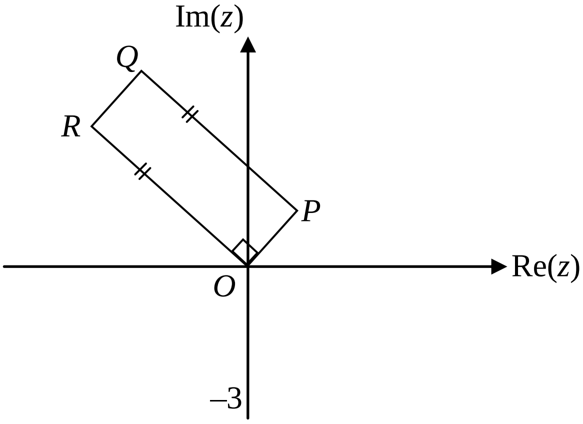
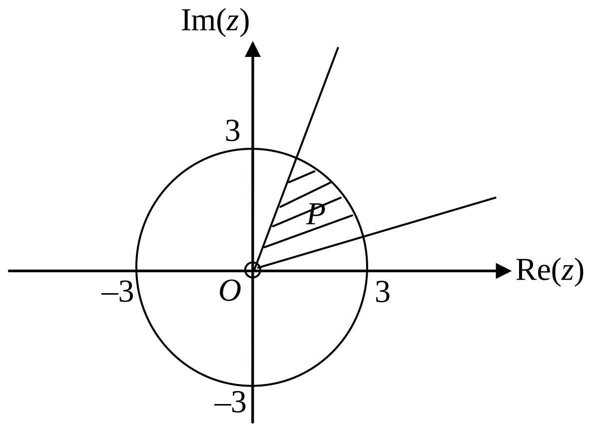
Qn	Solution
1	(i) Let θ be the acute angle between the 2 planes. $\cos\theta = \left\| \dfrac{\begin{pmatrix}1\\2\\1\end{pmatrix}\bullet\begin{pmatrix}2\\1\\-1\end{pmatrix}}{\sqrt{6}\sqrt{6}} \right\| = \dfrac{3}{6}$ $\theta = \dfrac{\pi}{3}$ (ii) $\overrightarrow{OB} = \begin{pmatrix}4\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\1\end{pmatrix}$ for some $\lambda \in \mathbf{R}$ $= \begin{pmatrix}4+\lambda\\-1+2\lambda\\2+\lambda\end{pmatrix}$ Since B is on π_1, $(4+\lambda)+2(-1+2\lambda)+(2+\lambda)=2$ $6\lambda=-2$ $\lambda=-\dfrac{1}{3}$ $\overrightarrow{OB} = \begin{pmatrix}\frac{11}{3}\\-\frac{5}{3}\\\frac{5}{3}\end{pmatrix} = \dfrac{1}{3}\begin{pmatrix}11\\-5\\5\end{pmatrix}$ Thus the coordinates of B is $\left(\dfrac{11}{3}, -\dfrac{5}{3}, \dfrac{5}{3}\right)$. (iii) Normal of $\pi_3 = \begin{pmatrix}1\\2\\1\end{pmatrix}\times\begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}-3\\3\\-3\end{pmatrix} = 3\begin{pmatrix}-1\\1\\-1\end{pmatrix}$ Cartesian equation of π_3: $\mathbf{r}\cdot\begin{pmatrix}-1\\1\\-1\end{pmatrix} = \begin{pmatrix}4\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\-1\end{pmatrix}$

	$-x+y-z=-7$ $x-y+z=7$ From G.C. the position vector of the point of intersection of the three planes is $\begin{pmatrix} 10/3 \\ -5/3 \\ 2 \end{pmatrix}$
2	 *OPQR* forms a rectangle with length *OR* which is 3 times the length of *OP*. (a) Let the position vector of complex number z^2 be $\overrightarrow{OS}$ $z=x+3xi$ $z^2=(x+3xi)^2=-8x^2+6x^2i$ $\therefore \overrightarrow{OS}=\begin{pmatrix}-8x^2\\6x^2\end{pmatrix}$ $z+3iz=x+3xi+3i(x+3xi)$ $=-8x+6xi$ $\therefore \overrightarrow{OQ}=\begin{pmatrix}-8x\\6x\end{pmatrix}$ $\therefore \overrightarrow{OS}=x\begin{pmatrix}-8x\\6x\end{pmatrix}=x\overrightarrow{OQ}$ Since x is a constant and $\therefore \overrightarrow{OS}=x\overrightarrow{OQ}$, $\therefore$ *S* is collinear with the origin and the point *Q*.

(b)



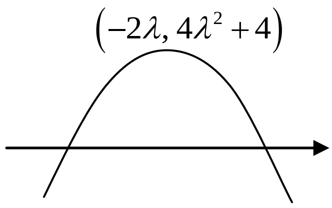
Area of region P = $\frac{1}{2}(3)^2\left(\tan^{-1}3-\tan^{-1}\frac{1}{3}\right)\approx 4.17$

3

(i) $\mathrm{f}(x)=4-x^2-4\lambda x$

$$=-\left[(x+2\lambda)^2-(2\lambda)^2\right]+4$$

$$=-(x+2\lambda)^2+4\lambda^2+4$$



Since horizontal line y = 0 cuts the graph $y=\mathrm{f}(x)$ twice, f is not a one-one function. $\therefore$ f does not have an inverse.

(ii) Largest value of $k = -2\lambda$

Let $y=-(x+2\lambda)^2+4\lambda^2+4$

$$x=-2\lambda\pm\sqrt{4\lambda^2+4-y}$$

Since $x<-2\lambda$, $x=-2\lambda-\sqrt{4\lambda^2+4-y}$

$$\therefore \mathrm{f}^{-1}: x\mapsto -2\lambda-\sqrt{4\lambda^2+4-x},\quad x<4\lambda^2+4$$

(iii) $R_{\mathrm{f}^{-1}}=(-\infty,-2\lambda)$

$D_{\mathrm{g}}=(-\infty,1)$

Given $\lambda>-\frac{1}{2}$, $\therefore R_{\mathrm{f}^{-1}}=(-\infty,-2\lambda)\subset(-\infty,1)=D_{\mathrm{g}}$

$\therefore \mathrm{gf}^{-1}$ exists

$\therefore R_{\mathrm{gf}^{-1}}=(\ln(1+2\lambda),\infty)$

4

(i) $u_2 = \frac{5}{3^2}, u_3 = \frac{7}{3^3}, u_4 = \frac{9}{3^4}$

(ii) $u_n = \frac{2n+1}{3^n}$

(iii) Let P_n be the statement "$u_n = \frac{2n+1}{3^n}, n \in \mathbb{Z}^+$"

When $n = 1$,

LHS = $u_1 = 1$, RHS = $\frac{2+1}{3} = 1$

Since LHS = RHS, P_1 is true.

Assume P_k is true for some values of $k \in \mathbb{Z}^+$

ie $u_k = \frac{2k+1}{3^k}$

When $n = k + 1$

Want to show: $u_{k+1} = \frac{2k+3}{3^{k+1}}$

$$
\begin{aligned}
\text{LHS} &= u_{k+1} \\
&= \frac{1}{3^{k+1}}\left(3^k u_k + 2\right) \\
&= \frac{1}{3^{k+1}}\left(3^k\left(\frac{2k+1}{3^k}\right) + 2\right) \\
&= \frac{2k+3}{3^{k+1}} \\
&= \text{RHS}
\end{aligned}
$$

$\therefore P_{k+1}$ is true

By Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.

5

(a) $\frac{\mathrm{dx}}{\mathrm{dt}} \alpha \sqrt{x}$

$\frac{\mathrm{dx}}{\mathrm{dt}} = k\sqrt{x}, \quad k < 0$

$\int x^{-\frac{1}{2}} \, \mathrm{d}x = \int k \, \mathrm{d}t$

$2x^{\frac{1}{2}} = kt + c$

When $t = 0$, $x=200$ $\Leftrightarrow C = 2\sqrt{200}$

When $x = 100$, $\frac{\mathrm{dx}}{\mathrm{dt}} = -1 \Leftrightarrow k = \frac{-1}{10}$

Hence, $2x^{\frac{1}{2}} = -\frac{1}{10}t + 2\sqrt{200}$ (shown)

	When $x = 100$, $2\sqrt{100} = -\frac{1}{10}t + 2\sqrt{200}$ $\Leftrightarrow t = 82.84$ The container has been leaking for 83 min. (b) $v = x - y$ $\Rightarrow \frac{dv}{dx} = 1 - \frac{dy}{dx}$ $\frac{dy}{dx} + \left(1 + (x-y)^2\right)\cos x = 1$ $1 - \frac{dv}{dx} + \left(1 + v^2\right)\cos x = 1$ $-\frac{dv}{dx} = -\left(1 + v^2\right)\cos x$ $\int \frac{1}{1+v^2}\ dv = \int \cos x\ dx$ $\tan^{-1} v = \sin x + C$ $\tan^{-1}(x-y) = \sin x + C$ $(x-y) = \tan(\sin x + C)$ $y = x - \tan(\sin x + C)$
6	(i) Simple random sample might not be representative if no manager is chosen. (ii) To obtain a sample of 40 staff members, we draw ***random*** samples from each category with sample size in the same proportion as the size of each category in the company. (table below) Advantage: Each staff category is represented proportionately.
7	(i) Let μ be the population mean decrease in cholesterol level $H_0 : \mu = 25$ $H_1 : \mu > 25$ (one-tailed test) Under H_0 , (since σ^2 is unknown and n is small,) the test statistic is $T = \frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t(n-1)$

Table for 6(ii):

	Managers	Technicians	Factory workers
Sample size	$\frac{60}{500} \times 40$ $=4.8$ ≈ 5	$\frac{140}{500} \times 40$ $=11.2$ ≈ 11	$\frac{300}{500} \times 40$ $=24$

	where $\mu = 25$, $s = 14.51645$, $n = 12$, $\bar{t} = 35$ i.e. $T \sim t(11)$ Level of significance : 0.05 From G.C., the p-value = 0.0180475 Conclusion: Since p value= 0.0180475 < 0.05 (significance level), we **reject H_0** and conclude that **at the 5% level, there is <u>significant evidence</u> to conclude the mean decrease in LDL level is more than 25**. (ii) The decrease in LDL level in the underlying population follows a normal distribution. (iii) $H_0 : \mu = 25$ $H_1 : \mu > 25$ (one-tailed test) Under H_0, the test statistic is $Z = \dfrac{\overline{X} - \mu}{S/\sqrt{n}} \sim N(0,1)$ approximately (by CLT) where $\mu = 25$, $n = 100$, $s = 14.51645$ Level of significance: 5% = 0.05 Critical Region (or Rejection Region): Z > 1.64485 Coffee company not promoting the new coffee product $\Rightarrow$ H_0 is not rejected , $\Rightarrow \dfrac{\bar{x} - 25}{14.51645/\sqrt{100}} < 1.64485$ $\Rightarrow \bar{x} < 27.3877$
8a	(i) P(Score is 4) = $\dfrac{3C2}{7C3} = \dfrac{3}{35}$ (ii) P(Score is 6 given that one of them is 2) $= \dfrac{P(\text{Score is 6 and one of them is 2})}{P(\text{one of the number is 2})}$ $= \dfrac{4C1}{6C2}$ $= \dfrac{4}{15}$
b	(i) Using M1 method: <table><tr><td>N mber of digits</td><td>Number of integers</td></tr><tr><td>1</td><td>3C1 = 3</td></tr><tr><td>2</td><td>3P2 = 6</td></tr><tr><td>3</td><td>3P3 = 6</td></tr></table> Using M2 Method: Number of integers = $4 + 4^2 + 4^3 = 84$ Hence, total number of integers in set A =3+6+6+84 = 99

	(ii) P(sum of 2 integers is even) = P(both integer is even) + P(both integer is odd) $= \frac{15}{99} \times \frac{14}{98} + \frac{84}{99} \times \frac{83}{98}$ $= \frac{57}{77}$
9	P(Player wins a prize in a game) = p^4 (i) Let X be r.v. "number of winners within the first ten games". i.e. $X \sim B(10, p^4)$ P(at least one winner within the first ten games) ≥ 0.9 $P(X \geq 1) \geq 0.9$ $1 - P(X = 0) \geq 0.9$ $P(X = 0) \leq 0.1$ $(1 - p^4)^{10} \leq 0.1$ $1 - p^4 \leq 0.1^{\frac{1}{10}}$ $p^4 \geq 0.205672$ $0.673 \leq p \leq 1$(correct to 3 s.f) (ii) Let Y be the r.v "no. of prizes won out of 100 games". $Y \sim B(100, (0.7)^4)$ $E(Y) = 100(0.7)^4 = 24.01$ $Var(Y) = 100(0.7)^4(1-(0.7)^4) = 18.245199$ Let $\bar{Y} = \frac{Y_1 + Y_2 + ... + Y_{60}}{60}$ Since n = 60 is large, by Central Limit Theorem, $\bar{Y} \sim N\left(24.01, \frac{18.245199}{60}\right)$ approximately. $\therefore \quad P(\bar{Y} \leq 24) \approx 0.493$ <u>Alternative solution</u> Let Y be the r.v "no. of prizes won out of 100 games". $Y \sim B(100, (0.7)^4)$ $np = 100(0.7)^4 = 24.01 \quad > 5$ $nq = 100(1-(0.7)^4) = 75.99 > 5$ $Y \sim N(24.01, 18.24511)$ approximately Let $\bar{Y} = \frac{Y_1 + Y_2 + ... + Y_{60}}{60}$ $\bar{Y} \sim N\left(24.01, \frac{18.245199}{60}\right)$ approximately.

$\therefore \quad P\left(\overline{Y} \le 24\right) \overset{cc}{\to} P\left(\overline{Y} < 24 + \frac{0.5}{60}\right) \approx 0.499$

10

(i) Meteors are seen singly, randomly and independently. There is a uniform (mean) rate of occurrence of meteor sightings.

(ii) Let X be the r.v. “number of meteors seen by Jess in 5 minutes’.

$$X \sim Po(6.5)$$

$P(X > 5) = 1 - P(X \le 5)$

$= 1 - 0.36904068 = 0.631$

(iii) Let Y be the r.v. “number of meteors seen by Jess in 1 hour”.

$$Y \sim P(78)$$

Since $\lambda = 78 > 10$, $Y \sim N(78, 78)$ approximately

$P(Y \le 100) \overset{cc}{\to} P(Y < 100.5)$

$= 0.995$

(iv) Let W be the r.v. number of meteors seen by Jess in t minutes

$$W \sim Po(1.3t)$$

$P(W > 2) \ge 0.96$

$1 - P(W \le 2) \ge 0.96$

$1 - e^{1.3t}\left(1 + 1.3t + \frac{(1.3t)^2}{2}\right) \ge 0.96$

$e^{1.3t}\left(1 + 1.3t + \frac{(1.3t)^2}{2}\right) - 0.04 \le 0$

From GC, $t \ge 5.076$. Hence, smallest possible integer $t = 6$ mins.

11

(i) $\overline{x} = 52.75, \ \overline{y} = \frac{356 + \alpha}{8}$

From the given regression line,

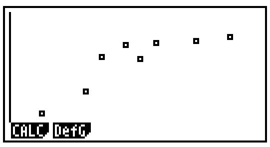
$$\frac{356 + \alpha}{8} = 0.9978(52.75) - 4.0104$$

Therefore, $\alpha = 32.9884 \approx 33$

(ii) (a) $r = 0.866$

(b) $r = 0.907$

(iii)



From the scatter diagram of y on x, the scatterplot seems to follow a logarithmic/non-linear model. As the product moment correlation coefficient of the

	regression line of y on $\ln x$ is closer to 1, $y = a+b\ln x$ is a better model. (iv) From GC, $y = -163.23278+ 53.7472975\ln x$ When $y=55$, depth, x = 58.0mm (to 3 s.f) As $y = 55$ is within the range of the data set of y and r = 0.905 is close to 1, the estimate is reliable.
12	(i) Let X be the r.v. "the volume (in ml) of coke dispensed into a cup". $X \sim \text{N}(\mu, 20^2)$ $\text{P}(X>500) < 0.001$ $\text{P}(Z>\frac{500-\mu}{20}) < 0.001$ $\frac{500-\mu}{20} > 3.09023231$ $\mu < 438.19535$ $\Rightarrow$ max $\mu = 438.19$ml (ii) Let C be the r.v. "volume (ml) of a cup of coke" $C \sim \text{N}(475, 20^2)$ $\text{P}(C>500) = 0.10565$ Let W be the number of cups of coke that will overflow out of 120 cups $W \sim B(120, 0.10565)$ Since $n = 120 > 30$ and $np = 12.675 > 5$ and $nq = 107.332 > 5$ $W \sim N(12.675, 11.33857)$ approximately $\text{P}(W \geq 10) \xrightarrow{cc} \text{P}(W > 9.5)$ $= 0.827$ (iii) Let L be the r.v. "volume (in ml) of a cup of lemon tea". $L \sim \text{N}(450, 30^2)$ $\text{P}(C_1+C_2+C_3+C_4 > 2(L_1+L_2))$ $= \text{P}(C_1+C_2+C_3+C_4-2(L_1+L_2) > 0)$ $= \text{P}(M > 0)$ $M \sim \text{N}\left(4(475)-2(2)(450),\ 4(20^2)+4(2)(30^2)\right)$ i.e. $M \sim \text{N}(100,\ 8800)$ $\text{P}(M > 0) \approx 0.857$