Name:	Index No.:	Class:

PRESBYTERIAN HIGH SCHOOL

ADDITIONAL MATHEMATICS **4049/02**
Paper 2

26 August 2021 Thursday 2 hrs 15 min

2021 SECONDARY FOUR EXPRESS
PRELIMINARY EXAMINATIONS

Marking Scheme

This question paper consists of **17** printed pages (including this cover page) and **1** blank page.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Binomial Theorem

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{r} a^{n-r} b^r + \cdots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \frac{n(n-1)\ldots\ldots\ldots\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1.$$

$$\sec^2 A = 1 + \tan^2 A.$$

$$\mathrm{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for $\Delta\, ABC$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

$$\Delta = \frac{1}{2} ab \sin C.$$

Answer all questions in the space provided.

1 The equation of a curve is $y = x\left(4+x^2\right)^3$.

(i) Find an expression for $\frac{dy}{dx}$. [3]

$$\frac{dy}{dx} = x \times 3\left(4+x^2\right)^2 (2x) + \left(4+x^2\right)^3 \times 1$$

M1, M1

$$\frac{dy}{dx} = \left(4+x^2\right)^2\left(6x^2+4+x^2\right)$$

$$\frac{dy}{dx} = \left(4+x^2\right)^2\left(7x^2+4\right)$$

A1

(ii) Explain whether y is an increasing or decreasing function. [2]

For all x, $\left(4+x^2\right)^2 > 0$ and $\left(7x^2+4\right) > 0$, $\frac{dy}{dx} = \left(4+x^2\right)^2\left(7x^2+4\right) > 0$ M1

y is a increasing function. A1

2 The equation of the curve is $y=(a+3)x^2+ax+1$, where a is a constant.

(a) When $a=4$, find the set of values of x for which $y-4>0$. [3]

$y-4>0$

$7x^2+4x+1-4>0$ M1

$7x^2+4x-3>0$

$(x+1)(7x-3)>0$ M1

$x<-1 \; or \; x>\frac{3}{7}$ A1

(b) **(i)** Find the range of values of a for which the curve has no real roots. [3]

$b^2-4ac<0$

$a^2-4(a+3)(1)<0$ M1

$a^2-4a-12<0$

$(a+2)(a-6)<0$ M1

$-2<a<6$ A1

(ii) Hence explain why the curve cannot lies completely below the *x*-axis. [1]

For curve to lie below *x*-axis, $a<-3$, but $-2<a<6$, hence the curve cannot lies completely below the *x*-axis

3 The expression $f(x)=x^3+ax+b$, where *a* and *b* are constants, is exactly divisible by $x-2$ and leaves a remainder of 30 when divided by $x-3$.

(i) Find the value of *a* and of *b*. [4]

$f(x)=x^3+ax+b$

when $f(-2)=0$,

$(-2)^3+a(-2)+b=0$ M1

$-8-2a+b=0$ ------ (1)

when $f(3)=30$,

$(3)^3+a(3)+b=30$ M1

$27+3a+b=30$ ------ (2)

(2) − (1),

$35+5a=30$

$5a=-5$

$a= \quad 1$ A1

$b=8-2$

$b=6$ A1

(ii) Determine by showing all necessary working, the number of real root(s) of the equation $f(x)=0$. [4]

(ii) $f(x)=x^3-x+6$

$f(x)=(x+2)(x^2-2x+3)$ M1, M1

Discriminant of $(x^2-2x+3) = (-2)^2-4(1)(3)=-8<0$

There is no real solution of (x^2-2x+3) M1

$f(x)=0$ has only one real roots A1

4 **(i)** Prove the identity $\dfrac{1}{\sec A+1}+\dfrac{1}{\sec A-1}=2\operatorname{cosec} A\cot A$. [4]

$$\frac{1}{\sec A+1}+\frac{1}{\sec A-1}$$

$$=\frac{\sec A-1+\sec A+1}{\sec^2 A-1}$$ M1

$$=\frac{2\sec A}{\tan^2 A}$$ M1

$$=\frac{2\sec A}{\tan A}\times\frac{1}{\tan A}$$

$$=\frac{2}{\sin A}\times\cot A$$ M1

$$=2\operatorname{cosec} A\cot A$$ AG1

(ii) Hence find all the angles between 0 and 2π for which $\dfrac{1}{\sec A+1}+\dfrac{1}{\sec A-1}=\dfrac{4}{3}$. [5]

$$\frac{1}{\sec A+1}+\frac{1}{\sec A-1}=\frac{4}{3}$$

$$2\cos ecA\cot A=\frac{4}{3}$$

$$2\left(\frac{1}{\sin A}\right)\left(\frac{\cos A}{\sin A}\right)=\frac{4}{3}$$ M1

$$\frac{\cos A}{\sin^2 A}=\frac{2}{3}$$

$$3\cos A=2\sin^2 A$$ M1

$$3\cos A=2\left(1-\cos^2 A\right)$$ M1

$$3\cos A=2-2\cos^2 A$$

$$2\cos^2 A+3\cos A-2=0$$

$$(2\cos A-1)(\cos A+2)=0$$

$\cos A=0.5$ *or* $\cos A=-2$ (rejected) M1

$A=\frac{\pi}{3},2\pi-\frac{\pi}{3}$

$A=\frac{\pi}{3},\frac{5\pi}{3}$ A1

5 It is given that $f(x)$ is such that $f'(x)=\cos 2x+\sin 3x$.

Given that $f(\pi)=0$, show that $f''(x)+9f(x)=a+b\sin 2x$ where *a* and *b* are constants. [7]

$f'(x)=\cos 2x+\sin 3x$

$f''(x)=-\sin 2x\times 2+\cos 3x\times 3$ M1, M1

$f''(x)=-2\sin 2x+3\cos 3x$

$f(x)=\int \cos 2x+\sin 3x\,dx$

$f(x)=\frac{\sin 2x}{2}-\frac{\cos 3x}{3}+c$ M1, M1

$0=\frac{\sin 2(\pi)}{2}-\frac{\cos 3(\pi)}{3}+c$ M1

$0=0-\frac{-1}{3}+0+c$

$c=-\frac{1}{3}$

$f(x)=\frac{\sin 2x}{2}-\frac{\cos 3x}{3}-\frac{1}{3}$

$f''(x)+9f(x)=-2\sin 2x+3\cos 3x+9\left(\frac{\sin 2x}{2}-\frac{\cos 3x}{3}-\frac{1}{3}\right)$ M1

$=-2\sin 2x+3\cos 3x+\left(\frac{9\sin 2x}{2}-3\cos 3x-3\right)$

$=-3+\frac{5\sin 2x}{2}$ A1

6 The table below shows the experimental values of two variables x and y.

x	1	2	3	4	5	6
y	63	127	258	510	1000	2100
$\lg y$	1.80	2.10	2.41	2.71	3.00	3.32

It is known that x and y are related by an equation of the form $y = \frac{b^x}{10^a}$, where a and b are constants.

(i) By plotting $\lg y$ against x, obtain a straight line graph to represent the above data. [3]

(ii) Use your graph to estimate the value of a and of b. [3]

(ii) Use your graph to find the value of x when $y = 160$. [1]

(iii) Explain how would you use the graph to find the value of x for which $(10b)^x = 10^{a+1}$. [2]

$(10b)^x = 10^{a+1}$

$10^x b^x = 10^a \times 10$

$\frac{b^x}{10^a} = \frac{10}{10^x}$ M1

$y = 10^{1-x}$

$\lg y = \lg\left(10^{1-x}\right)$ $\}$

$\lg y = 1 - x$

Draw the line $\lg y = 1 - x$ and find the *x*-coordinate of the point of intersection. A1

(i) P2 for all points correct and axis
C1 mark for straight line

(ii)

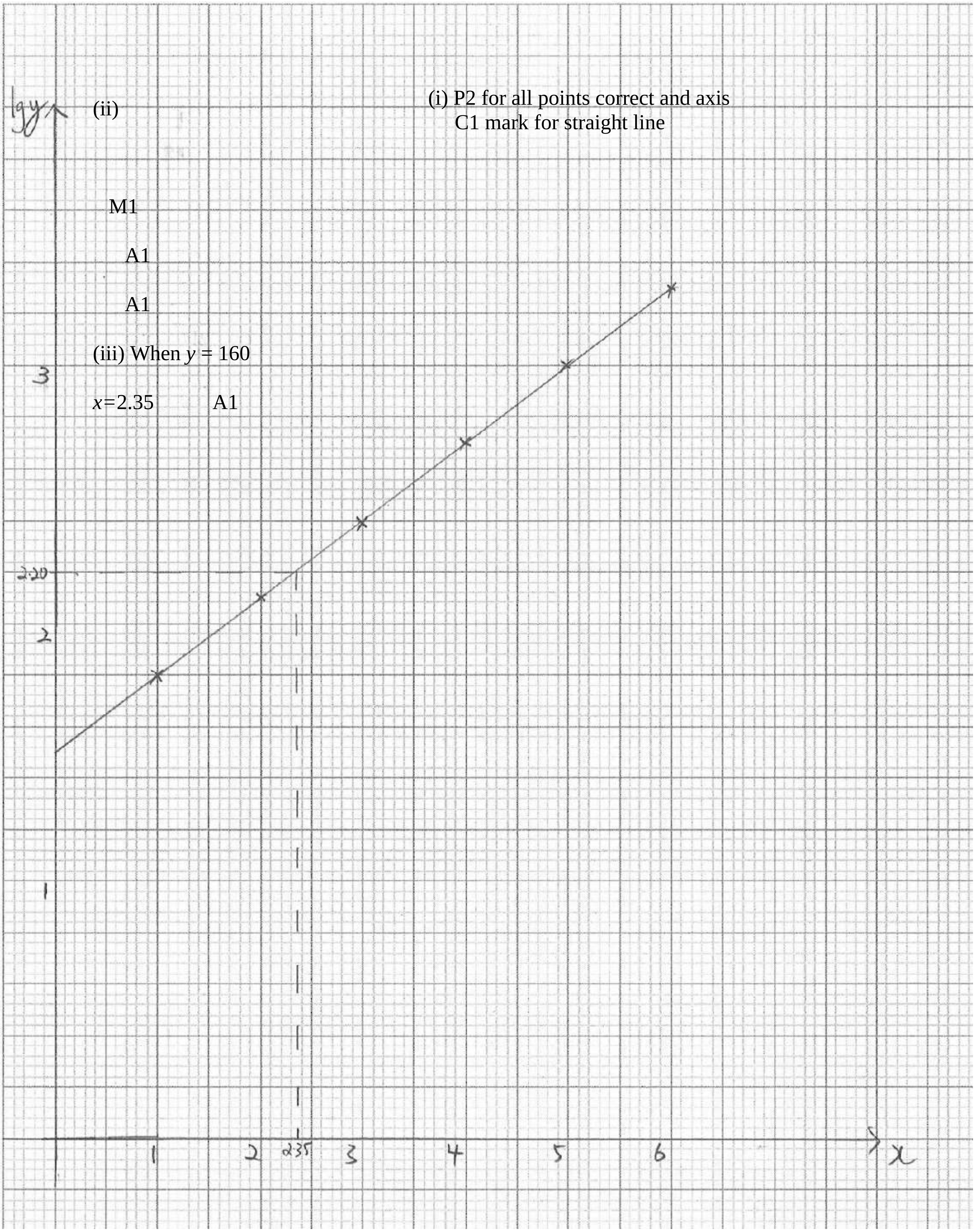
M1

A1

A1

(iii) When y = 160

x=2.35 A1



7 **(i)** Express $\dfrac{5x^2-12x-3}{(x-2)\left(x^2+3\right)}$ in partial fractions. [5]

$$\frac{5x^2-12x-3}{(x-2)\left(x^2+3\right)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+3}$$

M1

$$5x^2-12x-3=A\left(x^2+3\right)+(Bx+C)(x-2)$$

Let $x = 2$

$$5(2)^2-12(2)-3=A\left(2^2+3\right)$$

M1 for at least one correct method

$$-7=7A$$

$$-1=A$$

Let $x = 0$

$$-3=-1\left(0^2+3\right)+(C)(0-2)$$

$$-3=-3-2C$$

$$0=C$$

A1 for at least one correct
A2 for all 3 correct

Let $x = 1$

$$5(1)^2-12-3=-1\left(1^2+3\right)+(B)(1-2)$$

$$-10=-4-B$$

$$6-B$$

$$\frac{5x^2-12x-3}{(x-2)\left(x^2+3\right)}=-\frac{1}{(x-2)}+\frac{6x}{\left(x^2+3\right)}$$

A1

(ii) Differentiate $\ln\left(x^2+3\right)$ with respect to x. [1]

$$\frac{d\left[\ln\left(x^2+3\right)\right]}{dx} = \frac{1}{x^2+3} \times 2x$$

$$= \frac{2x}{x^2+3}$$

B1

(iii) Using the results from **(i)** and **(ii)**, determine $\int \frac{5x^2-12x-3}{(x-2)\left(x^2+3\right)}dx$. [4]

$$\int \frac{5x^2-12x-3}{(x-2)\left(x^2+3\right)}dx = \int -\frac{1}{x-2} + \frac{6x}{x^2+3}dx$$

M1

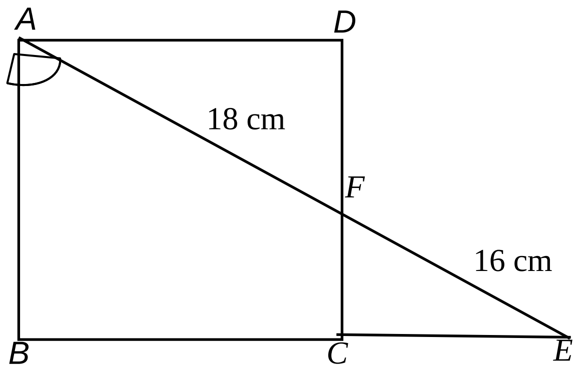
$$\int \frac{5x^2-12x-3}{(x-2)\left(x^2+3\right)}dx = \int -\frac{1}{x-2}dx + 3\int \frac{2x}{x^2+3}dx$$

M1

$$\int \frac{5x^2-12x-3}{(x-2)\left(x^2+3\right)}dx = -\ln(x-2) + 3\ln(x^2+3) + c$$

A1, A1

8 The diagram shows a rectangle *ABCD*. A line through *A* intersects *CD* at *F* and *BC* produced at *E*. It is given that $\angle BAE = \theta°$, *AF* =18 cm and *FE* =16 cm.



(i) Show that the perimeter, *P* cm, of the rectangle is given by
$P = 68\cos\theta + 36\sin\theta$. [3]

In ΔAFD, $\cos\theta = \dfrac{DF}{18}$ — $\cos\theta$ and $\sin\theta$ of either triangle

$DE = 18\cos\theta$

$\sin\theta = \dfrac{AF}{18}$ M1

$AF = 18\sin\theta$

In ΔFCE, $\cos\theta = \dfrac{FC}{16}$

$FC = 16\cos\theta$ M1

$\sin\theta = \dfrac{CE}{16}$

$CE = 16\sin\theta$

Therefore $P = 2(AB + AD)$

$P = 2(18\cos\theta + 16\cos\theta + 18\sin\theta)$

$P = 68\cos\theta + 36\sin\theta$ (shown) AG1

(ii) Express P in the form $R\cos(\theta-\alpha)$. [3]

$R=\sqrt{68^2+36^2}=\sqrt{5920}$ M1

$\tan\alpha=\dfrac{36}{68}$ M1

$\alpha=27.9°$

$P=\sqrt{5920}\cos(\theta-27.9°)$ A1

(iii) Find the value of (when the perimeter is 62 cm.

[2]

$\sqrt{5920}\cos(\theta-27.9°)=62$ M1

$\cos(\theta-27.9°)=\dfrac{62}{\sqrt{5920}}$

$\theta-27.89°=36.31$

$\theta=64.2°$

A1

9 A circle C_1 has equation $x^2 + y^2 - 6x + 4y = 12$.

(i) Find the radius and the coordinates of the centre of C_1. [3]

Centre = $\left(\frac{-6}{-2}, \frac{4}{-2}\right)$

Centre = $(3, -2)$ B1

Radius $= \sqrt{3^2 + (-2)^2 - (-12)}$ M1 for attempt to find centre or radius

Radius $= 5$ A1

(ii) Show that the equation of the tangent to the circle at the point $(7, -5)$ is $3y = 4x + k$ where *k* is an integer. [3]

Gradient of the from the point of tangent to centre $= \frac{-5-(-2)}{7-3}$ M1

Gradient of the tangent= $-\frac{1}{\left(-\frac{3}{4}\right)} = \frac{4}{3}$ M1

$y - (-5) = \frac{4}{3}(x - 7)$

$3y + 15 = 4x - 28$

$3y = 4x - 43$ A1

(iii) Another circle C_2 has centre $(-8, 4)$ and radius 7 cm. Find the shortest distance between the 2 circles. [2]

Distance between the 2 centre $= \sqrt{(3-(-8))^2 + (-2-4)^2} = \sqrt{157}$ M1

Shortest distance $= \sqrt{157} - 5 - 7 = 0.530$ A1

10 **(a)** **Without using a calculator,** find the value of 6^x given that $12^{x-2} = 3^{4-x}$. [4]

$12^{x-2} = 3^{4-x}$

$12^x \times 12^{-2} = 3^4 \times 3^{-x}$ M1

$\frac{12^x}{3^{-x}} = \frac{3^4}{12^{-2}}$ M1

$36^x = 9^2 \times 12^2$

$\left(6^x\right)^2 = (9 \times 12)^2$ M1

$6^x - 108$ A1

(b) Solve the equation $\log_2 x = 6 + \log_{16} x$. [3]

$\log_2 x = 6 + \log_{16} x$

$\log_2 x = 6 + \frac{\log_2 x}{\log_2 16}$ M1 (Change of base)

$\log_2 x = 6 + \frac{\log_2 x}{4}$

$\frac{3\log_2 x}{4} = 6$ M1

$\log_2 x = 8$

$x - 2^8$

$x = 256$ A1

(c) Solve the equation $\lg(x-4)+2\lg 3 = 1+\lg\left(\frac{x}{2}\right)$. [3]

$\lg(x-4)+2\lg 3 = 1+\lg\left(\frac{x}{2}\right)$

$\lg(x-4)+\lg 3^2 = \lg 10+\lg\left(\frac{x}{2}\right)$ M1 (Power rule)

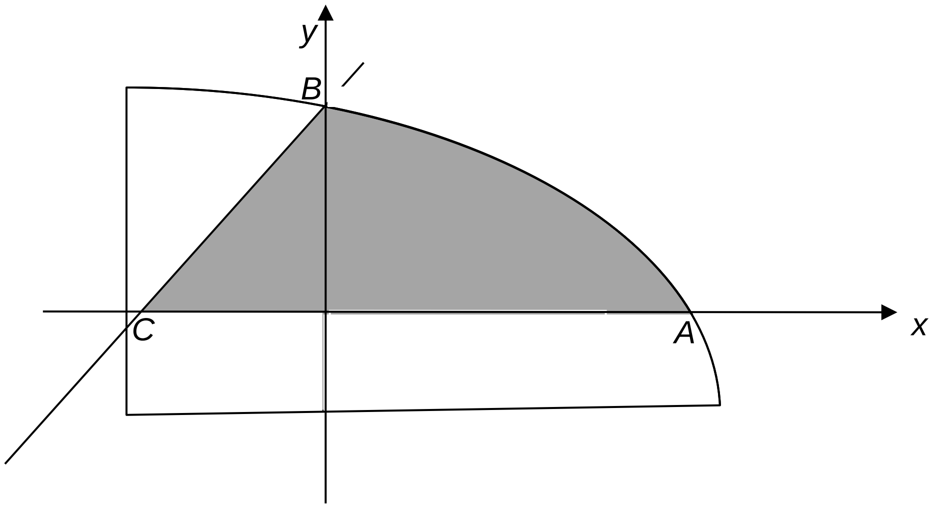
$\lg 9(x-4) = \lg 10\left(\frac{x}{2}\right)$ M1 (Product rule)

$9x-36=5x$

$4x=36$

$x=9$ A1

11



The diagram shows part of the curve $y = 27 + \frac{8}{(x-2)^3}$ which crosses the axes at points *A* and at *B*

(i) Find the coordinates of *A* and of *B*. [2]

$$When\ x = 0, y = 27 + \frac{8}{(0-2)^3} = 26$$

$B(0, 26)$ B1

$When\ y = 0,$

$$0 = 27 + \frac{8}{(x-2)^3}$$

$$-27 = \frac{8}{(x-2)^3}$$

$$(x-2)^3 = \frac{8}{-27}$$

$$x - 2 = -\frac{2}{3}$$

$$x = \frac{4}{3}$$

$\left(\frac{4}{3}, 0\right)$ B1

The normal to the curve at B cuts the x-axis at C.

Find

(ii) the equation of the normal, [3]

$$\frac{dy}{dx} = -\frac{24}{(x-2)^4}$$ M1

$$\frac{dy}{dx} = -\frac{24}{(0-2)^4} = -\frac{3}{2}$$ M1

Gradient of normal $= \frac{2}{3}$

Equation of normal is $y = \frac{2}{3}x + 26$ A1

(iii) the area of the shaded region. [4]

$$0 = \frac{2}{3}x + 26$$

$$-\frac{2}{3}x = 26$$

$$x = -39$$ M1

Area of shaded region $= \frac{1}{2}(39)(26) + \int_0^{\frac{4}{3}} 27 + 8(x-2)^{-3}\,dx$ M1 for either

$$= 507 + \left[27x + \frac{8}{-2}(x-2)^{-2}\right]_0^{\frac{4}{3}}$$ M1 for integration

$$= 507 + \left[27\left(\frac{4}{3}\right) - 4\left(\frac{4}{3}-2\right)^{-2} - \left(27(0) - 4(0-2)^{-2}\right)\right]$$

$$= 507 + [36 - 9 + 1]$$

$$= 535 \text{ unit}^2$$ A1

END OF PAPER