RAFFLES INSTITUTION
H2 Mathematics (9758)
2016 Year 5

Tutorial 3C
Vectors III: Equations of Planes

Section A (Basic Questions)

1 (a) The Cartesian equation of the plane π is $3x-y+4z=-2$. State the equation of π in scalar product form.

(b) The equation of the plane π is $\mathbf{r}\cdot(-\mathbf{i}+\mathbf{j}+2\mathbf{k})=4$. Write down the Cartesian equation of π, and find a vector equation of π, involving two parameters.

(c) The vector equation of the plane π is $\mathbf{r}=\begin{pmatrix}0\\1\\-3\end{pmatrix}+s\begin{pmatrix}0\\1\\-1\end{pmatrix}+t\begin{pmatrix}2\\-1\\0\end{pmatrix}$, $s,t\in\mathbb{R}$. Find the equation of π in both the scalar product form and the Cartesian form.

[(a) $\mathbf{r}\cdot(3\mathbf{i}-\mathbf{j}+4\mathbf{k})=-2$, (b) $-x+y+2z=4$, $\mathbf{r}=(-4\mathbf{i})+\lambda(\mathbf{i}+\mathbf{j})+\mu(2\mathbf{i}+\mathbf{k})$, $\lambda,\mu\in\mathbb{R}$
(c) $\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}+2\mathbf{k})=-4$, $x+2y+2z=-4$]

2 For each of the following cases, find the equation of the plane π_1 in scalar product form.

(a) π_1 contains the point $A(2,-4,7)$ which is the foot of perpendicular from the origin to π_1.

(b) π_1 contains the point $B(1,-2,0)$ and is parallel to the plane $\pi_2: 2x-y+3z=-1$.

(c) π_1 passes through the points $C(2,3,3)$, $D(0,1,0)$, and $E(-1,5,1)$.

(d) π_1 contains the lines $l_1: x+3=y-1, z=-1$ and $l_2: \mathbf{r}=(-3-t)\mathbf{i}+\mathbf{j}-(1+t)\mathbf{k}$, $t\in\mathbb{R}$.

(e) π_1 contains the line $l: \mathbf{r}=(-2\mathbf{i}+5\mathbf{j}-3\mathbf{k})+\lambda(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$, $\lambda\in\mathbb{R}$ and is perpendicular to the plane $\pi_2: \mathbf{r}\cdot(7\mathbf{i}+4\mathbf{j}+5\mathbf{k})=2$.

(f) π_1 passes through the points $F(1,1,1)$ and $G(3,2,2)$ and is perpendicular to the plane π_2 that is parallel to the vectors $-\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $2\mathbf{i}-\mathbf{j}+\mathbf{k}$.

[(a) $\mathbf{r}\cdot(2\mathbf{i}-4\mathbf{j}+7\mathbf{k})=69$ (b) $\mathbf{r}\cdot(2\mathbf{i}-\mathbf{j}+3\mathbf{k})=4$ (c) $\mathbf{r}\cdot(2\mathbf{i}+\mathbf{j}-2\mathbf{k})=1$
(d) $\mathbf{r}\cdot(-\mathbf{i}+\mathbf{j}+\mathbf{k})=3$ (e) $\mathbf{r}\cdot(-3\mathbf{i}+4\mathbf{j}+\mathbf{k})=23$ (f) $\mathbf{r}\cdot(-\mathbf{i}+\mathbf{j}+\mathbf{k})=1$]

3 (a) Find the point of intersection of the line $\mathbf{r}=\begin{pmatrix}1\\0\\0\end{pmatrix}+\mu\begin{pmatrix}0\\1\\2\end{pmatrix}$ and the plane $\mathbf{r}\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k})=5$.

(b) Show that the line $l: \dfrac{x-2}{1}=\dfrac{y-3}{1}=\dfrac{z}{-1}$ is contained in the plane $\pi: 3x-y+2z=3$.

[(a) $(1,3,6)$]

Section B (Standard Questions)

4 **(a)** Find the perpendicular distance from the origin to the plane $\mathbf{r}\cdot(4\mathbf{i}+4\mathbf{j}+7\mathbf{k})=9$.

(b) Show that the planes $2x-y+3z=7$ and $-4x+2y-6z=3$ are parallel and find the distance between them.

(c) Find the angle between the planes $x+3y-6z=-2$ and $x+5y+2z=8$.

[(a) 1 unit (b) $\frac{17\sqrt{14}}{28}$ units (c) 83.8°]

5 **NJC Prelim 9740/2008/01/Q10**

A line l is given by the equation $\mathbf{r}=2\mathbf{i}+\mathbf{k}+\lambda(\mathbf{i}+2\mathbf{j}-3\mathbf{k})$, $\lambda\in\mathbb{R}$ and the point P has position vector $2\mathbf{i}-\mathbf{j}-3\mathbf{k}$ relative to the origin. N is a point on l such that the line PN is perpendicular to l.

(i) Show that $\overrightarrow{PN}=\frac{1}{7}(5\mathbf{i}+17\mathbf{j}+13\mathbf{k})$. [2]

(ii) The plane Π_1 has equation $\mathbf{r}\cdot(3\mathbf{i}+\mathbf{k})=7$. Verify that l lies in Π_1 and find the perpendicular distance from point P to Π_1. [4]

(iii) A second plane Π_2 contains l and P. Using your answers above, or otherwise, determine the acute angle between Π_1 and Π_2. [2]

(iv) A third plane Π_3 has equation $\mathbf{r}\cdot(-\mathbf{j}+5\mathbf{k})=22$. Determine the position vector of the point of intersection between planes Π_1, Π_2 and Π_3. [2]

[(ii) $\frac{4}{\sqrt{10}}$ (iii) 23.8° (iv) $\begin{pmatrix}1\\-2\\4\end{pmatrix}$]

6 **9740/2014/01/Q9**

Planes p and q are perpendicular. Plane p has equation $x+2y-3z=12$. Plane q contains the line l with equation $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-3}{4}$. The point A on l has coordinates $(1,-1,3)$.

(i) Find a cartesian equation of q. [4]

(ii) Find a vector equation of the line m where p and q meet. [4]

(iii) B is a general point on m. Find an expression for the square of the distance AB. Hence, or otherwise, find the coordinates of the point on m which is nearest to A [5]

[(i) $-x+2y+z=0$ (ii) $\mathbf{r}=\begin{pmatrix}6\\3\\0\end{pmatrix}+\mu\begin{pmatrix}4\\1\\2\end{pmatrix}$, $\mu\in\mathbb{R}$ (iii) $21\mu^2+36\mu+50$; $\left(\frac{18}{7},\frac{15}{7},-\frac{12}{7}\right)$]

7 PJC Prelim 9740/2011/01/Q12 (modified)

The position vectors of the points A, B, C are $\begin{pmatrix}0\\2\\-3\end{pmatrix}$, $\begin{pmatrix}3\\3\\-3\end{pmatrix}$, $\begin{pmatrix}-9\\2\\6\end{pmatrix}$ respectively.

(i) Given that A, B, C lies on plane Π_1, show that the equation of Π_1 is $\mathbf{r}.\begin{pmatrix}1\\-3\\1\end{pmatrix}=-9$. [2]

The equations of the plane Π_2 and the line l are $\mathbf{r}.\begin{pmatrix}4\\-2\\1\end{pmatrix}=-6$ and $\mathbf{r}=\begin{pmatrix}1\\1\\4\end{pmatrix}+\mu\begin{pmatrix}-1\\-2\\3\end{pmatrix}, \mu\in\mathbb{R}$

respectively.

(ii) Find the position vector of the foot of the perpendicular from A to the plane Π_2 [3]

(iii) Find a vector equation of the line of intersection between Π_1 and Π_2. [2]

(iv) The plane Π_3 has the Cartesian equation $ax+by+cz=3$, where a, b and c are constants.

Π_1, Π_2 and Π_3 intersect in a line, and the point with position vector $\begin{pmatrix}5\\-12\\2\end{pmatrix}$ lies on Π_3.

Find a, b and c. Hence find the acute angle between Π_3 and l. [5]

[(ii) $\frac{1}{21}\begin{pmatrix}4\\40\\-62\end{pmatrix}$, (iii) $r=\begin{pmatrix}0\\3\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\3\\10\end{pmatrix}, \lambda\in\mathbb{R}$, (iv) $a=3, b=1, c=0$, $25.0°$]

8 9740/2013/02/Q4

The planes p_1 and p_2 have equations $\mathbf{r}.\begin{pmatrix}2\\-2\\1\end{pmatrix}=1$ and $\mathbf{r}.\begin{pmatrix}-6\\3\\2\end{pmatrix}=-1$ respectively, and meet in the line l.

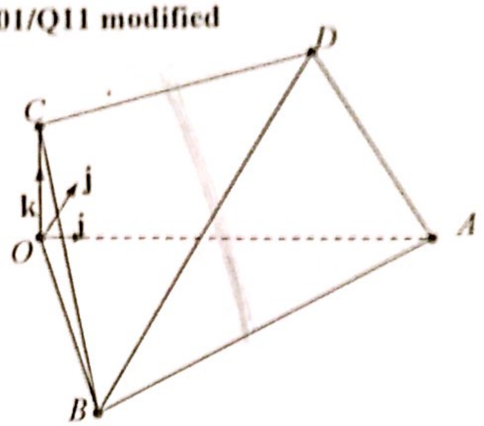
(i) Find the acut e angle between p_1 and p_2. [3]

(ii) Find a vector equation for l. [4]

(iii) The point A (4, 3, c) is equidistant from the planes p_1 and p_2. Calculate the two possible values of c. [6]

[(i) 40.4° (ii) $\underset{\sim}{r}=\begin{pmatrix}-1/6\\-2/3\\0\end{pmatrix}+\lambda\begin{pmatrix}7/6\\5/3\\1\end{pmatrix}, \lambda\in\mathbb{R}$ (iii) $c=\frac{35}{13}$ or -49]

9 HCI Prelim 9740/2010/01/Q11 modified



The diagram above shows part of the structure of a modern art museum designed by Marcus, with a horizontal base OAB and vertical wall $OADC$. Perpendicular unit vectors **i**, **j**, **k** are such that **i** and **k** are parallel to OA and OC respectively.

The walls of the museum BCD and ABD can be described respectively by the equations

$$\mathbf{r}\cdot\begin{pmatrix}-1\\-5\\6\end{pmatrix}=36 \quad\text{and}\quad \mathbf{r}=\begin{pmatrix}14\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}5\\4\\0\end{pmatrix}+\mu\begin{pmatrix}-1\\0\\4\end{pmatrix},\ \text{where } \lambda,\mu\in\mathbb{R}.$$

(i) Write down the distance of A from O. [1]
(ii) Find the vector equation of the intersection line of the two walls BCD and ABD. [3]
(iii) Marcus wishes to repaint the inner wall ABD. Find the area of this wall. [3]

Suppose Marcus wishes to divide the structure into two by adding a partition such that it intersects with the walls BCD and ABD at a line. This partition can be described by the equation $2x-7y+\alpha z=\beta$, where $\alpha,\beta\in\mathbb{R}$.
(iv) Find the values of α and β. [2]
[1]

[(i) $OA=14$ (ii) $\underset{\sim}{r}=\begin{pmatrix}4\\-8\\0\end{pmatrix}+\gamma\begin{pmatrix}1\\1\\1\end{pmatrix},\gamma\in\mathbb{R}$ (iii) 51.8 (iv) $\alpha=5,\ \beta=64$]

10 AJC Prelim 9234/2006/01/Q8(modified)

The plane π contains the origin and is parallel to vectors $-\mathbf{i}+\mathbf{k}$ and $\mathbf{i}+2\mathbf{j}$. A point P with position vector $3\mathbf{i}+\mathbf{j}+2\mathbf{k}$ is reflected about the plane π.

(i) Find the position vector of the foot N of the perpendicular from P to the plane π. Hence, show that the position vector of P', the reflection of P about π is $-\mathbf{i}+3\mathbf{j}-2\mathbf{k}$. [4]

A line l passes through the point P and is parallel to $\mathbf{i}-\mathbf{j}+3\mathbf{k}$.

(ii) Find the position vector of the intersection point between l and π. [3]
(iii) Hence or otherwise, find the equation of the line l', the reflection of line l about π. [2]

[(i) $\overrightarrow{ON}=\begin{pmatrix}1\\2\\0\end{pmatrix}$ (ii) $\begin{pmatrix}2\\2\\-1\end{pmatrix}$ (iii) $\underset{\sim}{r}=\begin{pmatrix}-1\\3\\-2\end{pmatrix}+\gamma\begin{pmatrix}3\\-1\\1\end{pmatrix},\gamma\in\mathbb{R}$]

Section C (Extension/Challenging Questions)

11 9234/ 2003/01/Q9

The line l_1 passes through the point A with position vector $\mathbf{i}-\mathbf{j}-2\mathbf{k}$ and is parallel to the vector $3\mathbf{i}-4\mathbf{j}-2\mathbf{k}$. The variable line l_2 passes through the point

$$(1+5\cos t)\mathbf{i}-(1+5\sin t)\mathbf{j}-14\mathbf{k}, \text{ where } 0 \le t \le 2\pi,$$

and is parallel to that vector $15\mathbf{i}+8\mathbf{j}-3\mathbf{k}$. The points P and Q are on l_1 and l_2 respectively, and PQ is perpendicular to both l_1 and l_2.

(i) Find the length of PQ in terms of t.

(ii) Hence show that the lines l_1 and l_2 do not intersect, and find the maximum length of PQ as t varies.

(iii) The plane π_1 contains the line l_1 and PQ; the plane π_2 contains l_2 and PQ. Find the angle between the planes π_1 and π_2.

[**(i)** $\frac{1}{13}(144-20\cos t-15\sin t)$, **(ii)** 13 units, **(iii)** 78.2°]

12 9234/19 89/01/Q12

With respect to an origin O, the points A, B and C, which are not coplanar with O, have position vectors **a**, **b**, and **c** respectively. The points L, M and N are on the line segment OA, AB, and OC respectively, and are such that $OL = LA$, $2AM = MB$ and $ON = 3NC$. The line MN meets the plane LBC at the point Q.

(i) Write down, in terms of **a**, **b**, and **c**, the position vectors of L, M and N.

(ii) Show that the position vector of any point on MN can be expressed in the form $\frac{2}{3}t\mathbf{a}+\frac{1}{3}t\mathbf{b}+\frac{3}{4}(1-t)\mathbf{c}$.

(iii) Show that the position vector of any point in the plane LBC can be expressed in the form $\frac{1}{2}(1-\lambda-\mu)\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$.

(iv) Hence, or otherwise, find the numerical value of $\dfrac{NQ}{QM}$.

[**(i)** $\overrightarrow{OL}=\frac{1}{2}\mathbf{a}$, $\overrightarrow{OM}=\frac{1}{3}(2\mathbf{a}+\mathbf{b})$, $\overrightarrow{ON}=\frac{3}{4}\mathbf{c}$, **(iv)** $\dfrac{3}{8}$]

13 9234/1991 /01/Q11

The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The plane π, with vector equation $\mathbf{r} = \mathbf{b} + \lambda\mathbf{u} + \mu\mathbf{v}$, where λ and μ are real parameters, contains B and does not contain A.

(i) Show that the perpendicular distance of A from π is p, where $p = \dfrac{|(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{b}-\mathbf{a})|}{|\mathbf{u}\times\mathbf{v}|}$.

(ii) The perpendicular from A to π meets π at C, and D is the point on AB such that CD is perpendicular to AB. Show that $AD = \dfrac{p^2}{AB}$ and hence, or otherwise, show that the position vector of D is $\mathbf{a} + \left(\dfrac{p}{|\mathbf{b}-\mathbf{a}|}\right)^2 (\mathbf{b}-\mathbf{a})$.

In the case where $\mathbf{a} = -\mathbf{i}+7\mathbf{j}+8\mathbf{k}$, $\mathbf{b} = 2\mathbf{i}+7\mathbf{j}+5\mathbf{k}$, $\mathbf{u} = \mathbf{i}-2\mathbf{j}+2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i}+2\mathbf{j}+2\mathbf{k}$, find the value of p, and show that $\overrightarrow{CD} = \dfrac{8\sqrt{2}}{9}\mathbf{x} + \dfrac{4}{9}\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are the unit vectors of $\overrightarrow{CB}$ and $\overrightarrow{CA}$ respectively.

[(ii) $p = 4$]

Section D (Self Practice Questions)

1 JJC Promo 9740/2006/Q9

The plane has vector equation $\mathbf{r} = 9\mathbf{i}+3\mathbf{k}+\alpha(\mathbf{i}+5\mathbf{j}+2\mathbf{k})+\beta(3\mathbf{i}-2\mathbf{j}+6\mathbf{k})$, where α and β are real parameters. The line through O perpendicular to Π meets Π at P. Find the position vector of P.

Given that Q is the point with coordinates $(1, 5, 7)$, find the length of the projection of PQ onto the plane Π.

The plane Π_1 has equation $\mathbf{r}.(-2\mathbf{i}+13\mathbf{j}+25\mathbf{k}) = 11$. Find a vector equation of the line of intersection of Π and Π_1.

[$\overrightarrow{OP} = 6\mathbf{i}-3\mathbf{k}$; $\sqrt{70}$; $\mathbf{r} = \begin{pmatrix} 15/2 \\ 2 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 13 \\ -48 \\ 26 \end{pmatrix}, \mu \in \mathbb{R}$]

2 AJC Mid year Paper 9740/2007/Q9

The equation of the plane Π_1 is $x+y-2z=3$.

(i) Find the vector equation of the line l_1, which lies in both the plane Π_1 and the yz plane.

(ii) Another plane Π_2 contains the line l_2 with equation $x=1, \dfrac{y+1}{2} = z$ and is perpendicular to Π_1. Find the equation of the plane Π_2 in scalar product form.

Determine whether l_1 lies on Π_2.

[(i) $\mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$ (ii) $\mathbf{r}\cdot\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} = 6$]

3 RI Y6 Common Test 9740/2007/Q8 modified

(i) The plane Π_1 and the line l_1 have equations $\mathbf{r}\cdot\begin{pmatrix}2\\1\\2\end{pmatrix}=4$ and $\mathbf{r}=\begin{pmatrix}0\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-2\end{pmatrix}$ where $\lambda\in\mathbb{R}$, respectively. Show that l_1 lies on Π_1.

(ii) Another plane Π_2 contains l_1 and is perpendicular to Π_1. Find an equation of Π_2 in the form $\mathbf{r}\cdot\mathbf{n}=p$.

(iii) A third plane Π_3 has a normal parallel to $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and passes through the origin. Write down an equation for Π_3.

[(ii) $\mathbf{r}\cdot\begin{pmatrix}2\\-2\\-1\end{pmatrix}=-5$ (iii) $\mathbf{r}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=0$]

4 TJC Mid Year Paper 9740/2007/Q7

The line ℓ whose vector equation is $\mathbf{r}=5\mathbf{i}+2\mathbf{j}-3\mathbf{k}+\alpha(7\mathbf{i}-2\mathbf{j}-3\mathbf{k})$ passes through the point A with position vector $5\mathbf{i}+2\mathbf{j}-3\mathbf{k}$. The plane π whose vector equation is $\mathbf{r}=-5\mathbf{i}+2\mathbf{j}+2\mathbf{k}+\beta(7\mathbf{i}-2\mathbf{j}-3\mathbf{k})+\gamma(9\mathbf{i}-\mathbf{j}+\mathbf{k})$ contains the point B with position vector $-5\mathbf{i}+2\mathbf{j}+2\mathbf{k}$.

(i) Find the position vector of the point P on the line segment AB such that $AP:PB=4:1$.

(ii) The plane π_1 contains the line ℓ and the point P.

Write down a vector equation of the line of intersection of π and π_1.

Find the vector equation of π_1 and the angle between π and π_1.

Hence find the ratio of the perpendicular distances from P to the line ℓ and from P to the plane π.

[(i) $\begin{pmatrix}-3\\2\\1\end{pmatrix}$ (ii) $\underset{\sim}{r}=\begin{pmatrix}5\\2\\-3\end{pmatrix}+s\begin{pmatrix}7\\-2\\-3\end{pmatrix}+t\begin{pmatrix}-2\\0\\1\end{pmatrix}$ or $\underset{\sim}{r}\cdot\begin{pmatrix}2\\1\\4\end{pmatrix}=0$; 90^0; 4:1]

5 IJC Prelim 9740/2008/02/Q4 modified

The line l has equation $\mathbf{r}=\mathbf{j}+\mathbf{k}+\lambda(\mathbf{i}-2\mathbf{j}+\mathbf{k}), \lambda\in\mathbb{R}$. The plane π_1 has equation $x+2y+3z=5$. The point A on l is given by $\lambda=2$ and the point B has position vector $2\mathbf{i}+\mathbf{j}+4\mathbf{k}$.

(i) Show that the line l lies in the plane π_1. [2]

(ii) Find the acute angle between the line AB and the plane π_1. [4]

(iii) The plane π_2 is perpendicular to the plane π_1 and parallel to the line l, and contains the point B. Find the equation of π_2. [3]

[(ii) 45.5^{o} (iii) $\mathbf{r}\cdot\begin{pmatrix}4\\1\\-2\end{pmatrix}=1$]

THE END