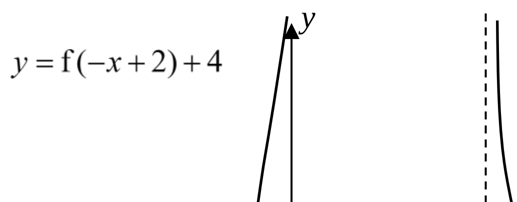
(a)

$$y = \frac{12x+11}{2x+1} = 6 + \frac{5}{2x+1}$$

$$y = \frac{1}{2x-3} \;\rightarrow\; y = \frac{1}{2(x+2)-3} = \frac{1}{2x+1} \;\rightarrow\; y = \frac{5}{2x+1} \;\rightarrow\; y = \frac{5}{2x+1} + 6$$

Translate by 2 units in the negative x-direction. (This can be at any step.)
Scale parallel to the y-axis by scale factor of 5.
Translate by 6 units in the positive y-direction.

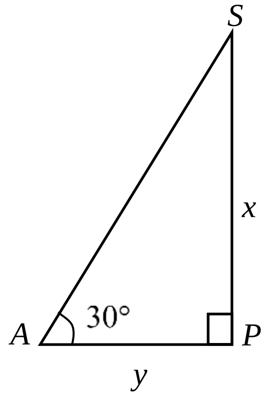
(b)



Step 1 : Translate by 2 units in the negative x-direction.
Step 2 : Reflect about y-axis.
Step 3 : Translate by 4 units in the positive y-direction.

	Step 1		Step 2		Step 3	
$(0,-5)$	$\rightarrow$	$(-2,-5)$	$\rightarrow$	$(2,-5)$	$\rightarrow$	$(2,-1)$
$(6,-4)$	$\rightarrow$	$(4,-4)$	$\rightarrow$	$(-4,-4)$	$\rightarrow$	$(-4,0)$
$x=-2$	$\rightarrow$	$x=-4$	$\rightarrow$	$x=4$	$\rightarrow$	$x=4$
$x=2$	$\rightarrow$	$x=0$	$\rightarrow$	$x=0$	$\rightarrow$	$x=0$
$y=-2$	$\rightarrow$	$y=-2$	$\rightarrow$	$y=-2$	$\rightarrow$	$y=2$

Q9



Let $AP = y$.

$$\frac{x}{y} = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$y = \sqrt{3}x$$

Since $AP = BQ = y$, $PQ = 20 - 2\sqrt{3}x$ cm.

Let A denote the area of rectangle $PQRS$.

$$A = x\left(20 - 2\sqrt{3}x\right) = 20x - 2\sqrt{3}x^2$$

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 20 - 4\sqrt{3}x\,.$$

To find maximum of minimum values, we let $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$.

$20 - 4\sqrt{3}x = 0$

Hence, we get $x = \frac{5}{\sqrt{3}} = \frac{5}{3}\sqrt{3}$

$\frac{\mathrm{d}^2 A}{\mathrm{d}x^2} = -4\sqrt{3} < 0$.

Area of $PQRS$ is maximum when $x = \frac{5}{3}\sqrt{3}$.

Hence, $PS = \frac{5}{3}\sqrt{3}$ cm, $PQ = 20 - 2\sqrt{3}\left(\frac{5}{3}\sqrt{3}\right) = 10$ cm.

Area of rectangle $PQRS$ = $\frac{50}{3}\sqrt{3}$ cm^2.

Q10

(i)

$\overrightarrow{OC} = \frac{2}{3}\mathbf{a}$, $\overrightarrow{OD} = \frac{4}{3}\mathbf{b}$

(ii)

$\overrightarrow{BC} = \frac{2}{3}\mathbf{a} - \mathbf{b}$, $\overrightarrow{AD} = \frac{4}{3}\mathbf{b} - \mathbf{a}$

E is on Line BC: $\overrightarrow{OE} = \mathbf{b} + \lambda\left(\frac{2}{3}\mathbf{a} - \mathbf{b}\right)$, for a value of λ

E is on Line AD: $\overrightarrow{OE} = \mathbf{a} + \mu\left(\frac{4}{3}\mathbf{b} - \mathbf{a}\right)$, for a value of μ

Consider $\overrightarrow{OE} = \mathbf{b} + \lambda\left(\frac{2}{3}\mathbf{a} - \mathbf{b}\right) = \mathbf{a} + \mu\left(\frac{4}{3}\mathbf{b} - \mathbf{a}\right)$

$$\frac{2\lambda}{3}\mathbf{a} + (1-\lambda)\mathbf{b} = (1-\mu)\mathbf{a} + \frac{4\mu}{3}\mathbf{b}$$

By comparing the coefficients:

$\frac{2}{3}\lambda = 1 - \mu$ ------------Equation (1)

$1 - \lambda = \frac{4}{3}\mu$ ------------Equation (2)

Solving: $\lambda = -3$, $\mu = 3$

Thus $\overrightarrow{OE} = \mathbf{b} + (-3)\left(\frac{2}{3}\mathbf{a} - \mathbf{b}\right) = \mathbf{a} + 3\left(\frac{4}{3}\mathbf{b} - \mathbf{a}\right) = 4\mathbf{b} - 2\mathbf{a}$ (Shown)

(ii)

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \frac{4}{3}\mathbf{b} - \frac{2}{3}\mathbf{a}$, $\overrightarrow{CE} = \overrightarrow{OE} - \overrightarrow{OC} = 4\mathbf{b} - \frac{8}{3}\mathbf{a}$

Area of triangle CDE

$= \frac{1}{2}\left|\overrightarrow{CD} \times \overrightarrow{CE}\right|$

$= \frac{1}{2}\left|\left(\frac{4}{3}\mathbf{b} - \frac{2}{3}\mathbf{a}\right) \times \left(4\mathbf{b} - \frac{8}{3}\mathbf{a}\right)\right|$

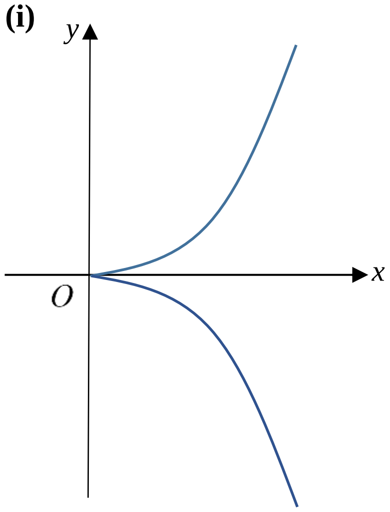
$= \frac{1}{2}\left|\left(\frac{4}{3}\mathbf{b} \times 4\mathbf{b}\right) - \left(\frac{4}{3}\mathbf{b} \times \frac{8}{3}\mathbf{a}\right) - \left(\frac{2}{3}\mathbf{a} \times 4\mathbf{b}\right) + \left(\frac{2}{3}\mathbf{a} \times \frac{8}{3}\mathbf{a}\right)\right|$

$= \frac{1}{2}\left|-\left(\frac{4}{3}\mathbf{b} \times \frac{8}{3}\mathbf{a}\right) - \left(\frac{2}{3}\mathbf{a} \times 4\mathbf{b}\right)\right|$ since $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ and $\mathbf{b} \times \mathbf{b} = \mathbf{0}$

$= \frac{1}{2}\left|\frac{32}{9}(\mathbf{a} \times \mathbf{b}) - \frac{8}{3}(\mathbf{a} \times \mathbf{b})\right| = \frac{1}{2}\left|\frac{8}{9}(\mathbf{a} \times \mathbf{b})\right| = \frac{4}{9}\left|\mathbf{a} \times \mathbf{b}\right|$ where $k = \frac{4}{9}$

Q11

(i)



(ii)

$x = 3t^2$ $\qquad y = 6t^3$

$\frac{dx}{dt} = 6t$ $\qquad \frac{dy}{dt} = 18t^2$

$\frac{dy}{dx} = 3t$

Since tangent is parallel to the line $y = 4 - 2x$,

$\frac{dy}{dx} = 3t = -2$

$t = -\frac{2}{3}$

Hence, $x = 3\left(-\frac{2}{3}\right)^2 = \frac{4}{3}$, $\qquad y = 6\left(-\frac{2}{3}\right)^3 = -\frac{16}{9}$ $\qquad P\left(\frac{4}{3}, -\frac{16}{9}\right)$

(iii)

At $Q\left(\frac{4}{3},\frac{16}{9}\right)$,

$$y=6t^3=\frac{16}{9}\Rightarrow t=\frac{2}{3}$$

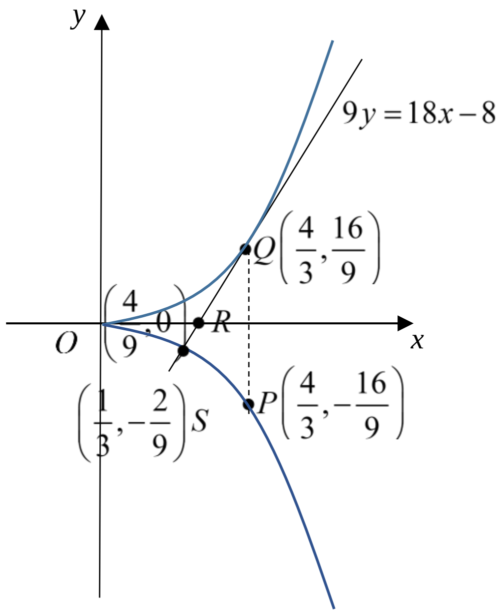
$$\frac{\mathrm{d}y}{\mathrm{d}x}=3t=3\left(\frac{2}{3}\right)=2$$

Equation of tangent at Q :

$$y-\frac{16}{9}=2\left(x-\frac{4}{3}\right)$$

$$y=2x-\frac{8}{9}$$

$9y=18x-8$ (shown)



(iv)

At R : $y=0$

$$0=18x-8\Rightarrow x=\frac{4}{9}$$

$$R\left(\frac{4}{9},0\right)$$

Area of triangle PQR

$$=\frac{1}{2}\left(\frac{4}{3}-\frac{4}{9}\right)\left[\frac{16}{9}-(-\frac{16}{9})\right]$$

$$=\frac{128}{81}\text{ units}^2$$

(v)

$$9y=18x-8$$

$$9\left(6t^3\right)=18\left(3t^2\right)-8$$

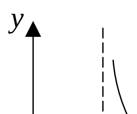
$$27t^3-27t^2+4=0$$

$t=\frac{2}{3}$ (N.A as Point Q) or $t=-\frac{1}{3}$

$$x=3\left(-\frac{1}{3}\right)^2=\frac{1}{3},\qquad y=6\left(-\frac{1}{3}\right)^3=-\frac{2}{9}$$

$$S\left(\frac{1}{3},-\frac{2}{9}\right)$$

Q12



(i)

Any horizontal line $y = k$ cuts the graph of $y = \mathrm{f}(x)$ at most **once**. Hence **f is one-one** and thus f^{-1} exists. (Shown)

(ii)

Let $y = \dfrac{2x}{x^2 - 1}$

$x^2 y - 2x - y = 0$

$x = \dfrac{2 \pm \sqrt{(-2)^2 - 4y(-y)}}{2y}$

$\therefore\ x = \dfrac{1 \pm \sqrt{1+y^2}}{y}$

Since $x > 1$ and $y > 0$, $x = \dfrac{1+\sqrt{1+y^2}}{y}$.

Therefore, $\mathrm{f}^{-1}(x) = \dfrac{1+\sqrt{1+x^2}}{x}$.

$\mathrm{D}_{\mathrm{f}^{-1}} = (0, \infty)$

Alternative

$x^2 y - 2x - y = 0$

$x^2 - \dfrac{2}{y}x - 1 = 0$

$\left(x - \dfrac{1}{y}\right)^2 - \dfrac{1}{y^2} - 1 = 0$

$x = \dfrac{1}{y} \pm \sqrt{\dfrac{1}{y^2} + 1}$

(iii)

$y = x$

Consider $\mathrm{f}(x) = x$

$\dfrac{2x}{x^2 - 1} = x$

$x^3 - 3x = 0$

$x(x^2 - 3) = 0$

$\therefore x = \sqrt{3},\quad 0$ (reject since $x > 1$) or $-\sqrt{3}$(reject since $x > 1$)

Q13

(i)

For p_1: Consider $\begin{pmatrix}3\\-1\\7\end{pmatrix}\times\begin{pmatrix}1\\-3\\12\end{pmatrix}=\begin{pmatrix}9\\-29\\-8\end{pmatrix}$. Then $\mathbf{r}.\begin{pmatrix}9\\-29\\-8\end{pmatrix}=\begin{pmatrix}4\\-1\\3\end{pmatrix}.\begin{pmatrix}9\\-29\\-8\end{pmatrix}=41$.

For p_2: $\mathbf{r}.\begin{pmatrix}1\\3\\0\end{pmatrix}=1$

For p_1: $9x-29y-8z=41$ -----------Equation 1

For p_2: $x+3y=1$ -----------------Equation 2

Using GC, let $z=\mu$

Then $\mathbf{r}$ = $\begin{pmatrix}19/7\\-4/7\\0\end{pmatrix}+\mu\begin{pmatrix}3/7\\-1/7\\1\end{pmatrix}=\frac{1}{7}\begin{pmatrix}19\\-4\\0\end{pmatrix}+\lambda\begin{pmatrix}3\\-1\\7\end{pmatrix}$, where $\lambda\in\mathbb{R}$.

(ii)

For p_1: $\mathbf{r}.\begin{pmatrix}9\\-29\\-8\end{pmatrix}=41$. For p_2: $\mathbf{r}.\begin{pmatrix}1\\3\\0\end{pmatrix}=1$

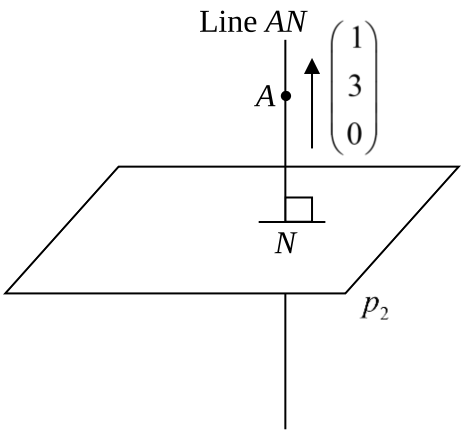
Acute angle between p_1 and p_2:

$$\cos\theta=\frac{\left|\begin{pmatrix}1\\3\\0\end{pmatrix}\bullet\begin{pmatrix}9\\-29\\-8\end{pmatrix}\right|}{\sqrt{10}\sqrt{986}}=\frac{|-78|}{\sqrt{9860}}=\frac{78}{\sqrt{9860}}$$

$\therefore\ \theta=38.2^{\circ}$ (nearest to 0.1°)

(iii)

Let the foot of perpendicular from A to p_2 be N.



Line AN: $\mathbf{r}$ = $\begin{pmatrix}5\\-4\\15\end{pmatrix}+\alpha\begin{pmatrix}1\\3\\0\end{pmatrix}$ where $\alpha\in\mathbb{R}$

$\overrightarrow{ON}=\begin{pmatrix}5+\alpha\\-4+3\alpha\\15\end{pmatrix}$ for a value of α

Since the point N lies on p_2,

$$\begin{pmatrix} 5+\alpha \\ -4+3\alpha \\ 15 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = 1 \Rightarrow 5+\alpha-12+9\alpha = 1$$. Solving, $\alpha = 0.8$

$$\therefore \overrightarrow{ON} = \begin{pmatrix} 5.8 \\ -1.6 \\ 15 \end{pmatrix}.$$

The coordinates of the foot of perpendicular from A to $p_2 = (5.8, -1.6, 15)$.

(iv)

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -4 \\ 4 \\ -17 \end{pmatrix}$$

Length of projection of AB onto p_2

$$= \frac{1}{\sqrt{10}} \left| \overrightarrow{AB} \times \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right|$$

$$= \frac{1}{\sqrt{10}} \left| \begin{pmatrix} -4 \\ 4 \\ -17 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right| = \frac{1}{\sqrt{10}} \left| \begin{pmatrix} 51 \\ -17 \\ -16 \end{pmatrix} \right|$$

= 17.7 units (to 3 s.f.)