Section A: Pure Mathematics [40 marks]

1 **Topic: Differential Equations**

Solution

(i) Given $u = x^2 + y^2$, differentiate with respect to x:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$x + y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}x} \text{ (I)}$$

Substitute (I) & $u = x^2 + y^2$ and into D.E:

$$\frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}x} = \sqrt{u} :$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2\sqrt{u} \text{ (shown)}$$

Hence,

$$\frac{1}{\sqrt{u}}\frac{\mathrm{d}u}{\mathrm{d}x} = 2$$

$$u^{-\frac{1}{2}}\frac{\mathrm{d}u}{\mathrm{d}x} = 2$$

Integrate both sides with respect to x:

$$\int u^{-\frac{1}{2}}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int 2\,\mathrm{d}x$$

$$\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2x + C,$$ where C is an arbitrary constant

Solution

$$2u^{\frac{1}{2}} = 2x + C$$

$$\sqrt{u} = x + \frac{C}{2}$$

$$\sqrt{x^2 + y^2} = x + D, \text{ where } D = \frac{C}{2}$$

(ii)

$$\sqrt{x^2 + y^2} = x + D$$

$$x^2 + y^2 = (x + D)^2$$

$$y^2 + x^2 = x^2 + 2Dx + D^2$$

$$y^2 = 2Dx + D^2$$

$$y = \pm\sqrt{2Dx + D^2}$$

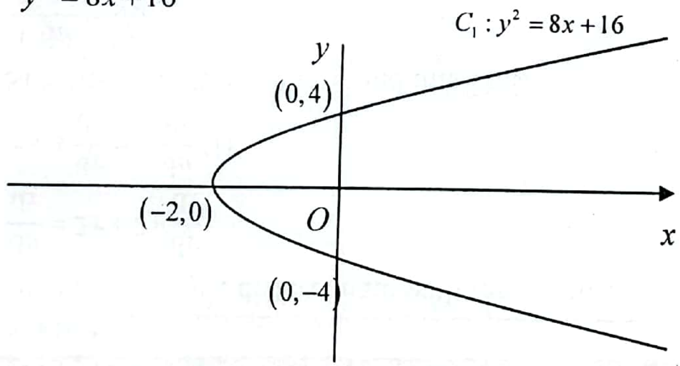
When $x = -2, y = 0$

$$0 = -4D + D^2$$

$$D(D-4) = 0$$

$D = 0$ (rej.) or $D = 4$

$$y^2 = 8x + 16$$



(iii) The equation of line of symmetry is $y = 0$.

Solution

Using Geometric series,

$$1+\omega+\omega^2=\frac{\omega^3-1}{\omega-1}=0 \qquad (\text{as } \omega^3=1)$$

Or

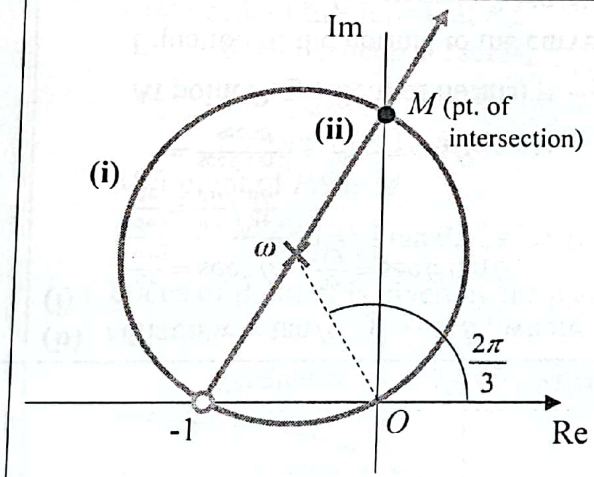
$$\omega^3-1=0$$

Since ω is a root of $x^3-1=0$, $(\omega-1)(\omega^2+\omega+1)=0$

Since $\omega\neq 1, \omega^2+\omega+1=0$

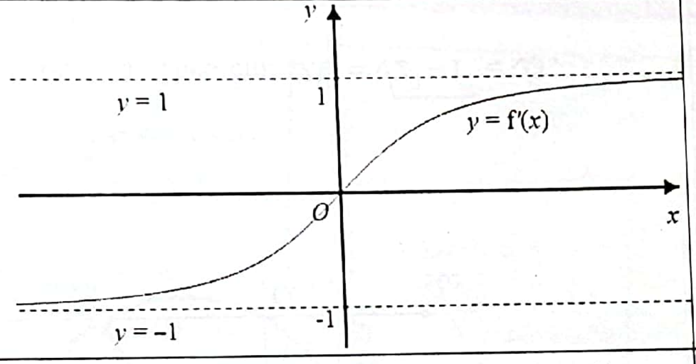
Or

Any acceptable method



Using Pythagoras Theorem, $OM=\sqrt{2^2-1^2}=\sqrt{3}$.

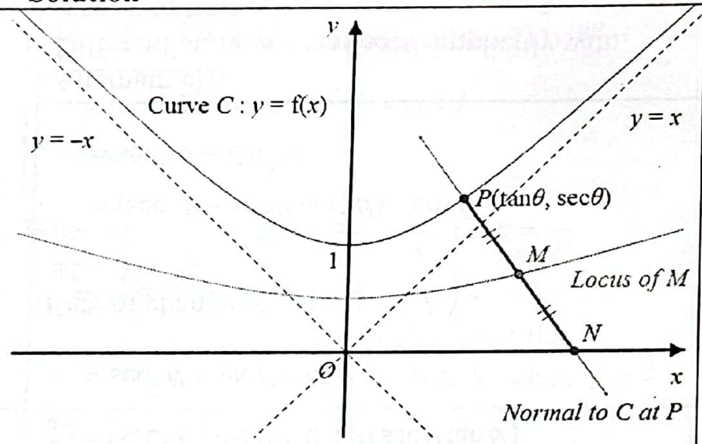
$\therefore$ The complex number is $\sqrt{3}\mathrm{i}$.

3	Topic: Application of Differentiation					
			Solution			
			(a) 			
			(b) (i) Given $x = \tan\theta$, $y = \sec\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. $\frac{dx}{d\theta} = \sec^2\theta$, $\frac{dy}{d\theta} = \sec\theta\tan\theta$, $\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$ $= \frac{\sec\theta\tan\theta}{\sec^2\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta$ At point P, gradient of normal $= -\frac{1}{\sin\theta}$. Equation of the normal to the curve at P : $y - \sec\theta = -\frac{1}{\sin\theta}(x - \tan\theta)$, $y - \frac{1}{\cos\theta} = -x\frac{1}{\sin\theta} + \frac{1}{\cos\theta}$, $y = -x\ \text{cosec}\,\theta + 2\sec\theta$. (shown)			

3 **Topic: Application of Differentiation**

Solution

(b) (ii)



x-intercept of the normal at P :

$$0 = -x\ \text{cosec}\theta + 2\sec\theta,$$

$$x = 2\frac{\sec\theta}{\text{cosec}\theta} = 2\tan\theta.$$

$\therefore$ Point N is $(2\tan\theta,\ 0)$.

Mid-point of PN is M

$$\left(\frac{x_P + x_N}{2},\ \frac{y_P + y_N}{2}\right) = \left(\tfrac{3}{2}\tan\theta,\ \tfrac{1}{2}\sec\theta\right).$$

Locus of point M is given by the parametric equations

$$x = \tfrac{3}{2}\tan\theta,\ y = \tfrac{1}{2}\sec\theta,\ \text{where } -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}.$$

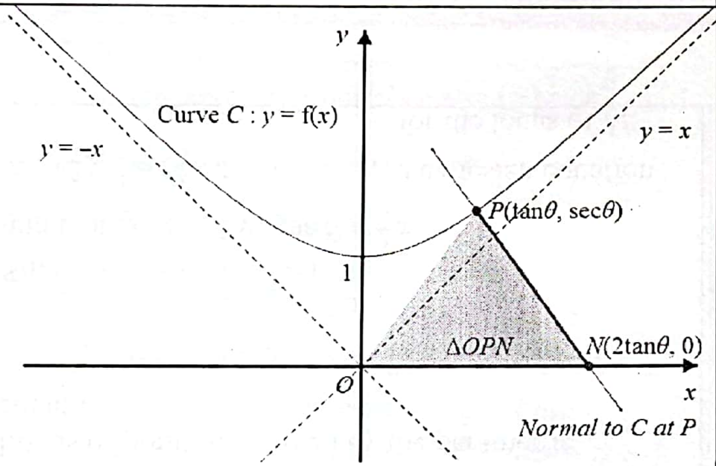
Since $\sec^2\theta - \tan^2\theta = 1$,

and that $\sec\theta = 2y$, $\tan\theta = \frac{2}{3}x$,

$\therefore\ (2y)^2 - (\frac{2}{3}x)^2 = 1$, a cartesian equation for the locus of M.

Solution

(b)(iii)



A, area of $\Delta OPN = \frac{1}{2}(ON)\left(\text{Height of } P \text{ w.r.t. } x\text{-axis}\right)$

$= \frac{1}{2}(2\tan\theta)(\sec\theta) = \tan\theta\sec\theta$

($ON = 2\tan\theta$, from (b)(ii) assuming $\theta > 0$.)

$$\frac{dA}{d\theta} = \left(\sec^2\theta\right)\sec\theta + \tan\theta\left(\sec\theta\tan\theta\right)$$

$$= \sec^3\theta + \sec\theta\tan^2\theta$$

Rate of change of area of ΔOPN,

$$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$$

$$= (\sec^3\theta + \sec\theta\tan^2\theta) \times \cos\theta$$

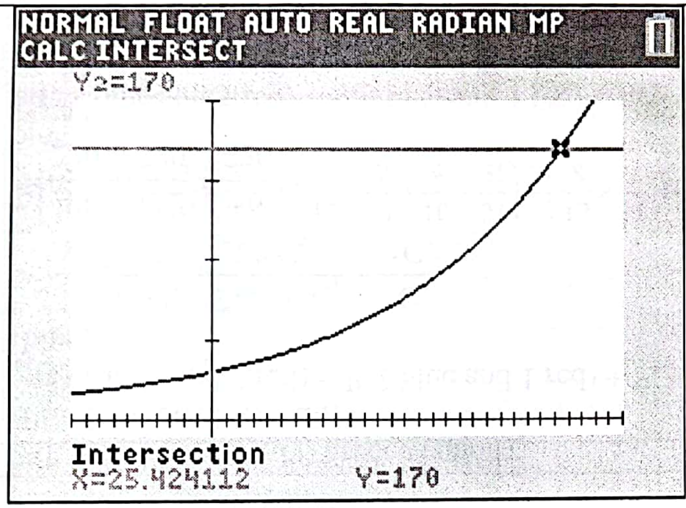
$$= \sec^2\theta + \tan^2\theta$$

<u>Alternatively</u>,

Differentiating $A = \tan\theta\sec\theta$ implicitly with respect to time t,

3			Topic: Application of Differentiation			
			Solution			
			$\frac{dA}{dt} = \left[\left(\sec^2\theta\right)\sec\theta + \tan\theta\left(\sec\theta\tan\theta\right)\right] \times \frac{d\theta}{dt}$ $= (\sec^3\theta + \sec\theta\tan^2\theta) \times \cos\theta$ $= \sec^2\theta + \tan^2\theta$ When $\theta = \frac{\pi}{6}$, $\sec\theta = \frac{1}{\cos\theta} = \frac{2}{\sqrt{3}}$, $\tan\theta = \frac{1}{\sqrt{3}}$ $\therefore \quad \frac{dA}{dt} = \sec^2\theta + \tan^2\theta = \frac{4}{3} + \frac{1}{3} = \frac{5}{3}$, rate of change of the area of ΔOPN when $\theta = \frac{\pi}{6}$.			

4	APGP					
			Solution			
			(i) $S_{15} \text{ of } B = \frac{15}{2}(2(2.4)+14d)$ $= 36+105d$			
			(ii) $S_{15} \text{ of } A = \frac{2.4\left((1.2)^{15}-1\right)}{1.2-1}$ $=172.88$ >170 Yes, Adam can achieve his target.			
			(iii) $U_{15} \text{ of } A > U_{15} \text{ of } B$ $U_{15} \text{ of } A - U_{15} \text{ of } B > 0$ $2.4(1.2)^{14} - (2.4+14d) > 0$ $d < 2.02957$ $\max d = 2.02 (2\text{dp})$			
			$2.4(1.2)^{14} - (2.4+14(2.02)) = 0.134$ (shown)			
			(iv) $\text{New } S_{13} \text{ of } A = \frac{2.4\left(\left(1+\frac{r}{100}\right)^{13}-1\right)}{\left(1+\frac{r}{100}\right)-1} = 170$ From GC,			

4		APGP				
			Solution  $x = 25.4\%$			

Section B: Statistics [60 marks]

5 Topic: Probability

Solution

(i) P(at least one blue ball)

$= 1 - P(\text{no blue balls})$

$= 1 - P(3 \text{ red balls})$

$$= 1 - \frac{{}^{4}C_{3}}{{}^{12}C_{3}}$$

$$= 1 - \frac{4}{220}$$

$$= \frac{54}{55} \quad \text{(Shown)}$$

<u>Alternative Method</u>

P(at least one blue ball)

$= 1 - P(3 \text{ red balls})$

$$= 1 - \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10}$$

$$= \frac{54}{55} \quad \text{(Shown)}$$

<u>Alternative Method (Direct Method)</u>

P(at least one blue ball)

= P(1 blue and 2 red) + P(2 blue and 1 red) + P(3 blue)

$$= \frac{{}^{8}C_{1} \times {}^{4}C_{2}}{{}^{12}C_{3}} + \frac{{}^{8}C_{2} \times {}^{4}C_{1}}{{}^{12}C_{3}} + \frac{{}^{8}C_{3}}{{}^{12}C_{3}}$$

$$= \frac{48}{220} + \frac{112}{220} + \frac{56}{220}$$

$$= \frac{54}{55} \quad \text{(Shown)}$$

Solution

Alternative Method

P(at least one blue ball)

= P(1 blue and 2 red) + P(2 blue and 1 red) + P(3 blue)

$$=\left(\frac{8}{12}\times\frac{4}{11}\times\frac{3}{10}\times\frac{3!}{2!}\right)+\left(\frac{8}{12}\times\frac{7}{11}\times\frac{4}{10}\times\frac{3!}{2!}\right)+\left(\frac{8}{12}\times\frac{7}{11}\times\frac{6}{10}\right)$$

$$=\frac{48}{220}+\frac{112}{220}+\frac{56}{220}$$

$$=\frac{54}{55} \quad \text{(Shown)}$$

(ii) P(at least one of each colour drawn)

= 1 – P(3 red) – P(3 blue)

$$= 1 - \frac{^{4}C_3}{^{12}C_3} - \frac{^{8}C_3}{^{12}C_3}$$

$$= 1 - \frac{4}{220} - \frac{56}{220}$$

$$= \frac{8}{11} \text{ or } \approx 0.727$$

Alternative Method

P(at least one of each colour drawn)

= P(1 blue and 2 red) + P(2 blue and 1 red)

$$=\frac{^{8}C_1 \times {^{4}C_2}}{^{12}C_3}+\frac{^{8}C_2 \times {^{4}C_1}}{^{12}C_3}$$

$$=\frac{48}{220}+\frac{112}{220}$$

$$=\frac{8}{11} \text{ or } \approx 0.727$$

(iii) There are 8 balls with “0” and 4 balls with “1”

P(sum is at least two)

= P(1, 1, 0) + P(1, 1, 1)

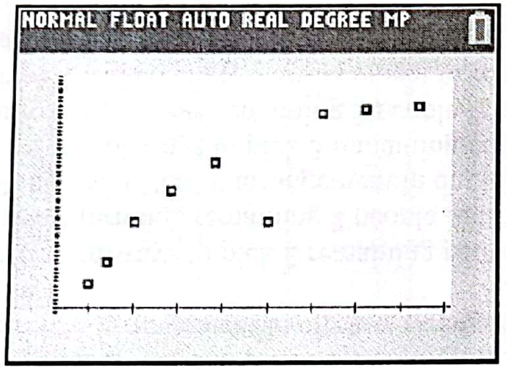
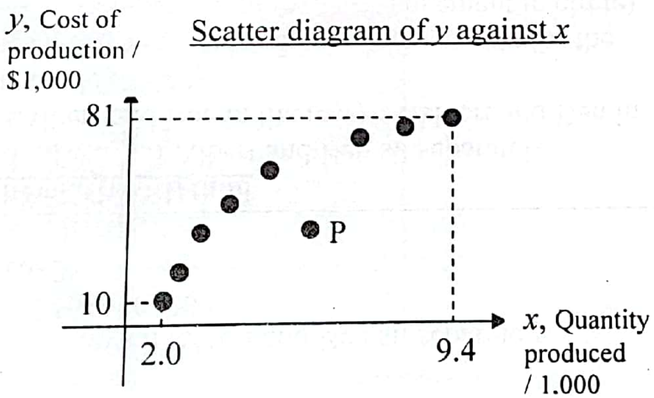
5	Topic: Probability					
			Solution			
			$= \frac{{}^{4}C_2 \times {}^{8}C_1}{{}^{12}C_3} + \frac{{}^{4}C_3}{{}^{12}C_3}$			
			$= \frac{48}{220} + \frac{4}{220}$ $= \frac{13}{55}$ or ≈ 0.236			
			<u>Alternative Method</u> P(sum is at least two) = P(1, 1, 0) + P(1, 1, 1) $= \left(\frac{4}{12} \times \frac{3}{11} \times \frac{8}{10} \times \frac{3!}{2!}\right) + \left(\frac{4}{12} \times \frac{3}{11} \times \frac{2}{10}\right)$ $= \frac{13}{55}$ or ≈ 0.236			

6	Topic: Binomial Distribution					
			Solution			
			(i) $X \sim B(60, 0.03)$ Since $n = 60 > 50$ is large, $p = 0.03 < 0.1$ is small and $np = 1.8 < 5$, $X \sim \text{Po}(1.8)$ approximately $X_1 + X_2 \sim \text{Po}(3.6)$ approximately $P(X_1 + X_2 > 6) = 1 - P(X_1 + X_2 \le 6)$ ≈ 0.073273 $= 0.0733$ (3s.f.)			
			(ii) Since sample size = 50 is large, $\bar{X} \sim \text{N}\left(1.8, \frac{1.746}{50}\right)$ approximately by Central Limit Theorem $P(\bar{X} > 2) \approx 0.14225$ $= 0.142$ (3s.f.)			

7			Topic: Permutation and Combination			
			(i) No. of ways with no restriction = (no. of ways to separate 9 people into a group of 4 and a group of 5) × (no. of ways to arrange the row of 4 people) × (no. of ways to arrange the circle of 5 people) $= {}^9C_4 \times 4! \times {}^5C_5 \times (5-1)!$ = 72 576			
			Alternative Method No. of ways with no restriction = (no. of ways to separate 9 people into a group of 5 and a group of 4) × (no. of ways to arrange the circle of 5 people) × (no. of ways to arrange the row of 4 people) $= {}^9C_5 \times (5-1)! \times {}^4C_4 \times 4!$ [Note: ${}^9C_4 = {}^9C_5$] = 72 576			
			(ii) No. of ways if Albert and Ben sit together = (Albert and Ben in the row) + (Albert and Ben in the circle) = (no. of ways to pick 2 remaining people for the row × arrange row people × A and B swap × arrange circle people) + (no. of ways to pick 3 remaining people for the circle × arrange circle people × A and B swap × arrange row people) $= ({}^7C_2 \times 3! \times 2 \times (5-1)!) + ({}^7C_3 \times (4-1)! \times 2! \times 4!)$			
			= 6048 + 10080 = 16128			
			(iii) No. of ways if Albert and Ben both sit on the couch			

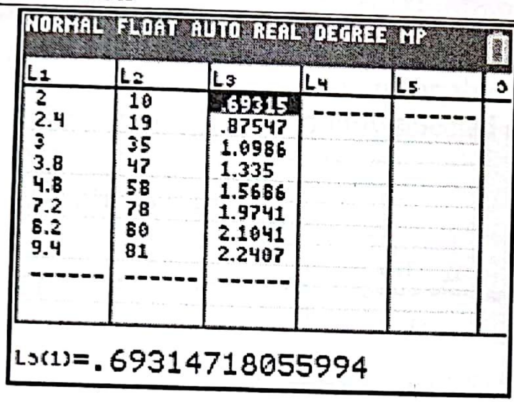
or both sit at the table
= (Albert and Ben in the row) + (Albert and Ben in the circle)
= (no. of ways to pick 2 remaining people for the row × arrange row people × arrange circle people)
+ (no. of ways to pick 3 remaining people for the circle × arrange row people × arrange circle people)
$({}^7C_2 \times 4! \times (5-1)!) + ({}^7C_3 \times 4! \times (5-1)!)$
= 32256
No. of ways if Albert and Ben sit separately
= 32256 – 16128
= 16128

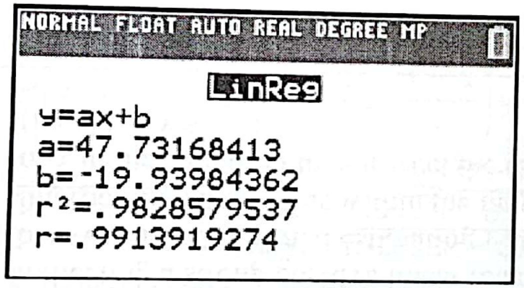
Alternative Method
No. of ways if Albert and Ben sit separately
= (Albert and Ben in the row) + (Albert and Ben in the circle)
= (no. of ways to pick 2 remaining people for the row × arrangement in row × arrangement in circle)
+ (no. of ways to pick 3 remaining people for the circle × arrangement in circle × arrangement in row)
= (no. of ways to pick 2 remaining people for the row × [arrange remaining 2 people in row × slot in Albert and Ben] × arrangement in circle)
+ (no. of ways to pick 3 remaining people for the circle × [arrange remaining 3 people in circle × slot in Albert and Bert] × arrangement in row)
$= ({}^7C_2 \times [\,2! \times {}^3C_2 \times 2!\,] \times (5-1)!) + ({}^7C_3 \times [\,(3-1)! \times {}^3C_2 \times 2!\,) \times 4!)$
= 10080 + 6048
= 16128

8	Topic: Correlation and Regression
	Solution
(i)	GC screenshot :  
(ii)	P = (6.0, 35).
(iii)	The better model is $y = a + b \ln x$ since the set of data points in the scatter diagram exhibits a non-linear trend in which y increases at a decreasing rate as x increases, rather than a linear trend in which y increases at a constant rate with x.

Solution

(iv)





$r_{(\ln x),y} = 0.991$.

Line of regression y on $\ln x$.

$y = (47.73168413\ldots) \ln x + (-19.93984362\ldots)$

$\Rightarrow y = 47.7 \ln x - 19.9$

If a fixed cost is included in y, the value of $r_{(\ln x),y} = 0.991$ is unchanged, as there is no change in the strength of the linear relationship between ($\ln x$) and y, under translation.

However, as a constant value M is added to the y value of each data point, the regression line would be also translated in the direction of the positive y-axis by M units,

i.e. with resultant regression line equation being

			Solution			
			$y = 47.7 \ln x - 19.9 + M$ or $y - M = 47.7 \ln x - 19.9$			
		(v)	When $x = 6$, $y = (47.73168413\ldots) \ln(6) - 19.93984362\ldots$ $= 65.5838534\ldots$ ≈ 65.6 Hence, the estimated cost of production is \$ 65,600 (to 3 s.f). This value is <u>reliable</u>, since <u>$r = 0.991$ is close to 1</u> indicating a strong positive linear relationship between y and $\ln x$, and <u>estimating y at $x = 6$ is an interpolation</u> as $x = 6$ is within the range of values of x in the data used to construct the regression line (i.e. $2.0 \le x \le 9.4$).			

Solution

(i) Let X minutes be the random variable denoting the finishing time of a randomly selected runner in the race.

$X \sim N(\mu, \sigma^2)$

$$P(X \le 40) = 0.1 \Rightarrow P\left(Z \le \frac{40-\mu}{\sigma}\right) = 0.1$$

$$\Rightarrow \frac{40-\mu}{\sigma} = -1.2816$$

$$\Rightarrow \mu - 1.2816\sigma = 40 \quad (1)$$

$$P(X > 60) = 0.35 \Rightarrow P\left(Z \le \frac{60-\mu}{\sigma}\right) = 0.65$$

$$\Rightarrow \frac{60-\mu}{\sigma} = 0.38532$$

$$\Rightarrow \mu + 0.38532\sigma = 60 \quad (2)$$

Solving (1) and (2), $\mu \approx 55.377 = 55.4 \text{(3s.f.)}$ and $\sigma \approx 11.998 = 12.0 \text{(3s.f.)}$

(ii) For $P(X \le a) = 0.2 \Rightarrow a \approx 45.300 = 45.3 \text{(3s.f.)}$

Maximal timing is 45.3 minutes.

[Accept $a \approx 45.279$ for 5s.f. intermediate]

(iii) Let Y be the number of runners, out of 12, who receive a medal.

$Y \sim B(12, 0.2)$

$P(Y > 4) = 1 - P(Y \le 4) \approx 0.072555 = 0.0726$

(iv) $P(\text{slowest runner finishes within 1 hour} \mid \text{all do not receive medal})$

$$= \frac{P(\text{slowest runner finishes within 1 hour and all do not receive medal})}{P(\text{all do not receive medal})}$$

9			Topic: Normal Distribution			
			Solution			
			$= \frac{P(\text{all runners finish within 1 hour and do not receive medal})}{(1-0.2)^{12}}$ $= \frac{P(45.279 \le X_1 \le 60 \text{ and } 45.279 \le X_2 \le 60 \text{ and ... and } 45.279 \le X_{12} \le 60)}{(1-0.2)^{12}}$ $= \frac{[0.65-0.2]^{12}}{(0.8)^{12}}$ $= \left(\frac{0.45}{0.8}\right)^{12}$ $\approx 0.0010034 = 0.00100(3\text{s.f.})$ Method 2: Let A be the r.v. denoting the number of runners who finish in an hour and do not receive medal, out of 12. $A \sim B(12, 0.65-0.2)$ $A \sim B(12, 0.45)$ Required probability $= \frac{P(A=12)}{P(Y=0)}$			

10 **Topic: Sampling Methods Hypothesis Testing**

Solution

a(i) This method is quota sampling and is not representative of the student population as there is no consideration of the size of each sports team and each club and society relative to the student population, and the student from each group is selected non-randomly i.e. only Captains and Presidents selected.

a(ii) Stratified sampling.
Divide school population by year/level (strata), calculate the proportion of each strata relative to the population and select the respective number randomly.

b(i) Let X cm be the vertical jump height of a randomly selected volleyball player.

$H_0 : \mu = 40$ vs $H_1 : \mu > 40$

b(ii) Under H_0, test statistics: $T = \dfrac{\overline{X} - \mu}{s / \sqrt{n}} \sim t(n-1)$

Where $\mu = 40, \overline{x} = 42.1, n = 7, s^2 = \dfrac{7}{6}k^2$

To reject H_0, p-value < 0.1

$$\Rightarrow P(T > t) < 0.1$$

$$\Rightarrow P\left(T > \frac{42.1 - 40}{\sqrt{7/6}\ \alpha / \sqrt{7}}\right) < 0.1$$

$$\Rightarrow P\left(T \le \frac{2.1}{k/\sqrt{6}}\right) > 0.9$$

$$\Rightarrow \frac{2.1}{k/\sqrt{6}} > 1.4398$$

10 **Topic: Sampling Methods Hypothesis Testing**

				Solution			
				$\Rightarrow k < \dfrac{2.1\sqrt{6}}{1.4398} \approx 3.5728$ Required set is $\{k \in □ : 0 < k < 3.57\}$ or $(0, 3.57)$			
			b(iii)	Since $k^2 = 15 \Rightarrow k = \sqrt{15} > 3.57$, we do not reject H_0 at 10% level of significance and conclude that there is insufficient evidence to support the claim that the training regime is effective.			

11			Topic: Poisson Distribution			
			Solution			
		(i)	Any two of the following 4 assumptions: (1) The calls are made independently of each other. (2) The probability of receiving two or more calls within a very short interval of time is negligible. OR Calls are received one at a time. (3) The average number of calls received over any time interval of the same duration within the day is constant. (4) The calls received occur randomly.			
		(ii)	Let W_n, X_n and Y_n be the random variables denoting the number of calls received in n hours pertaining to credit card queries, business banking queries and personal banking queries respectively. Then $W_n \sim \text{Po}(n\mu), X_n \sim \text{Po}(6n)$ and $Y_n \sim \text{Po}(7n)$ $8P(W_4 = 2) = P(W_1 = 2)$ where $W_1 \sim Po(\mu)$ and $\Rightarrow \frac{8e^{-4\mu}(4\mu)^2}{2!} = \frac{e^{-\mu}\mu^2}{2!} \Rightarrow e^{3\mu} = 128$ $\Rightarrow \mu = \frac{1}{3}\ln 128 = \frac{7}{3}\ln 2$			
		(iii)	Since W_2, X_2, and Y_2 are independent, $P(W_2 = 0 \mid W_2 + X_2 + Y_2 > 50) = \frac{P(W_2 = 0 \text{ and } W_2 + X_2 + Y_2 > 50)}{P(W_2 + X_2 + Y_2 > 50)}$ $= \frac{P(W_2 = 0 \text{ and } X_2 + Y_2 > 50)}{1 - P(W_2 + X_2 + Y_2 \le 50)}$			

11				Topic: Poisson Distribution			
				Solution			
				where $W_2 \sim \text{Po}\left(\frac{14}{3}\ln 2\right)$ and $W_2 + X_2 + Y_2 \sim \text{Po}\left(\frac{14}{3}\ln 2 + 26\right)$, $X_2 + Y_2 \sim \text{Po}(26)$ we have $= \frac{P(W_2 = 0)\cdot[1 - P(X_2 + Y_2 \le 50)]}{1 - P(W_2 + X_2 + Y_2 \le 50)}$ $\approx 0.0022359 = 0.00224$ (3s.f.)			
			(iv)	$X_2 \sim \text{Po}(12)$ and $Y_2 \sim \text{Po}(14)$. Since $\lambda_1 = 12 > 10$ and $\lambda_2 = 14 > 10$, $X_2 \sim \text{N}(12,12)$ approximately and $Y_2 \sim \text{N}(14,14)$ approximately Since X_2 and Y_2 are independent, $X_2 - Y_2 \sim \text{N}(-2, 26)$ approximately $P(X_2 > Y_2) = P(X_2 - Y_2 > 0)$ $\approx P(X_2 - Y_2 > 0.5)$ by continuity correction ≈ 0.31196 $= 0.312$ (3s.f.)			