RAFFLES INSTITUTION
2024 YEAR 6 PRELIMINARY EXAMINATION

CANDIDATE NAME

CLASS 24

MATHEMATICS **9758/01**
Paper 1 **3 hours**

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper. **You may use the blank pages on page 22, 23 and 24 if necessary and you are reminded to indicate the question number(s) clearly.**
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

For Examiner's Use Only					
Q1	Q2	Q3	Q4	Q5	Q6
/ 4	/ 5	/ 8	/ 8	/ 8	/ 7
Q7	Q8	Q9	Q10	Q11	TOTAL
/ 10	/ 12	/ 12	/ 14	/ 12	/ 100

This document consists of **21** printed pages and **3** blank pages.

1 A function f is defined by $\mathrm{f}(x) = ax^3 + bx^2 + cx + d$. The graph of $y = \mathrm{f}(x)$ passes through the points $(-3, 4)$ and $(1, 8)$. Given that the graph of $y = \dfrac{1}{\mathrm{f}(x)}$ has a turning point at $(2, \frac{1}{4})$, find the values of a, b, c and d. [4]

2 [The volume of a sphere with radius r is given by $\frac{4}{3}\pi r^3$ and the surface area of a sphere with radius r is given by $4\pi r^2$.]

(a) The volume of an expanding sphere is increasing at a constant rate of $5 \text{ cm}^3\text{s}^{-1}$. Show that, at any instant, the rate of increase of the surface area is $\dfrac{k}{r} \text{ cm}^2\text{s}^{-1}$, where r is the radius of the sphere and k is a constant to be determined. [3]

(b) Find the exact rate of change of surface area of the expanding sphere when the surface area is 20 cm^2. [2]

3 **(a)** Without using a calculator, solve exactly $\frac{x^2-x-1}{x+1} \le 1$. [4]

(b) Hence solve exactly $\frac{x^2+|x|-1}{1-|x|} \le 1$. [4]

4 **(a)** Find $\displaystyle\int \frac{\cos x}{\cos 3x + \cos x}\, \mathrm{d}x$. [3]

(b) Find $\displaystyle\int x \tan^{-1} x^2 \, \mathrm{d}x$. Hence find the exact value of $\displaystyle\int_{-1}^{1} \left| x \tan^{-1} x^2 \right| \mathrm{d}x$. [5]

5 **(a)** Using the formulae for $\sin(A \pm B)$, prove that

$$\sin(2r+1)\theta - \sin(2r-1)\theta = 2\cos 2r\theta \sin\theta.$$ [1]

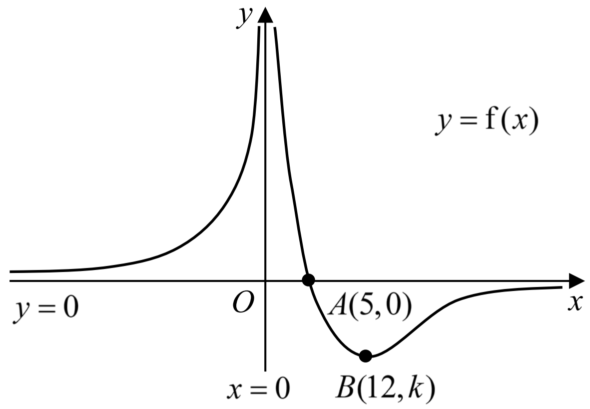
(b) Hence find a formula for $\sum_{r=1}^{n} \cos 2r\theta$, where $0 < \theta < \pi$, in terms of $\sin(2n+1)\theta$ and $\sin\theta$. [3]

(c) Using the formula found in part **(b)**, show that the sum of the series

$$\sin^2 10\theta + \sin^2 11\theta + \sin^2 12\theta + ... + \sin^2 20\theta, \text{ for } 0 < \theta < \pi$$

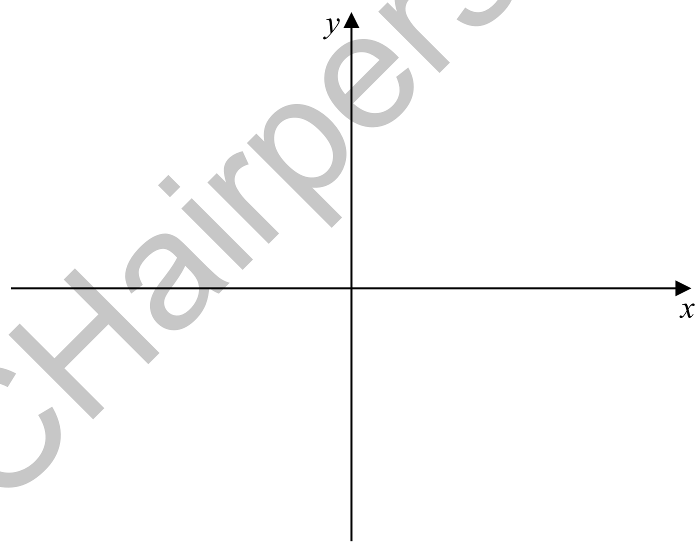
is $k - \dfrac{\sin(41\theta) - \sin(19\theta)}{4\sin\theta}$, where k is a constant to be determined. [4]

6 **(a)** The diagram below shows the graph of $y = f(x)$.

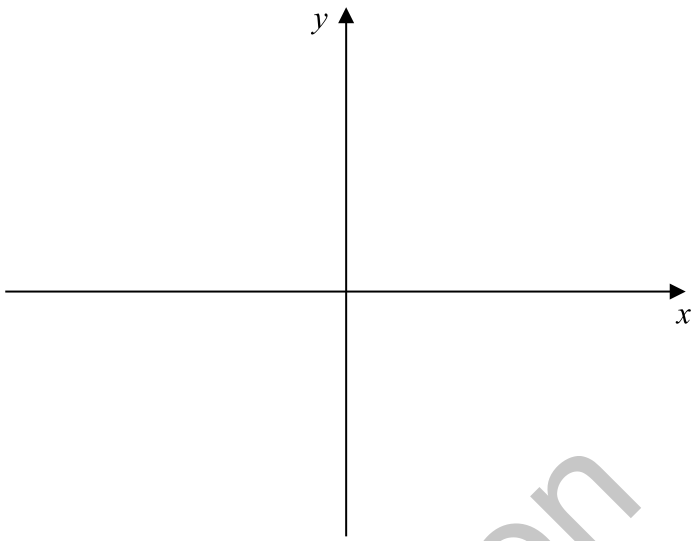


The graph cuts the x-axis at point $A(5,0)$. It has a turning point at $B(12,k)$, where $k<0$ and asymptotes with equations $x=0$ and $y=0$. On separate diagrams, sketch the graph of

(i) $y = 2f(x)+k$, stating the equations of any asymptotes and the coordinates of any turning point(s). [2]



(ii) $y = \mathrm{f}'(x)$, stating the equations of any asymptotes and the coordinates of any point(s) where the curve crosses the axes. [2]



(b) The graph with equation $y = \mathrm{g}(x)$, where $\mathrm{g}(x) = x(x-1)^2$ undergoes a single transformation and the equation of the resultant graph is $y = \mathrm{h}(x)$. Describe the transformation if

(i) $\mathrm{h}(x) = -x(x+1)^2$, [1]

(ii) $\mathrm{h}(x) = \dfrac{1}{8}x(x-2)^2$. [2]

7 An arithmetic series has first term a and common difference d, where $a > 0$ and $d \neq 0$. The first, sixth and ninth terms of the arithmetic series are consecutive terms of a geometric series.

(a) Show that $25d = -2a$. [2]

(b) The sum of the first n terms of the arithmetic series is denoted by S. Find the set of possible values of n for which S exceeds $6a$. [3]

(c) Find the common ratio of the geometric series, and deduce that the geometric series is convergent. [2]

(d) Hence find the smallest value of m such that the sum of the terms of the geometric series after, but not including, the mth term, is less than 1% of the sum to infinity. [3]

8 **Do not use a calculator in answering this question.**

(a) The complex number w is such that $w = a + \mathrm{i}b$, where a and b are non-zero real numbers. The complex conjugate of w is denoted by w^*. Given that

$$ww^* = 4 - 2\mathrm{i} + 2\mathrm{i}w^*,$$

find the two possible values of w. [4]

(b) The complex number z is given by $z = \dfrac{1-\sqrt{3}\,\mathrm{i}}{-1+\mathrm{i}}$.

(i) Find $\arg(z)$. [3]

(ii) Find z in cartesian form $x+\mathrm{i}y$. [2]

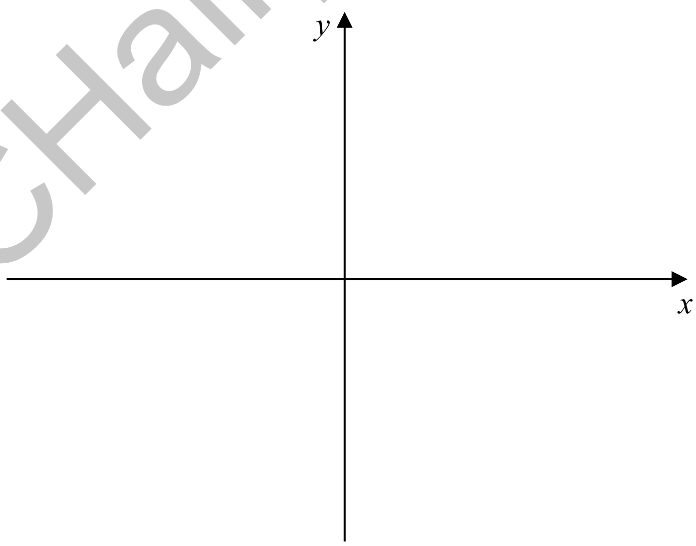
(iii) Hence find the value of $\tan\frac{\pi}{12}$ in the form $c+d\sqrt{3}$, where c and d are integers to be found. [3]

9 A curve C has equation $y = 2ax + \dfrac{b}{x}$ where a and b are non-zero real constants and $x \neq 0$.

(a) Using differentiation, determine whether C has any stationary points if $ab < 0$. [2]

It is now given that $b = \dfrac{1}{2}a$ where $a > 0$.

(b) Sketch C, stating the equations of any asymptotes, and the coordinates of any stationary points and points where C crosses the axes (if any). [3]



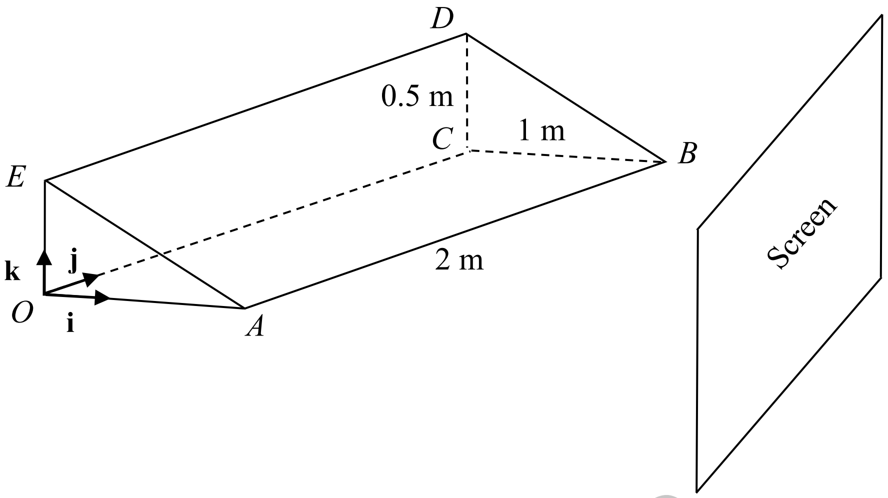
(c) State the range of values of k, in terms of a, for which the equation $2ax+\dfrac{b}{x}=kx$ has no real roots. [1]

(d) The region bounded by C, the axes, the lines $x=\dfrac{1}{2}$ and $y=4a$ is rotated about the y-axis through 2π radians. Show that the volume generated is given by

$$\frac{1}{2}a\pi+\frac{\pi}{8a^2}\int_{2a}^{4a} y^2-2a^2-y\sqrt{y^2-4a^2}\ \mathrm{d}y\,.$$

Hence, find, in terms of a and π, the exact volume generated. [6]

10 For an upcoming motor-car race, a spectators' gallery is to be set up near the racing track. As part of the preparations, a model of this gallery, shaped as a prism, is constructed.



The diagram above shows the model of the gallery with O as the origin and the unit vectors **i**, **j** and **k** are parallel to OA, OC and OE respectively. Points (x, y, z) are defined relative to O, where units are in metres.

It is given that $OA = CB = 1$ m, $OC = AB = ED = 2$ m and $OE = CD = 0.5$ m.

(a) Find a cartesian equation of the plane $ABDE$. [2]

A shelter is to be constructed above the plane *ABDE*. On the model, this shelter is a rectangular plane that intersects plane *ABDE* in the line *ED*.

(b) Given that the equation of the shelter is $-0.5x+z=h$, show that $h=0.5$. [1]

(c) Find the acute angle between the plane *ABDE* and the shelter. [2]

A spotlight is shone towards the gallery and the beam of light is in the form of a line *l* with cartesian equation $x-a=z,\ y=1$ for some real number *a*. The beam lands on a point *M* on the rectangular surface *ABDE*.

(d) Find in terms of *a*, the position vector of *M*, and hence find the range of values of *a*. [5]

(e) A large rectangular flat screen is to be placed in front of the gallery at a distance away. The screen can be taken to be part of a vertical plane with equation $12x+5y=d$, where $d>50$. Using $a=\frac{1}{2}$, find the value of d so that the shortest distance between point M and the screen is 4 m. [4]

11 A wafer fabrication company uses the floating-zone method to purify polysilicon ingots, each having a uniform cross-sectional area and a length of 200 cm. The method involves placing a polysilicon ingot with impurity concentration C_0 atoms/cm^3 on top of a single seed crystal. The polysilicon ingot is then heated externally by an RF coil, which locally melts the ingot. The impurities prefer to stay in the molten state than in the solid state and thus as the coil and the melt zone move upwards, a single crystal, that is purer, solidifies on top of the seed crystal. A schematic illustration of the method is shown below in Fig. 1 and Fig. 2.

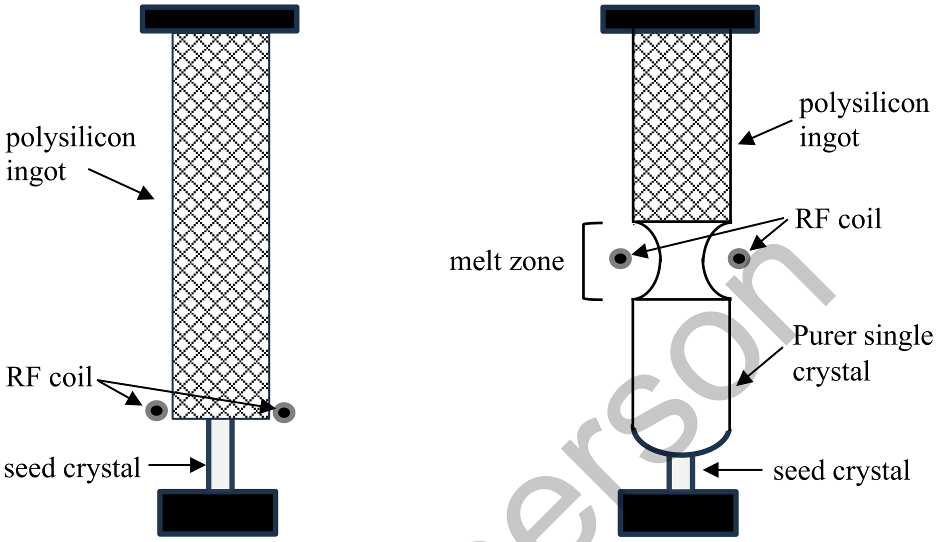


Fig. 1 Initial set-up Fig. 2 During purification process

For a ‘floating’ melt zone of length L cm, the concentration of impurities in the melt zone, C atoms/cm^3, and the distance moved by the RF coil, x cm, are related by the differential equation

$$\frac{\mathrm{d}C}{\mathrm{d}x} = \frac{1}{L}\left(C_0 - kC\right),$$

where k is a constant such that $0 < k < 1$.

The length of the “floating” melt zone, L cm, adopted by the company is 2 cm and $0 \le x \le 198$. It is also given that when $x = 0$, $C = C_0$.

(a) Solve the differential equation to find an expression for C in terms of C_0, k and x. [4]

(b) Sketch the graph of C against x. [2]

(c) Assume that $k = 0.3$ and that the RF coil moves upwards at a constant speed of 8 mm per hour. Find the time taken for the concentration of impurities in the melt zone to reach $2C_0$ and the rate of change of the concentration of impurities, in terms of C_0 at this instant. [5]

The company decides to change the length of the "floating" melt zone.

(d) Explain, with a reason, whether a shorter length is preferable over a longer one. [1]

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