ANGLO-CHINESE JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

Higher 2

/100

CANDIDATE NAME

TUTORIAL/ FORM CLASS

INDEX NUMBER

MATHEMATICS 9758/01

Paper 1

25 August 2020

3 hours

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your index number, class and name on all the work you hand in.
Write in dark blue or black pen.

You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

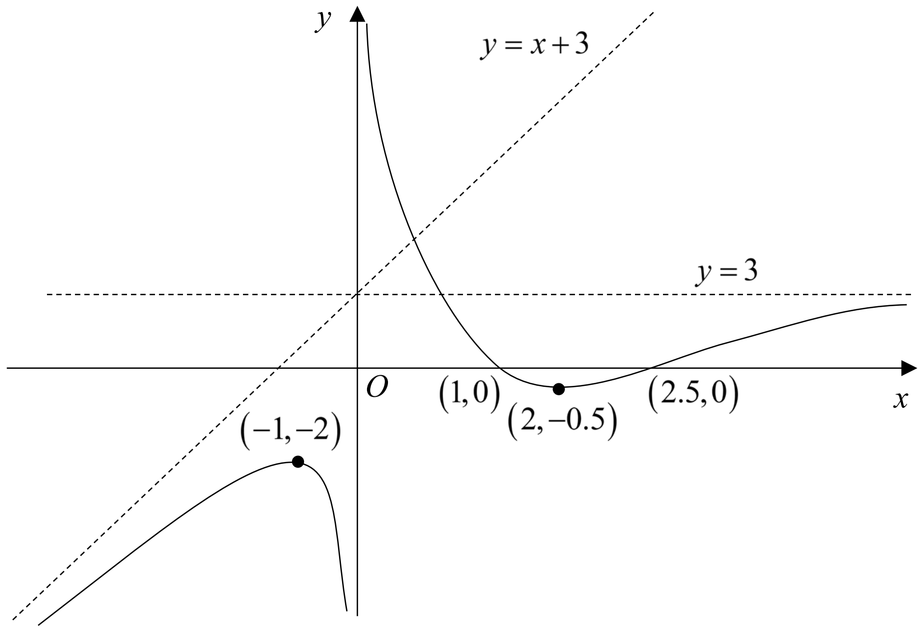
Question	Marks
1	/6
2	/6
3	/7
4	/8
5	/8
6	/9
7	/10
8	/10
9	/12
10	/12
11	/12

This document consists of __ printed pages.

1 **(i)** Use the substitution $u = 2x - 1$ to find $\displaystyle\int \frac{x}{\sqrt{1-(2x-1)^2}}\,\mathrm{d}x$. [4]

(ii) Hence find $\displaystyle\int \sin^{-1}(2x-1)\,\mathrm{d}x$. [2]

2



The diagram above shows the graph of $y = \mathrm{f}(x)$. The curve has turning points $(-1,-2)$ and $(2,-0.5)$. It cuts the x-axis at $x = 1$ and $x = 2.5$. The curve has asymptotes with equations $x = 0$, $y = 3$ and $y = x + 3$.

Sketch, on separate diagrams, the graphs of

(i) $y = \mathrm{f}'(x)$, [3]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

including the coordinates of the points where the graphs cross the axes, the turning points, and the equations of any asymptotes if possible.

3 It is given that $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \mathrm{e}^{3x}\sec 2x$.

(i) Show that $\mathrm{f}'(x) = y(3 + 2\tan 2x)$. [1]

(ii) Find the Maclaurin series for $\mathrm{f}(x)$, up to and including the term in x^2. [3]

(iii) Hence find the first two non-zero terms in the Maclaurin series for $(\mathrm{e}^{3x}\sec 2x)(3 + 2\tan 2x)$. [1]

(iv) Using the standard series from the List of Formulae (MF26), verify the result found in part **(ii)**. [2]

4 **(i)** Given that $a < 0$ and $b < -1$, find the roots of the equation $b|x - a| = x - ab$, where a and b are constants. [2]

(ii) If $a < b < 0$,

(a) write down a condition for $b|x - a| = x - ab$ to have a negative real root. [1]

(b) using the same condition from **(a)**, on the same axes, sketch the graphs of $y = x - ab$ and $y = b|x - a|$. [2]

Hence solve the inequality

$$b|x - a| \geq x - ab.$$ [1]

Deduce the set of values of x for which

$$-b|\mathrm{e}^x + a| \leq \mathrm{e}^x + ab.$$ [2]

5 **(i)** Let $\mathrm{f}(n) = \dfrac{3^n}{n}$, where n is a positive integer. Show that

$$\mathrm{f}(n+1) - \mathrm{f}(n) = \frac{3^n(2n-1)}{n(n+1)}.$$ [1]

(ii) Hence find $\displaystyle\sum_{n=1}^{N}\left(\frac{3^n}{n+1}\left(1 - \frac{1}{2n}\right)\right)$ in terms of N. [4]

(iii) Deduce $\displaystyle\sum_{n=1}^{N}\frac{3^n(2n+1)}{(n+1)(n+2)}$, leaving your answer in the form $a\left(\dfrac{3^{N+2}}{N+2} - b\right)$, where a and b are constants to be determined. [3]

6 The function f is defined by

$$\mathrm{f}: x \mapsto \begin{cases} 1-|3x-1| & \text{for } 0 \le x \le 2, \\ (x-2)^2-4 & \text{for } 2 < x \le 4. \end{cases}$$

It is given that $\mathrm{f}(x+4)=\mathrm{f}(x)$ for all real values of x.

(i) Sketch the graph of $y=\mathrm{f}(x)$ for $-1 \le x \le 9$, stating the coordinates of the end points.

Find the range of f. [4]

The function g is defined by

$$\mathrm{g}: x \mapsto \frac{1}{1+(x-1)^2}, \quad x \in \mathbb{R}, \ -2 \le x \le 2.$$

The graph of $y=\mathrm{h}(x)$ is obtained by applying the following transformations in succession to the graph of $y=\mathrm{g}(x)$:

(A) Reflection in the y-axis
(B) Stretch with scale factor 2 parallel to the x-axis
(C) Stretch with scale factor 3 parallel to the y-axis

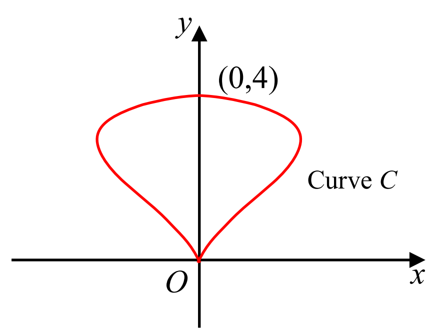
(ii) Find $\mathrm{h}(x)$ and state the domain of h. [3]

(iii) Prove that $\mathrm{hf}(x)$ exists and find its range. [2]

7 A hot air balloon company gives a sticker to each passenger who is going for a ride. The sticker is in the shape of a curve C given by the parametric equations

$$x = 2\cos\theta + \sin 2\theta, \quad y = 2 + 2\sin\theta, \quad \text{for } -\pi < \theta \le \pi.$$

The curve has a y-intercept at (0, 4).



(i) **Use a non-calculator method** to find the exact area of the sticker in terms of π. [5]

(ii) Prove that $x = y\cos\theta$. Deduce that the Cartesian equation of the curve C is

$$4x^2 = 4y^3 - y^4.$$ [3]

(iii) A model of a hot air balloon is made by rotating the curve C about the y-axis through π radians. Find the volume of the model. [2]

8 A hollow circular cylinder of diameter 42 mm is wrapped with n layers of thin film. The first layer of film is in contact with the outer surface of the cylinder and it has length 42π mm. The second layer is in contact with the first layer and has length $(42 + 2x)\pi$ mm, where x mm is the thickness of the thin film. There is no gap between any two layers. The n^{th} layer has length 124π mm.

(i) Show that the thickness of the film satisfies the equation $x(n - 1) = 41$. [2]

(ii) Hence find x, given that the total sum of the lengths of the n layers of film is 16766π mm. [3]

Another roll of film which has length 52580 mm is to be cut into n pieces of increasing lengths, where the lengths form a geometric progression with common ratio r. It is given that the sum of the lengths of the first ten pieces of film is k times the sum of the lengths of the first five pieces of film, where k is an unknown constant.

(iii) Show that $k = 1 + r^5$. [2]

(iv) Given that $k = 33$ and that the first piece of film is of length 50 mm, find the largest possible value of n. [3]

9 **(i)** Given that $\sin x + \sqrt{3}\cos x = R\sin\left(x + \frac{\pi}{3}\right)$, find R, where $R > 0$. [1]

(ii) The function f is defined by

$$\mathrm{f} : x \mapsto \sin x + \sqrt{3}\cos x, \quad x \in \mathbb{R},\ 0 \le x \le \pi.$$

Using a graphical method, explain why f^{-1} does not exist. [2]

(iii) The function g is defined by

$$\mathrm{g} : x \mapsto \mathrm{f}(x + a), \quad x \in \mathbb{R}, \quad 0 \le x \le \pi, \text{ where } a > 0.$$

State in exact form the minimum value of a such that g^{-1} exists. [1]

Using this value of a, sketch on the same diagram the graphs of $y = \mathrm{g}(x)$ and $y = \mathrm{g}^{-1}(x)$, indicating the equation of the line of symmetry and the coordinates of the endpoints. [3]

Determine also the solution to $\mathrm{gg}^{-1}(x) = \mathrm{g}^{-1}\mathrm{g}(x)$. [1]

(iv) The function h is defined by

$$\mathrm{h} : x \mapsto xe^{x^2}, \quad x \in \mathbb{R}.$$

Show that h is an increasing function. Hence, using the value of a found in **(iii)**, find the range of values of x for which $\mathrm{h}^{-1}\mathrm{g}(x) \le 0.5$. [4]

10 In the study of light, a straight line is used to model a ray of light. A scientist wishes to conduct an experiment using light rays.

Two light rays L_1 and L_2 have equations as follows.

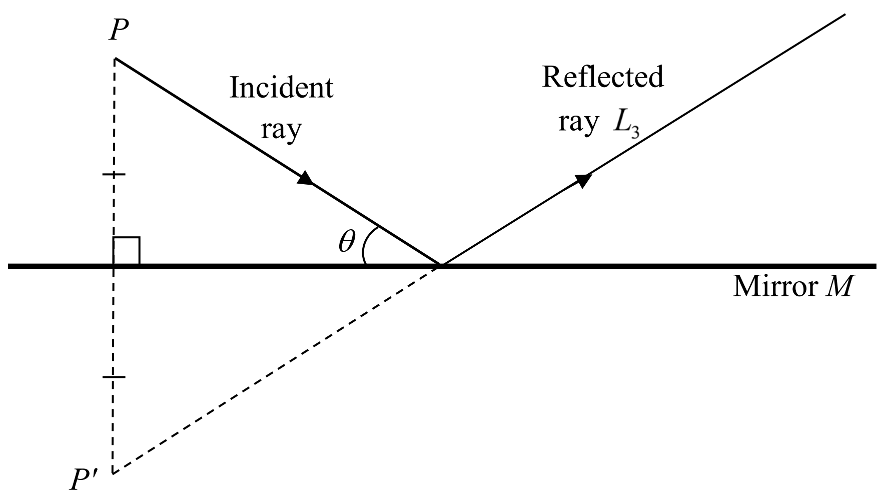
$$L_1: \ \mathbf{r} = \mathbf{i} + 2\mathbf{j} + s(-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}),$$

$$L_2: \mathbf{r} = 5\mathbf{j} - \frac{9}{2}\mathbf{k} + t(2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}),$$

where s and t are parameters.

(i) Show that lines L_1 and L_2 are skew lines. [2]

A scientist uses a light source to shine an incident ray of light from the point P with coordinates $(1,5,0)$ such that it makes an angle of incidence, θ, with a plane mirror M, as shown in the diagram below. M is modelled by a plane with equation $y - z - 2 = 0$. The scientist wishes to find θ such that the reflected light ray from the mirror, denoted by L_3, cuts both the skew lines L_1 and L_3.



(ii) Find the coordinates of P', the reflection of point P in M. [3]

(iii) Find the cartesian equation of the plane Π containing point P' and line L_1. [2]

(iv) Find the coordinates of the point of intersection of the plane Π and line L_2. [2]

(v) Hence find the cartesian equation of the reflected ray, line L_3, that cuts both the lines L_1 and L_2. [2]

(vi) Find the angle of incidence, θ, that the scientist should use. [1]

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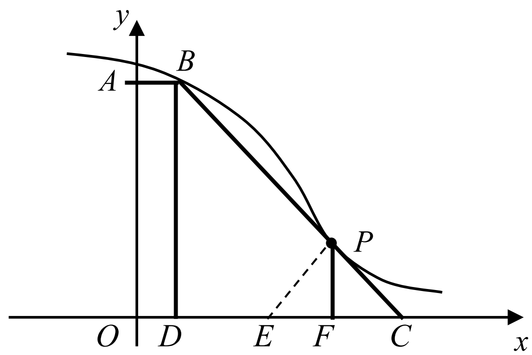

The diagram above shows part of a roller coaster track, made up by a curve with parametric equations

$$x = 3t^2 - 10t - 1, \qquad y = 16\tan^{-1} t^3 - 4t + 16, \qquad \text{for } -2 \le t \le 0,$$

with points *B* and *P* on the curve.

Rods are used to support the track. *AB* is a horizontal metal rod. Another straight metal rod, fixed from point *B* to *C*, is tangential to the track at point *P*. In order to further strengthen the support, vertical metal rods *BD* and *PF*, of length *m* metres and $4(5-\pi)$ metres respectively, are fixed.

(i) Find the *x*-coordinate of point *P*. [2]

Hence show that the equation of line *BC* is $5x + 4y + 16\pi - 140 = 0$. [3]

(ii) Find *m*, giving your answer to 3 decimal places. [3]

For regular maintenance, an extendable ladder, fixed at point *E*, is extended till it meets rod *BC*. If the ladder and rod *BC* meet at point *P*, then the length of the extended ladder, *l*, is at its shortest length. Find the exact value of *l*. [4]