EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2023

General Certificate of Education Advanced Level

Higher 2

CANDIDATE NAME	

CIVICS GROUP		INDEX NO.	

FURTHER MATHEMATICS 9649/01

Paper 1 [100 marks]

12 September 2023

3 hours

Additional Materials: Answer Booklet

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and index number on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

1 A curve has parametric equations

$$x = t^4 - 4\ln t, \qquad y = 4t^2, \qquad t > 0.$$

The arc of the curve from $t = 1$ to $t = k$, where $k > 1$, is rotated through one complete revolution about the x-axis.

Express the surface area generated as a polynomial in k.

Given that this surface area is 384π, find the value of k. [4]

2 The sequence $\{x_n\}$ is given by $x_1 = 1$ and

$$nx_n = 4\left(1 - \frac{1}{n}\right)^{n-1} x_{n-1} + 2n^{1-n} \qquad \text{for } n \geq 2.$$

By multiplying the recurrence relation throughout by n^{n-1}, use a suitable substitution to determine x_n as a function of n, simplifying your answer. [7]

3 The variables x and y are related by the differential equation

$$x^2 y \frac{\mathrm{d}y}{\mathrm{d}x} = x^3 + x^2 y - y^3, \qquad (*)$$

and it is given that $x > y > 0$.

(a) Use the substitution $y = ux$ to find the general solution of the differential equation. [5]

A particular solution of the differential equation (*) is such that $y = 1$ when $x = 2$.

(b) Use one step of the Euler method to calculate an approximate value of y when $x = 2.5$. [2]

(c) Given that the solution curve is concave downwards near the point $(2, 1)$, determine with a sketch whether the approximate value of y found in **(b)** is an over-estimate or under-estimate. [1]

4 The curve C has parametric equations

$$x = a(t - \sin t), \quad y = a(1 - \cos t)$$

where $0 \leq t \leq \dfrac{3\pi}{2}$ and a is a positive constant.

(i) The region R is bounded by the curve C, the x-axis and the line $x = a\left(\dfrac{3\pi}{2} + 1\right)$. The volume of the solid generated when R is rotated through 2π radians about the y-axis is given by V. Using the shell method, show that

$$V = 2\pi a^3 \int_0^b (t - \sin t)(1 - \cos t)^2 \, \mathrm{d}t,$$

where b is a constant to be determined. [2]

(ii) Find the exact value of V. [6]

5 **(i)** A 3×3 square matrix $\mathbf{A}$ is said to be skew symmetric if $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$.
Given that $\mathbf{A}$ is skew symmetric, show that

$$\det(\mathbf{A}) = 0.$$ [3]

(ii) Let T be the linear transformation such that

$$\mathrm{T}:\mathbb{R}^3 \to \mathbb{R}^3 \text{ and } \mathrm{T}(\mathbf{x}) = \mathbf{a}\times\mathbf{x}$$

where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and the multiplication is the usual vector product in $\mathbb{R}^3$.

Let $\mathbf{M}$ be the matrix representation of T.

(a) Find $\mathbf{M}$ in terms of a_1, a_2 and a_3. [2]

(b) Determine if $\mathbf{M}$ is invertible. [1]

(c) State $\ker(\mathrm{T})$ and $\mathrm{R}(\mathrm{T})$, the null space and range space of T respectively. Give a geometrical interpretation of your answers. [4]

6 Let $\mathbf{T}_\theta$ denote the matrix $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$.

(i) By finding $\mathbf{T}_\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $\mathbf{T}_\theta \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$, show that $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ are eigenvectors and write down their corresponding eigenvalues. [4]

(ii) Express $\mathbf{T}_\theta$ in the form $\mathbf{RDR}^{-1}$ where $\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. [1]

The matrix $\mathbf{T}_\theta$ represents the reflection of a position vector in $\mathbb{R}^2$ about the line through the origin that makes an angle θ with the positive x-axis.

(iii) Find the reflection of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ about the line $y = \sqrt{3}x$. [3]

(iv) The matrix $\mathbf{R}$ in **(ii)** represents the rotation of a position vector in $\mathbb{R}^2$ through an angle θ about the origin. By finding $\mathbf{T}_\alpha \mathbf{T}_\beta$, show that the product of two reflection matrices is a matrix that represents the rotation of a position vector in $\mathbb{R}^2$. [2]

7 It is given that $\mathrm{f}(x)=\dfrac{1}{x^2}-\sqrt{x-2}$.

(i) Show that the equation $\mathrm{f}(x)=0$ has a root α in the interval $[k,k+1]$ where k is an integer to be determined. [2]

In order to find an approximation β to α, two stages of the linear interpolation process is used on the interval $[k,k+1]$.

(ii) Find the value of β, correct to 3 significant figures. [2]

(iii) By considering $\mathrm{f}''(x)$ in the interval $[k,k+1]$, determine whether β is an under-estimate or an over-estimate of α. [3]

The Newton-Raphson iteration can be used to estimate α.

(iv) Explain why an initial approximation $x_0=\dfrac{2k+1}{2}$ would not work. [1]

The root α for the equation $\mathrm{f}(x)=0$ satisfies the equation $x=\mathrm{g}(x)$ where $\mathrm{g}(x)=2+\dfrac{1}{x^4}$.

(v) Use an iterative method based on the form $x_{n+1}=\mathrm{g}(x_n)$ with $x_0=k+1$ to find an approximation, γ to α, correct to 2 decimal places. Verify that your answer is accurate up to 2 decimal places. [3]

8 Consider a body moving along a straight line through a liquid medium. The body experiences a force $\mathrm{F}(t)$ in the direction of motion and a frictional force in the opposite direction. At lower speeds of motion, the frictional force is directly proportional to the velocity $v(t)$ of the motion.

According to Newton's second law of motion, for a body with constant mass *m*, the product of the body's mass and acceleration is equal to the net force on the body in the direction of motion. This basic model leads to the differential equation

$$m\frac{\mathrm{d}v}{\mathrm{d}t}=\mathrm{F}(t)-kv(t)$$

where *k* is a positive constant.

(a) Suppose $\mathrm{F}(t)=C$ where *C* is a positive constant. Given that the initial velocity of the body is v_0, find an expression for *v*. [4]

(b) Show that, after a long time, the velocity of the body tends to a limiting value that is independent of the initial velocity. [2]

(c) **(i)** Now suppose $\mathrm{F}(t)=C-qt$ where *C* and *q* are positive constants. Given that the body is initially at rest, show that

$$v=\left(\frac{C}{k}+\frac{qm}{k^2}\right)\left(1-\mathrm{e}^{-\frac{kt}{m}}\right)-\frac{qt}{k}.$$ [5]

(ii) Show that at the instant when $\mathrm{F}(t)=0$, the velocity of the body is positive. (You may use the fact that $\mathrm{e}^{-x}<\dfrac{1}{1+x}$ when $x>0$.) [2]

9 **(a)** The terms in a sequence satisfy the recurrence relation

$$4u_r = 2u_{r-1} - u_{r-2}.$$

(i) Find the general solution of this recurrence relation. [3]

(ii) Show that $u_{k+3} = \alpha u_k$ for all $k \geq 0$ where α is a constant to be determined. Find $\sum_{r=1}^{\infty} u_{3r-2}$ in terms of u_1. [4]

(b) A message is sent by transmitting a sequence of signals with each signal being one of the five different signals. Two of these signals require 1 microsecond each to transmit, while the other three signals require 2 microseconds each. It is assumed that the signals in a message are transmitted without additional time between signals.

Let a_n be the number of messages that can be transmitted within n microseconds using these five different signals, where n is a positive integer.

(i) Explain why the sequence $a_1, a_2, a_3, \ldots$ satisfies the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2}$. [2]

(ii) Solve the recurrence relation. [5]

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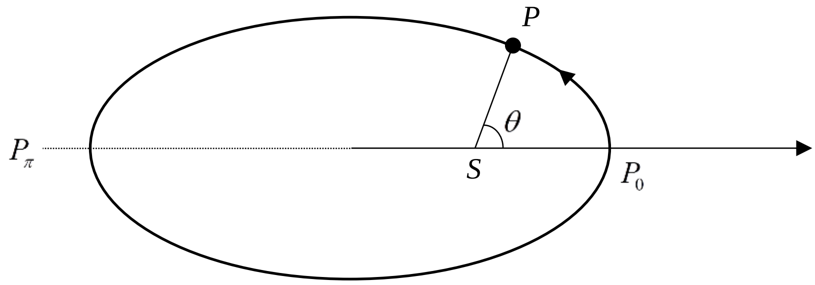

The diagram shows the elliptical orbit of a Planet *P* and its sun, *S*, which is at a focus of the ellipse. The point P_0, called the *perihelion*, is the point on the orbit when *P* is closest to *S*. The point P_π, called the *aphelion*, is the point on the orbit when *P* is furthest from *S*.

Taking *S* as the pole and the line SP_0 as the initial line, the equation of the orbit of *P* is

$$r = \frac{a}{1 + e\cos\theta},\ \theta \geq 0,$$

where *e* is the eccentricity of the ellipse, *a* is a positive constant and θ is in radians.

(i) Given that the distance between P_0 and P_π is 5.9 astronomical units (AU) and the distance between *S* and P_π is 2.15 times of that between *S* and P_0, determine the values of *a* and *e*. [5]

(ii) *Kepler's Second Law of Planetary Motion* states that the line segment *SP* sweeps out equal areas in equal times. Given that the orbital period of Planet *P* is 130 days, find the time *P* takes to travel from the point where $\theta = \frac{\pi}{2}$ to the point where $\theta = 3$. [4]

(iii) Find also the additional time *P* takes to travel from the point where $\theta = 3$ to P_π. [2]

Kepler's Third Law of Planetary Motion states that the orbital period of a planet, *T*, is related to the length of the semi-major axis of its orbit, *l*, by the equation

$$T^2 = kl^3,$$

where *k* is constant for planets orbiting the same sun.

Planet *Q* orbits around the same sun as Planet *P*. The equation of the orbit of *Q* is given by

$$25x^2 + 4y^2 - 50x_0x - 8y_0y + 25{x_0}^2 + 4{y_0}^2 - 100 = 0$$

where (x_0, y_0) are the coordinates of the centre of the orbit.

Use Kepler's Third Law to determine the orbital period of *Q*. [4]