H2 Mathematics 9758
Topic 11: Differentiation Applications
Tutorial Worksheet

1 Given that $y=3x^4-16x^3+24x^2-9$. Without the use of a GC,

(a) find the stationary point(s) of its graph and determine their nature.

$(0,-9)$ min pt, $(2,7)$ stat pt of inflex

(b) find the range of x, if any, for which the curve is

(i) strictly increasing, $0<x<2$ or $x>2$

(ii) concave downwards. $\frac{2}{3}<x<2$

(c) sketch the curve showing all the important features.

(a)

$y=3x^4-16x^3+24x^2-9 \quad \Rightarrow \quad \frac{dy}{dx}=12x^3-48x^2+48x=12x(x^2-4x+4)=12x(x-2)^2$

At stationary points, $\frac{dy}{dx}=0 \quad \Rightarrow \quad x=0, \quad 2$. Also, $\frac{d^2y}{dx^2}=36x^2-96x+48$

When $x=0, \quad y=-9$ and $\frac{d^2y}{dx^2}=48>0.$ So $(0,-9)$ is a minimum point.

When $x=2, \quad y=7$. Note that $x=2^-$, $\frac{dy}{dx}>0$, and, when $x=2^+$, $\frac{dy}{dx}>0$.

Hence, $(2, 7)$ is a stationary point of inflexion.

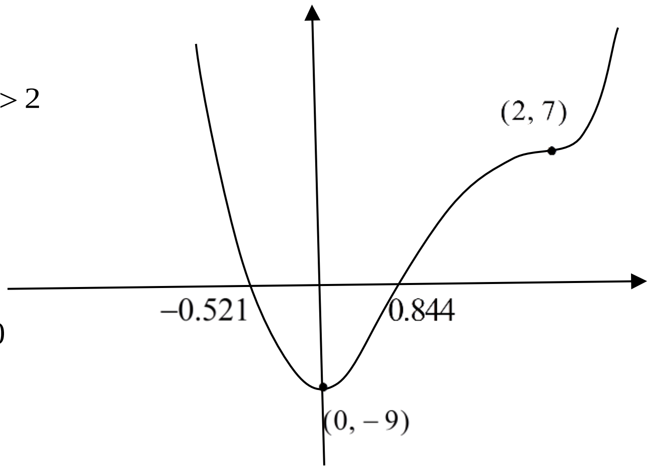
(b) & (c)

(i) For strictly increasing,

$$\frac{dy}{dx}>0 \quad \Rightarrow \quad 0<x<2 \text{ or } x>2$$

(ii) For concave downwards,

$$\frac{d^2y}{dx^2}=36x^2-96x+48$$
$$=12(3x^2-8x+4)$$
$$=12(3x-2)(x-2)<0$$
$$\Rightarrow \frac{2}{3}<x<2$$



2 N10/I/Q4

Given that $x^2-y^2+2xy+4=0$, find $\frac{dy}{dx}$ in terms of x and y.

For the curve with equation $x^2-y^2+2xy+4=0$, find the coordinates of each point at which the tangent is parallel to x-axis.

$\left(\sqrt{2},-\sqrt{2}\right), \left(-\sqrt{2},\sqrt{2}\right)$

$x^2-y^2+2xy+4=0$ Differentiate with respect to x $2x-2y\frac{dy}{dx}+2x\frac{dy}{dx}+2y=0$ $2x+2y=\frac{dy}{dx}(2y-2x)$ $\frac{dy}{dx}=\frac{2x+2y}{2y-2x}$ For tangents parallel to x-axis, $\frac{dy}{dx}=0$ $\frac{2x+2y}{2y-2x}=0 \quad \Rightarrow x=-y$	Substitute into equation of curve $(-y)^2-y^2+2(-y)y+4=0$ $-2y^2+4=0$ $y=\pm\sqrt{2}$ Hence the coordinates of the points whose tangent are parallel to the x-axis are $(\sqrt{2},-\sqrt{2})$ and $(-\sqrt{2},\sqrt{2})$

3 N2012/I/8

The curve C has equation

$$x-y=(x+y)^2$$

It is given that C has only one turning point.

(i) Show that $1+\frac{dy}{dx}=\frac{2}{2x+2y+1}$.

(ii) Hence, or otherwise, show that $\frac{d^2y}{dx^2}=-\left(1+\frac{dy}{dx}\right)^3$

(iii) Hence state, with a reason, whether the turning point is a maximium or a minimium.

(i)	$x-y=(x+y)^2$ differentiate wrt x: $1-\frac{dy}{dx}=2(x+y)\left(1+\frac{dy}{dx}\right)$ $(2x+2y+1)\frac{dy}{dx}=1-2x-2y$ $\frac{dy}{dx}=\frac{1-2x-2y}{2x+2y+1}$ $1+\frac{dy}{dx}=\frac{2x+2y+1+1-2x-2y}{2x+2y+1}=\frac{2}{2x+2y+1}$

(ii)	Differentiate wrt x: $\frac{d^2y}{dx^2}=\frac{-2}{(2x+2y+1)^2}\left(2+2\frac{dy}{dx}\right)$ $=-\frac{1}{2}\left(\frac{4}{(2x+2y+1)^2}\right)\times 2\left(1+\frac{dy}{dx}\right)$ $=-\left(1+\frac{dy}{dx}\right)^2\left(1+\frac{dy}{dx}\right)$ $=-\left(1+\frac{dy}{dx}\right)^3$
(iii)	When $\frac{dy}{dx}=0$, $\frac{d^2y}{dx^2}=-1<0$ Therefore, the turning point is a maximum point.

4 The parametric equations of a curve are $x=ct,\ y=\frac{c}{t^2}$, where c is a constant.

(i) Find the equation of the tangent to the curve at the point $P\left(cp,\frac{c}{p^2}\right)$.

$$p^3y+2x=3cp$$

(ii) Hence find the coordinates of the points Q and R where the tangent meets the x- and y-axes respectively.

$$Q\left(\frac{3cp}{2},\ 0\right),\ R\left(0,\ \frac{3c}{p^2}\right)$$

(iii) Find the Cartesian equation of the curve.

$$x^2y=c^3$$

(iv) Find a Cartesian equation of the locus of the mid-point of QR as p varies.

$$x^2y=\frac{27c^3}{32}$$

(i)	$x=ct \Rightarrow \frac{dx}{dt}=c$, and, $y=\frac{c}{t^2} \Rightarrow \frac{dy}{dt}=-\frac{2c}{t^3}$ $\Rightarrow \frac{dy}{dx}=\frac{-\frac{2c}{t^3}}{c}=-\frac{2}{t^3}$ At point $P, t=p$, we have $\frac{dy}{dx}=-\frac{2}{p^3}$.

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Thus, the equation of the tangent to the curve at the point $P\left(cp, \frac{c}{p^2}\right)$ is:

$$y-\frac{c}{p^2}=-\frac{2}{p^3}(x-cp) \quad \Rightarrow \quad p^3y-cp=-2x+2cp \quad \Rightarrow \quad p^3y+2x=3cp$$

(ii) This tangent meets the x- axes at point Q, to find the coordinates of Q, we let $y=0$,

$$p^3(o)+2x=3cp \quad \Rightarrow \quad x=\frac{3cp}{2}. \quad \text{That is } Q\left(\frac{3cp}{2},\ 0\right).$$

Also this tangent meets the y- axes at point R, to find the coordinates of R, we let $x=0$,

$$p^3y+2(0)=3cp \quad \Rightarrow \quad y=\frac{3c}{p^2}. \quad \text{That is } R\left(0,\ \frac{3c}{p^2}\right).$$

(iii) The Cartesian equation of the curve can be easily obtained by eliminating the parameter t from the given parametric equations

$$x=ct, \quad y=\frac{c}{t^2} \quad \Rightarrow \quad x^2y=(ct)^2\cdot\frac{c}{t^2}=c^3$$

The Cartesian equation of the curve is : $x^2y=c^3$

(iv)

To find a Cartesian equation of the locus of the mid-point of QR as p varies, we let (x, y) be the coordinate of the mid-point , then

$$x=\frac{\frac{3cp}{2}+0}{2}=\frac{3cp}{4}, \quad \text{and ,} \quad y=\frac{\frac{3c}{p^2}+0}{2}=\frac{3c}{2p^2}.$$

By eliminating the variable p, we have $x^2y=\left(\frac{3cp}{4}\right)^2\cdot\frac{3c}{2p^2}=\frac{27c^3}{32}$.

Therefore, the Cartesian equation of the locus of the mid-point of QR is: $x^2y=\frac{27c^3}{32}$.

5 The line l is tangent to the curve $y=1-\frac{1}{x}$, where $x\neq 0$, at the point where $x=a$.

(i) Show that the equation of l may be expressed in the form $a^2y-x=a^2-2a$.

(ii) The tangent line l passes through a fixed point (X,Y). Give a brief argument to explain why there cannot be more than 2 tangents passing through (X,Y).

(iii) Find the value of a for which the line l passes through the origin, and find the equation of l in this case.

$a=2$, $y=\frac{x}{4}$

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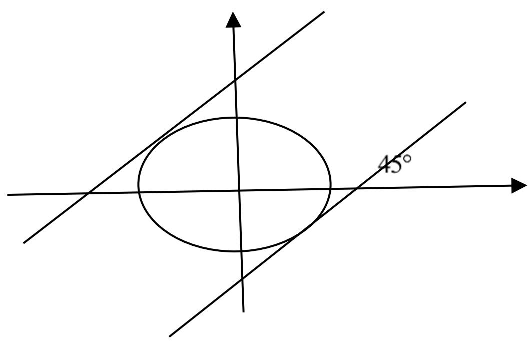
(i) $y=1-\frac{1}{x}=1-x^{-1}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x^2}$ At $x=a$, $y=1-\frac{1}{a}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{a^2}$ Equation of line *l*, $y-\left(1-\frac{1}{a}\right)=\frac{1}{a^2}(x-a)$ $y-1+\frac{1}{a}=\frac{x}{a^2}-\frac{1}{a}$ $a^2y-a^2+a=x-a$ $a^2y-x=a^2-2a$ (shown) $\cdots(1)$	(ii) Since for any fixed point (X,Y), $a^2Y-X=a^2-2a$ is a quadratic equation in *a*, there are at most 2 real solutions for *a*. Therefore, there cannot be more than 2 tangents passing through (X,Y). (iii) Substitute $x=0, y=0$ into (1), $a^2-2a=0$ $a(a-2)=0$ $a=0$ (rej) or $a=2$ Equation of line *l*, $2^2y-x=0$ $y=\frac{x}{4}$

6 Find the equations of the tangents to the curve with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad a,b>0$, which makes an angle of $45°$ with the positive *x*-axis, measured in an anticlockwise sense.

$y=x-\sqrt{a^2+b^2}$ or $y=x+\sqrt{a^2+b^2}$

At the points on the curve where the tangents make an angle of $45°$ with the positive *x* – axis, the gradient of the tangents = $\frac{dy}{dx}=\tan 45^\circ=1$.

Given equation of curve: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Differentiating w.r.t. *x*, $\frac{2x}{a^2}+\frac{2y\frac{dy}{dx}}{b^2}=0$

Substitute $\frac{dy}{dx}=1$ into the above, we have

$\frac{2x}{a^2}+\frac{2y\,(1)}{b^2}=0 \quad\Rightarrow y=-\frac{b^2x}{a^2}$

To find the coordinates of the points at which the gradient is $\frac{dy}{dx}=1$:

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Substitute $y=-\dfrac{b^2x}{a^2}$ into equation of the curve,

$$\frac{x^2}{a^2}+\frac{1}{b^2}\left(-\frac{b^2x}{a^2}\right)^2=1 \quad\Rightarrow\quad \frac{x^2}{a^2}+\frac{b^2x^2}{a^4}=1 \quad\Rightarrow\quad x=\pm\frac{a^2}{\sqrt{a^2+b^2}}$$

When $x=\dfrac{a^2}{\sqrt{a^2+b^2}}$, $y=-\dfrac{b^2x}{a^2}=-\dfrac{b^2}{\sqrt{a^2+b^2}}$.

When $x=-\dfrac{a^2}{\sqrt{a^2+b^2}}$, $y=-\dfrac{b^2x}{a^2}=\dfrac{b^2}{\sqrt{a^2+b^2}}$.

At $\left(\dfrac{a^2}{\sqrt{a^2+b^2}},-\dfrac{b^2}{\sqrt{a^2+b^2}}\right)$, the equation of the tangent is :

$$y-\left(-\frac{b^2}{\sqrt{a^2+b^2}}\right)=1\left(x-\frac{a^2}{\sqrt{a^2+b^2}}\right) \quad\Rightarrow\quad y=x-\sqrt{a^2+b^2}$$

At $\left(-\dfrac{a^2}{\sqrt{a^2+b^2}},\dfrac{b^2}{\sqrt{a^2+b^2}}\right)$, the equation of the tangent is :

$$y-\frac{b^2}{\sqrt{a^2+b^2}}=1\left(x-\left(-\frac{a^2}{\sqrt{a^2+b^2}}\right)\right) \quad\Rightarrow\quad y=x+\sqrt{a^2+b^2}$$

7 NJC Prelim 09/I/7 (Anchor Q – Connected rate of change)
In a triangle ABC, $AB=3$ cm and $AC=2$ cm. If angle BAC is increasing at a constant rate of 0.1 radians per second, find the rate of increase of the length BC at the instant where $BAC=\dfrac{\pi}{3}$ radians. 0.196 cm/s [4]

Let $BC=x$ cm and $\angle BAC=\theta$ radians.

By cosine rule, $x^2=3^2+2^2-2(3)(2)\cos\theta$

$$x^2=13-12\cos\theta \quad\Rightarrow\quad 2x\frac{dx}{dt}=12\sin\theta\frac{d\theta}{dt}$$

When $\theta=\dfrac{\pi}{3}$ and $\dfrac{d\theta}{dt}=0.1$, $x=\sqrt{13-12\cos\dfrac{\pi}{3}}=\sqrt{7}$,

we have $2\sqrt{7}\dfrac{dx}{dt}=12\left(\dfrac{\sqrt{3}}{2}\right)(0.1)$

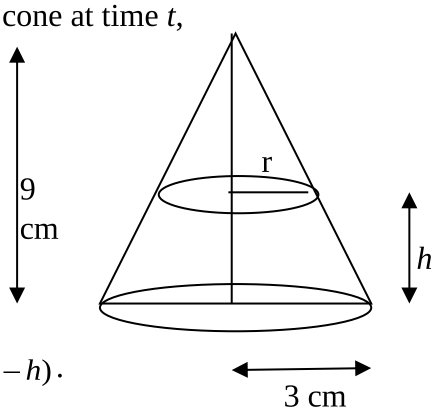
Hence, $\dfrac{dx}{dt} = \dfrac{0.3\sqrt{3}}{\sqrt{7}} \approx 0.196$ cm/s.

Therefore, the length BC is increasing at the rate of 0.196 cm/s.

8 HCI Prelim 08/I/3

A right circular cone with radius 3 cm and height 9 cm is initially full of water. Water is leaking from the circular base of the cone at a constant rate of 2 $cm^3 s^{-1}$, find the exact rate of change of the depth of water when the depth of water is 6 cm. $-\dfrac{2}{\pi}$ cm /s [5]

Let V cm^3 be the volume of water that leaked out of the cone at time t,
h cm be the depth of the water at time t , and,
r cm be the radius of the water surface at time t.



Given $\dfrac{dV}{dt} = 2 \text{ cm}^3 / s$, find $\dfrac{dh}{dt}$ when $h = 6$ cm.

$V = \dfrac{1}{3}\pi r^2 (9-h)$.

From the figure, we have, $\dfrac{9-h}{r} = \dfrac{9}{3}$. That is $r = \dfrac{1}{3}(9-h)$.

Then $V = \dfrac{1}{3}\pi\left(\dfrac{1}{3}(9-h)\right)^2 (9-h)$

$= \dfrac{1}{27}\pi(9-h)^3$.

Thus, $\dfrac{dV}{dt} = \dfrac{1}{27}\pi \cdot 3(9-h)^2(-\dfrac{dh}{dt})$

$= -\dfrac{1}{9}\pi(9-h)^2 \dfrac{dh}{dt}$

Substitute $\dfrac{dV}{dt} = 2 \text{ cm}^3 / s$ and $h = 6$ cm into the above,

we have $2 = -\dfrac{1}{9}\pi(9-6)^2 \dfrac{dh}{dt}$. So $\dfrac{dh}{dt} = -\dfrac{2}{\pi}$ cm /s.

Thus, when the depth of water is 6 cm, it is decreasing at the rate of $\dfrac{2}{\pi}$ cm /s.

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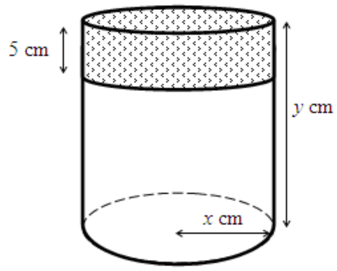
9 HCI Prelim 10/II/1 (Anchor Q – Maxima/ Minima)
A tin has a fitting cylindrical lid which overlaps its cylindrical body by 5 cm. When completely closed, it has base radius x cm and height y cm, as shown in the diagram. The body and the lid are made from thin metal sheet such that the difference in their radii is negligible. The total area of metal sheet used to make the tin and its lid is 400π cm^2.



Show that the volume V cm^3 of the tin is given by
$V = \pi x\left(200 - x^2 - 5x\right)$. [2]

If x varies, find using differentiation the values of x and y for which V has its maximum value. $x = \frac{20}{3}$, $y = \frac{55}{3}$ [4]

Surface area of the tin and lid $= 2\pi x^2 + 2\pi xy + 10\pi x = 400\pi \Rightarrow y = \dfrac{200 - x^2 - 5x}{x}$

Volume of the container = $V = \pi x^2\left(\dfrac{200 - x^2 - 5x}{x}\right) = \pi(200x - x^3 - 5x^2)$

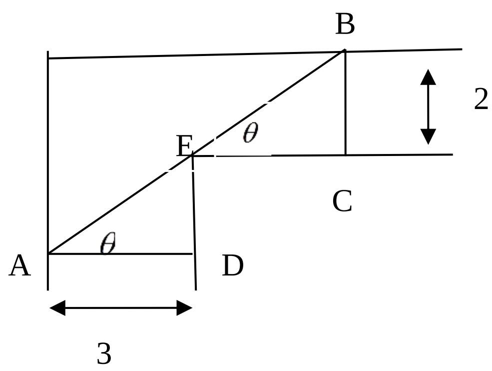
$$\frac{dV}{dx} = \pi\left(200 - 3x^2 - 10x\right)$$

Let $\dfrac{dV}{dx} = 0 \Rightarrow x = \dfrac{20}{3}$ or $x = -10$ (rejected)

$\dfrac{d^2V}{dx^2} = \pi\left(-6x - 10\right) < 0$ when $x = \dfrac{20}{3}$ $\Rightarrow$ V is maximum when $x = \dfrac{20}{3}$.

When $x = \dfrac{20}{3}$, we have $y = \dfrac{55}{3}$ (or $x = 6.67$, $y = 18.3$).

10 The figure shows two corridors meet at right angles and are 2 m and 3 m wide respectively. θ is the angle marked on the given figure and AB is a thin metal tube which must be kept horizontal and cannot be bent as it moves around the corner from one corridor to the other.



(a) Show that the length AB is given by
$L = 3\sec\theta + 2\operatorname{cosec}\theta$.

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(b) Show that $\frac{dL}{d\theta}=0$ when $\theta=\tan^{-1}\sqrt[3]{\frac{2}{3}}\approx 41.14^\circ$

(c) Find L when $\theta=\tan^{-1}\sqrt[3]{\frac{2}{3}}$ and comment on the significance of this value. 7.023

(a) $L=AE+EB$

In ΔADE, $\cos\theta=\frac{AD}{AE}\quad\Rightarrow\quad AE=3\sec\theta$

In ΔBCE, $\sin\theta=\frac{BC}{BE}\quad\Rightarrow\quad BE=2\operatorname{cosec}\theta$

$L=3\sec\theta+2\operatorname{cosec}\theta$

(b) $\frac{dL}{d\theta}=3\sec\theta\tan\theta-2\operatorname{cosec}\theta\cot\theta$

$\frac{dL}{d\theta}=3\cdot\frac{1}{\cos\theta}\frac{\sin\theta}{\cos\theta}-2\cdot\frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta}=\frac{3\sin^3\theta-2\cos^3\theta}{\sin^2\theta\cos^2\theta}$

Let $\frac{dL}{d\theta}=0$, we have ,

$3\sin^3\theta-2\cos^3\theta=0\quad\Rightarrow\quad\frac{\sin^3\theta}{\cos^3\theta}=\frac{2}{3}\quad\Rightarrow\quad\tan\theta=\sqrt[3]{\frac{2}{3}}$

$\theta=\tan^{-1}\sqrt[3]{\frac{2}{3}}=41.14^\circ$

(c) Also, when $\theta=\left(\tan^{-1}\sqrt[3]{\frac{2}{3}}\right)^-$, $\frac{dL}{d\theta}<0$;

and, when $\theta=\left(\tan^{-1}\sqrt[3]{\frac{2}{3}}\right)^+$, $\frac{dL}{d\theta}>0$.

Therefore, L is minimum.

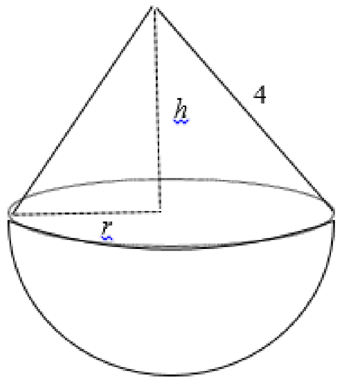
Minimum value of $L=3\sec\theta+2\operatorname{cosec}\theta=7.023$

L is minimized when $\theta=41.14^\circ$. At this time, L = 7.023 meters.

So if we ignore the width of the metal tube, then the greatest length of the tube that can be carried horizontally around the corner is 7.023 m.

11 N14/I/11

A toy manufacturer makes a toy which consists of a hemisphere of radius r cm joined to a circular cone of base radius r cm and height h cm. The manufacturer determines that the length of the slant edge of the cone must be 4 cm and that the total volume of the toy, V cm^3, should be as large as possible.



(i) Find a formula for V in terms of r. Given that $r = r_1$ is the value of r which gives the maximum value of V, show that r_1 satisfies the equation $45r^4 - 768r^2 + 1024 = 0$.

(ii) Find the two solutions to the equation in part (i) for which $r > 0$, giving your answers correct to 3 decimal places. 3.951, 1.207

(iii) Show that one of the solutions found in part (ii) does not give a stationary value of V. Hence write down the value of r_1 and find the corresponding value of h. 3.951, 0.625

(iv) Sketch the graph showing the volume of the toy as the radius of the hemisphere varies.

(i) $V = \frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 h$ and $r^2 + h^2 = 4^2 \Rightarrow V = \frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 \sqrt{16 - r^2}$

$$\Rightarrow \frac{dV}{dr} = \frac{2}{3}\pi(3r^2) + \frac{1}{3}\pi\left(2r\sqrt{16-r^2} + r^2 \cdot \frac{1}{2}(16-r^2)^{-\frac{1}{2}}(-2r)\right)$$

$$= 2\pi r^2 + \frac{1}{3}\pi\left(2r\sqrt{16-r^2} - \frac{r^3}{\sqrt{16-r^2}}\right)$$

$$= 2\pi r^2 + \frac{1}{3}\pi\left(\frac{2r(16-r^2) - r^3}{\sqrt{16-r^2}}\right)$$

$$\therefore \frac{dV}{dr} = 2\pi r^2 + \frac{\pi}{3}\left(\frac{32r - 3r^3}{\sqrt{16-r^2}}\right)$$

When $r = r_1$, V is maximum, then $\frac{dV}{dr} = 0$, we have, $2\pi r_1^2 + \frac{\pi}{3}\left(\frac{32r_1 - 3r_1^3}{\sqrt{16-r_1^2}}\right) = 0$

As $r_1 \neq 0$, dividing both sides by πr_1, we have

$$2r_1 + \frac{1}{3}\left(\frac{32 - 3r_1^2}{\sqrt{16-r_1^2}}\right) = 0 \Rightarrow \frac{6r_1\sqrt{16-r_1^2} + 32 - 3r_1^2}{3\sqrt{16-r_1^2}} = 0$$

$$\Rightarrow 6r_1\sqrt{16-r_1^2} = 3r_1^2 - 32 \quad -----(1)$$

Squaring both sides, $36r_1^2(16 - r_1^2) = 9r_1^4 - 2\cdot 3\cdot 32r_1^2 + 32^2$

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$\Rightarrow \quad 576r_1^2 - 36r_1^4 = 9r_1^4 - 192r_1^2 + 1024$

$\Rightarrow \quad 45r_1^4 - 768r_1^2 + 1024 = 0$

This show that r_1 satisfies the equation $45r^4 - 768r^2 + 1024 = 0$.

Alternative

$V = \frac{1}{3}\pi r^2 h + \frac{2}{3}\pi r^3$

Diff wrt r,

$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{2}{3}\pi rh + \frac{1}{3}\pi r^2 \frac{\mathrm{d}h}{\mathrm{d}r} + 2\pi r^2$

Since $h^2 = 16 - r^2$,

Diff wrt r,

$2h\frac{\mathrm{d}h}{\mathrm{d}r} = -2r$

$\frac{\mathrm{d}h}{\mathrm{d}r} = -\frac{r}{h}$

So $\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{2}{3}\pi rh + \frac{1}{3}\pi r^2\left(-\frac{r}{h}\right) + 2\pi r^2$

$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{\pi r}{3h}\left[2h^2 - r^2 + 6rh\right]$

$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{\pi r}{3h}\left[2h^2 - r^2 + 6rh\right]$

When $r = r_1$, $h = \sqrt{16 - r_1^2}$, $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$,

i.e. $0 = \frac{\pi r_1}{3\sqrt{16 - r_1^2}}\left[2(16 - r_1^2) - r_1^2 + 6r_1\sqrt{16 - r_1^2}\right]$

Since $r_1 > 0$,

$0 = 2(16 - r_1^2) - r_1^2 + 6r_1\sqrt{16 - r_1^2}$

$3r_1^2 - 32 = 6r_1\sqrt{16 - r_1^2}$

Squaring both sides,

$9r_1^4 - 192r_1^2 + 1024 = 36r_1^2\left(16 - r_1^2\right)$

$45r_1^4 - 768r_1^2 + 1024 = 0$ (shown)

(ii) Using GC, the roots, r_1, for the equation $45r^4 - 768r^2 + 1024 = 0$ are ± 3.95080, ± 1.20742.

As $r_1 > 0$, the roots are 3.951 (3 dp) and 1.207 (3 dp).

(iii) Substitute r_1 = 1.2074 into LHS & RHS of equation (1), i.e. $6r_1\sqrt{16 - r_1^2} = 3r_1^2 - 32$, we noted that

LHS > 0 but RHS < 0. Thus $r_1 = 1.207$ does not give a stationary value for V.

(Note: Students should be aware that "squaring both sides" or "multiplying both sides by somethings" will generate "extra" roots when solving an equation.)

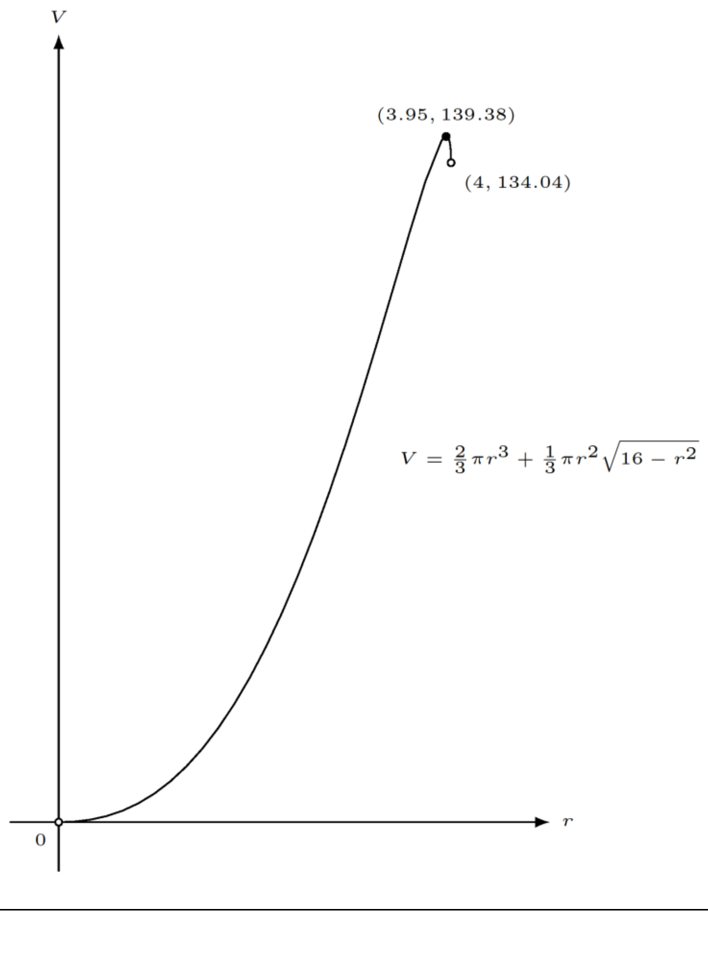
(Note: We need to use the higher accuracy values of r_1 to check☺

Sub $r_1 = 3.9508$, we get LHS=14.826=RHS,

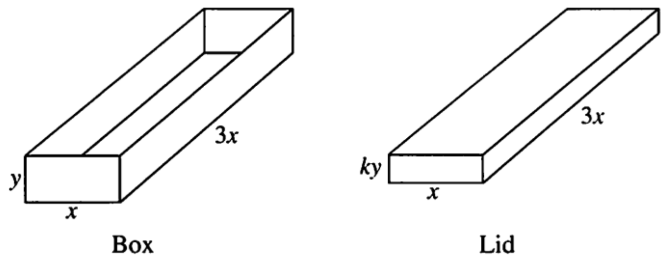
Thus, $r_1 = 3.9508$ satisfies equation (1).

When $r = 3.9508$, $h = \sqrt{4^2 - r^2} = \sqrt{16 - 3.9508^2} = 0.6254 = 0.625$ (3dp)

(iv) Using GC, the graph of $V = \frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2\sqrt{16 - r^2}$ is:



12 N10/I/9



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A company requires a box made of cardboard of negligible thickness to hold 300 cm^3 of powder when full. The length of the box is $3x\text{ cm}$, the width is $x\text{ cm}$ and the height is $y\text{ cm}$. The lid has depth $ky\text{ cm}$, where $0<k\le 1$ (see diagram).

(i) Use differentiation to find, in terms of k, the value of x which gives a minimum total external surface area of the box and the lid. $\boxed{x=\left[\frac{200}{3}(k+1)\right]^{\frac{1}{3}}}$

(ii) Find also the ratio of the height to the width, $\frac{y}{x}$, in this case, simplifying your answer. $\boxed{\frac{3}{2(k+1)}}$

(iii) Find the values between which $\frac{y}{x}$ must lie. $\boxed{\left[\frac{3}{4},\frac{3}{2}\right)}$

(iv) Find the value of k for which the box has square ends. $\boxed{\frac{1}{2}}$

i) Let volume be V and total surface area be A.

$$V=(3x)(x)(y)=3x^2y$$

$$300=3x^2y \quad \Rightarrow y=\frac{100}{x^2}$$

$$A=\underbrace{\left[2(xy)+2(3xy)+3x^2\right]}_{\text{Box}}+\underbrace{\left[2(kxy)+2(3kxy)+3x^2\right]}_{\text{Lid}}$$

$$=8xy+3x^2+8kxy+3x^2$$

$$=8(k+1)xy+6x^2$$

$$=8(k+1)x\left(\frac{100}{x^2}\right)+6x^2$$

$$A=800(k+1)\frac{1}{x}+6x^2$$

$$\frac{\mathrm{d}A}{\mathrm{d}x}=-\frac{800(k+1)}{x^2}+12x$$

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=\frac{1600(k+1)}{x^3}+12$$

Let $\frac{\mathrm{d}A}{\mathrm{d}x}=0$

$$-\frac{800(k+1)}{x^2}+12x=0$$

$$12x^3=800(k+1)$$

$$x^3=\frac{200}{3}(k+1)$$

$$x=\left[\frac{200}{3}(k+1)\right]^{\frac{1}{3}}$$

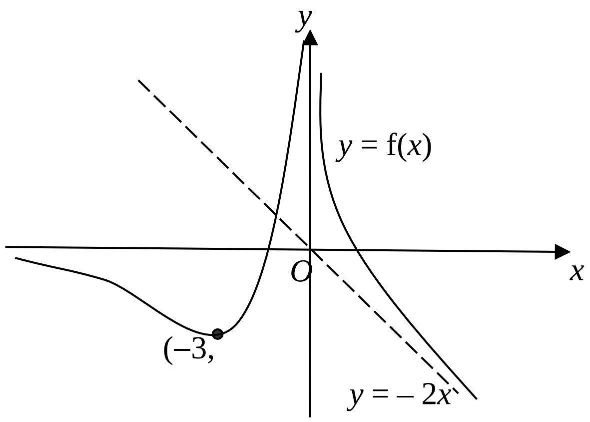
Since $x=\left[\frac{200}{3}(k+1)\right]^{\frac{1}{3}}>0$,

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=\frac{1600(k+1)}{\left[\frac{200}{3}(k+1)\right]^{\frac{1}{3}}}+12>0.$$

Therefore the area is minimum at $x=\left[\frac{200}{3}(k+1)\right]^{\frac{1}{3}}$.

ii)	$\dfrac{y}{x}=\dfrac{\frac{100}{x^2}}{x}=\dfrac{100}{x^3}$ At the case where surface area is minimum, $\dfrac{y}{x}=\dfrac{100}{\frac{200}{3}(k+1)}=\dfrac{3}{2(k+1)}$
iii)	Since $0<k\le 1$ $1<k+1\le 2$ $1>\dfrac{1}{k+1}\ge\dfrac{1}{2}$ $\dfrac{3}{2}\times 1>\dfrac{3}{2}\left(\dfrac{1}{k+1}\right)\ge\dfrac{3}{2}\times\dfrac{1}{2}$ $\dfrac{3}{4}\le\dfrac{y}{x}<\dfrac{3}{2}$
iv)	When the ends are square, $y=x$. $\dfrac{3}{2(k+1)}=1$ $k+1=\dfrac{3}{2}$ $k=\dfrac{1}{2}$

13 The diagram shows the graph of $y = \mathrm{f}(x)$, which has a minimum point at $(-3, -3)$.



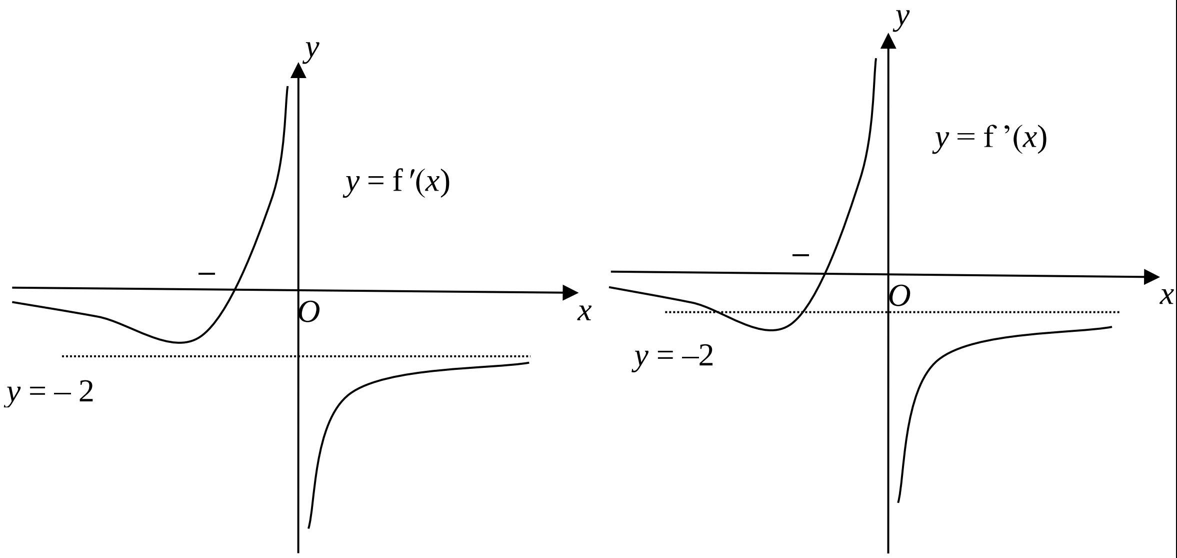
The curve is such that both the x and y-axis are asymptotes and $y = -2x$ is an oblique asymptote.

(a) State the range of values of x for which f is strictly decreasing.

(b) Sketch the graph of $y = \mathrm{f}'(x)$.

(a) When f is strictly decreasing, $x<-3$ or $x>0$.

(b) Below are the two possible graphs of $y = f'(x)$



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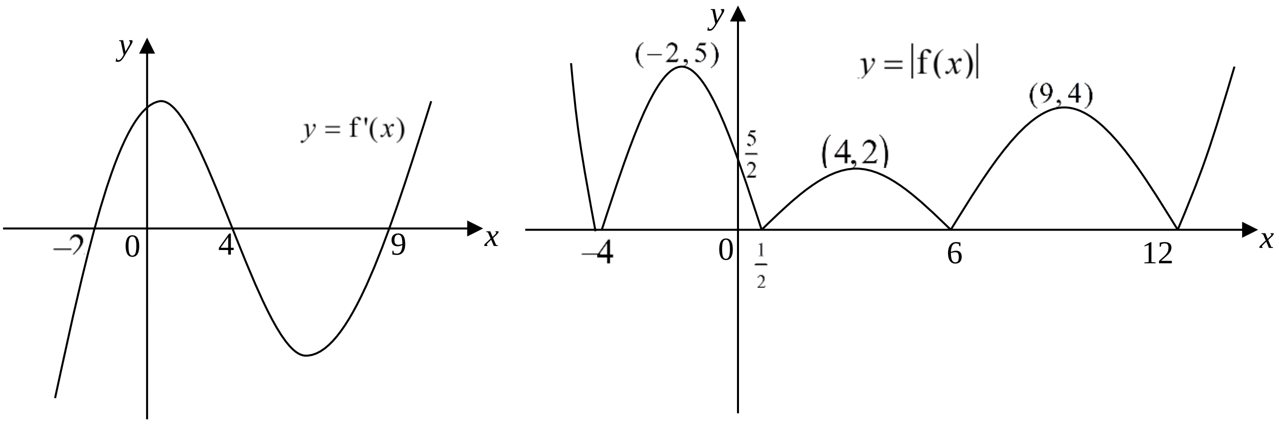
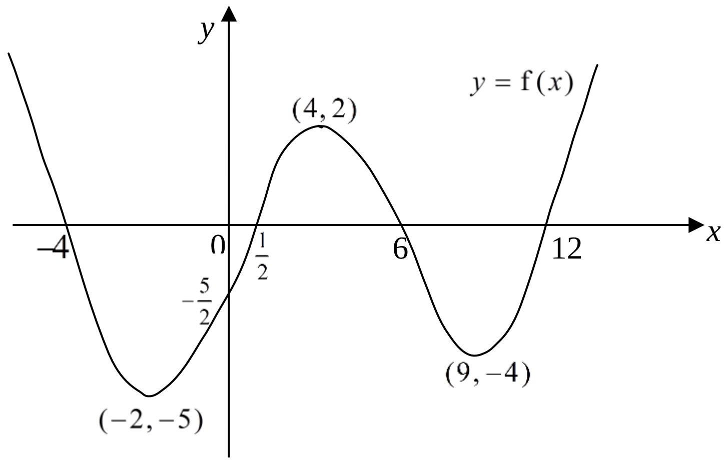
The given sketches show the graphs of $y = f'(x)$ and $y = |f(x)|$.



Sketch the graph of $y = f(x)$, showing clearly the stationary points and intercepts. [3]



Supplementary Questions

15 Find the x-coordinates of all the stationary points on the curve $y=\dfrac{x^3}{(x+2)^2}$, stating, with reasons, the nature of each point.

$x=-6$ max pt, $x=0$ stat pt of inflex

$$y=\frac{x^3}{(x+2)^2}$$

$$\frac{dy}{dx}=\frac{(x+2)^2(3x^2)-x^3.2(x+2)}{(x+2)^4}$$

$$=\frac{x^2[(3x+6)-2x]}{(x+2)^3}$$

$$=\frac{x^2(x+6)}{(x+2)^3}$$

When $\frac{dy}{dx}=0$, $x=0$ or $x=-6$.

x	0^-	0	0^+	$(-6)^-$	6	$(-6)^+$
$\frac{dy}{dx}$	+	0	+	+	0	–
Sketch Of tangent	/	—	/	/	—	\

Curve has a stationary point of inflexion at $x=0$ and a maximum point at $x=-6$

16 Find the coordinates and determine the nature of the turning points on the curve $y=3x^4-16x^3+18x^2$. Sketch the graph of $y = 3x^4 - 16x^3 + 18x^2$ and state the range of values of k for which the equation $3x^4 - 16x^3 + 18x^2 = k$ has precisely two real roots for x.

$k>5$ or $-27<k<0$

When $y=0$,	When $\frac{dy}{dx}=0$, $x = 0, 1, 3$ When $x = 0$, $y = 0$, $\frac{d^2y}{dx^2}=36>0$ (min)

$0 = 3x^4 - 16x^3 + 18x^2$

$0 = x^2\left(3x^2 - 16x + 18\right)$

$0 = x^2\left(3x^2 - 16x + 18\right)$

$\therefore x = 0$ or

$x = \dfrac{-(-16) \pm \sqrt{(-16)^2 - 4(3)(18)}}{6}$

$= \dfrac{16 \pm 2\sqrt{10}}{6}$

$= \dfrac{8 \pm \sqrt{10}}{3}$

$\dfrac{dy}{dx} = 12x^3 - 48x^2 + 36x$

$= 12x(x^2 - 4x + 3)$

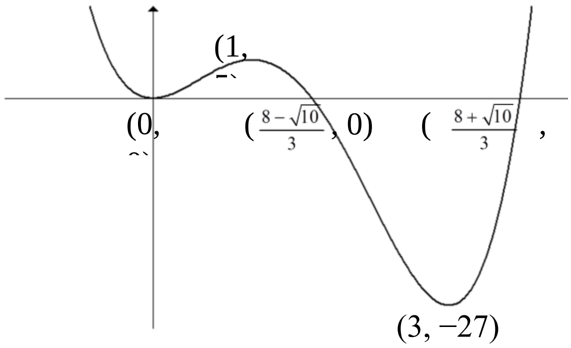
$= 12x(x-1)(x-3)$

$\dfrac{d^2y}{dx^2} = 36x^2 - 96x + 36$

$= 12(3x^2 - 8x + 3)$

When $x = 1$, $y = 5$, $\dfrac{d^2y}{dx^2} = -24 < 0$ (max)

When $x = 3$, $y = -27$, $\dfrac{d^2y}{dx^2} = 72 > 0$ (min)



From the graph, we can see that when $k > 5$ or $-27 < k < 0$, there will be exactly 2 intersection points.

17 Find by differentiation the x-coordinate of the stationary point of the curve $y = x^2 - k^2 \ln(\frac{x}{a})$ where k and a are positive constants, and determine the nature of the stationary point.

$\dfrac{k}{\sqrt{2}}$ min pt

$\ln(\frac{x}{a})$ is defined for $\frac{x}{a} > 0$.

Since $a > 0$ $\therefore x > 0$

$y = x^2 - k^2 \ln\left(\dfrac{x}{a}\right)$

$= x^2 - k^2\left(\ln x - \ln a\right)$

$\dfrac{dy}{dx} = 2x - k^2\dfrac{1}{x}$

At stationary point, $\dfrac{dy}{dx} = 0$

$x^2 = \dfrac{k^2}{2}$

$x = \dfrac{k}{\sqrt{2}}$ or $x = -\dfrac{k}{\sqrt{2}}$ (rej)

$\dfrac{d^2y}{dx^2} = 2 + \dfrac{k^2}{x^2}$

$0=2x-\frac{k^2}{x}$ $0=2x^2-k^2$	When $x=\frac{k}{\sqrt{2}}$, $\frac{d^2y}{dx^2}=2+\frac{k^2}{\left(\frac{k}{\sqrt{2}}\right)^2}$ $=2+2$ $=4>0$ The point at $x=\frac{k}{\sqrt{2}}$ is a minimum point.

18 A curve C has parametric equations

$$x=\sin^3\theta\ ,\qquad y=3\sin^2\theta\cos\theta,\qquad \text{for } 0\le\theta\le\frac{1}{2}\pi.$$

(i) Show that $\frac{dy}{dx}=2\cot\theta-\tan\theta$.

(ii) Show that C has a turning point when $\tan\theta=\sqrt{k}$, where k is an integer to be determined. Find in non-trigonometric form, the exact coordinates of the turning point and explain why it is a maximum. $\left(\frac{2\sqrt{6}}{9},\frac{2\sqrt{3}}{3}\right)$

(i) $x=\sin^3\theta \Rightarrow \frac{dx}{d\theta}=3\sin^2\theta\cos\theta$, and

$y=3\sin^2\theta\cos\theta \Rightarrow \frac{dy}{d\theta}=3\sin^2\theta(-\sin\theta)+3\cos\theta(2\sin\theta\cos\theta)=-3\sin^3\theta+6\sin\theta\cos^2\theta$

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{-3\sin^3\theta+6\sin\theta\cos^2\theta}{3\sin^2\theta\cos\theta}=\frac{-3\sin^3\theta}{3\sin^2\theta\cos\theta}+\frac{6\sin\theta\cos^2\theta}{3\sin^2\theta\cos\theta}=-\tan\theta+2\cot\theta$$

$\therefore\ \frac{dy}{dx}=2\cot\theta-\tan\theta$ (Shown)

(ii) At the stationary point, $\frac{dy}{dx}=0$, then $2\cot\theta-\tan\theta=0 \Rightarrow 2\cot\theta=\tan\theta$

Hence, $2=\tan^2\theta \Rightarrow \tan\theta=\sqrt{2}$ as $0\le\theta\le\frac{1}{2}\pi$, so $\tan\theta>0$

Note that:

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(2\cot\theta-\tan\theta\right)=\frac{d}{d\theta}\left(2\cot\theta-\tan\theta\right)\cdot\frac{d\theta}{dx}$$

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$$=\left(-2\operatorname{cosec}^2\theta-\sec^2\theta\right)\cdot\frac{1}{3\sin^2\theta\cos\theta}=-\frac{2\operatorname{cosec}^2\theta+\sec^2\theta}{3\sin^2\theta\cos\theta}<0$$

Hence, the curve C has a maximum point when $\tan\theta=\sqrt{k}$, where $k = 2$.

When $\tan\theta=\sqrt{2}$, $x=\sin^3\theta=\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^3=\frac{2}{3}\cdot\frac{\sqrt{2}}{\sqrt{3}}=\frac{2\sqrt{6}}{9}$,

and $y=3\sin^2\theta\cos\theta=3\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right)=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$. Maximum point is $\left(\frac{2\sqrt{6}}{9},\frac{2\sqrt{3}}{3}\right)$.

19 N14/II/1 (Anchor Q – Tangent/ Normal)

A curve C has parametric equations

$$x=3t^2,\ y=6t.$$

(i) Find the value of t at the point on C where the tangent has gradient 0.4. 2.5[3]

(ii) The tangent at the point $P\ (3p^2,\ 6p)$ on C meets the y-axis at the point D. Find the Cartesian equation of the locus of the mid-point of PD as p varies. $y^2=\frac{27}{2}x$ [4]

(i)	$x=3t^2 \Rightarrow \frac{dx}{dt}=6t$ $y=6t \Rightarrow \frac{dy}{dt}=6$ $\therefore \frac{dy}{dx}=\frac{6}{6t}=\frac{1}{t}$	At the point where the tangent has gradient 0.4, $\frac{dy}{dx}=0.4$ $\Rightarrow \frac{1}{t}=0.4$ $\Rightarrow t=2.5$
(ii)	Equation of tangent at P: $y-6p=\frac{1}{p}\left(x-3p^2\right)$ At D, $x=0$, $y-6p=\frac{1}{p}\left(0-3p^2\right)$ $y=-3p+6p$ $=3p$ P: $\left(3p^2,\ 6p\right)$ D: $\left(0,\ 3p\right)$	

Topic 11:Differentiation Applications

	Midpoint of PD is $\left(\frac{3p^2}{2}, \frac{6p+3p}{2}\right)$ i.e. $\left(\frac{3p^2}{2}, \frac{9p}{2}\right)$ As p varies, $x=\frac{3p^2}{2}$ and $y=\frac{9p}{2}$ $\Rightarrow p=\frac{2}{9}y$ $x=\frac{3}{2}\left(\frac{2}{9}y\right)^2$ $=\frac{2}{27}y^2$ Eqn of locus of midpoint of PD is $y^2=\frac{27}{2}x$

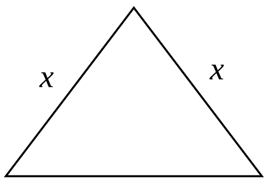
20 ACJC Prelim 11/I/7

An isosceles triangle has fixed base of length b cm. The other 2 equal sides of the triangle are each decreasing at the constant rate of 3 cm per second. How fast is the area changing when the triangle is equilateral? Leave your answer in terms of b. $-\sqrt{3}b$

Let x be the length of each of the other 2 sides of the triangle.

Area, $A = \frac{1}{2}b\times\text{height}$

$$=\frac{1}{2}b\sqrt{x^2-\left(\frac{b}{2}\right)^2}$$



$$\Rightarrow \frac{dA}{dx}=\frac{1}{4}b\left(x^2-\frac{b^2}{4}\right)^{-\frac{1}{2}}(2x)$$

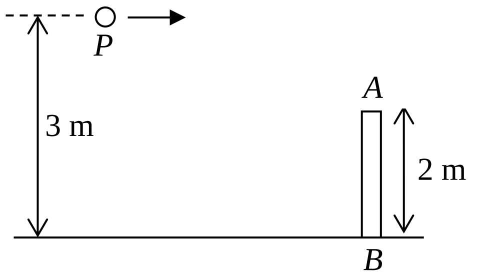
Now, $\frac{dA}{dt}=\frac{dA}{dx}\times\frac{dx}{dt}$

$$=\frac{1}{4}b\left(x^2-\frac{b^2}{4}\right)^{-\frac{1}{2}}(2x)\frac{dx}{dt}$$

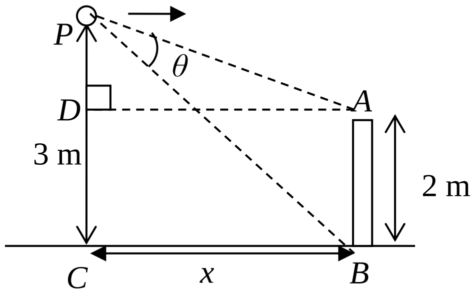
When $x=b$, $\frac{dx}{dt}=-3$, $\frac{dA}{dt}=\frac{1}{4}b\left(b^2-\frac{b^2}{4}\right)^{-\frac{1}{2}}(2b)(-3)$

$$\Rightarrow \frac{dA}{dt}=\frac{-\frac{3}{2}b^2}{\sqrt{\frac{3}{4}b^2}}=-\sqrt{3}b\ \text{cm}^2/\text{s}$$

21 The given diagram shows a vertical pole *AB* of length 2 m on a horizontal ground, with points *A* and *B* representing the top and bottom ends respectively. A moving point *P* approaches the pole on a horizontal path of 3 m above the ground and at a constant speed of 2 m/s. If θ is the angle *APB*, find the rate of change of θ with respect to time when *P* is at a horizontal distance of 8 m from the pole.



0.0514 rad s^{-1}



$$\frac{dx}{dt} = -2 \text{ ms}^{-1}.$$

It is negative since P is moving towards the pole and distance is decreasing.

$$\tan \angle APD = \frac{x}{1} \quad \tan \angle BPC = \frac{x}{3}$$

$$\tan \angle APD = x$$

$$\theta = \angle APD - \angle BPC$$

$$\theta = \tan^{-1}(x) - \tan^{-1}\left(\frac{x}{3}\right)$$

$$\frac{d\theta}{dx} = \frac{1}{1+x^2} - \frac{1}{1+\left(\frac{x}{3}\right)^2} \cdot \frac{1}{3}$$

$$= \frac{1}{1+x^2} - \frac{3}{9+x^2}$$

$$\frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt}$$

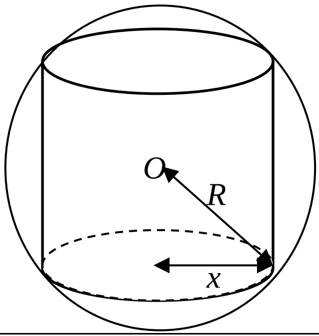
$$\frac{d\theta}{dt} = -2\left(\frac{1}{1+x^2} - \frac{3}{9+x^2}\right) \text{rads}^{-1}$$

$$\left.\frac{d\theta}{dt}\right|_{x=8 \text{ m}} = (-2)\left[\frac{1}{1+8^2} - \frac{3}{9+8^2}\right] \text{ rads}^{-1}$$

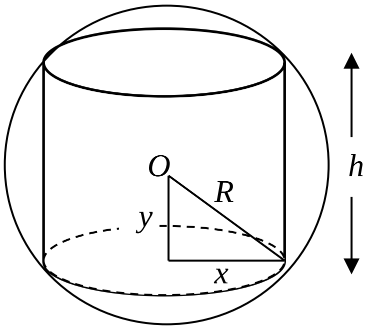
$$\left.\frac{d\theta}{dt}\right|_{x=8 \text{ m}} = (-2)\left[\frac{1}{65} - \frac{3}{73}\right] \text{ rads}^{-1}$$

$$\left.\frac{d\theta}{dt}\right|_{x=8 \text{ m}} = 0.0514 \text{rads}^{-1}$$

22 A cylinder of radius x is inscribed in a fixed sphere of internal radius R, the circumferences of the circular ends being in contact with the inner surface of the sphere. Show that $A^2 = 16\pi^2 x^2\left(R^2 - x^2\right)$, where A is the curved surface area of the cylinder. Prove that, if x is made to vary, the maximum value of A is obtained when the height of the cylinder is equal to its diameter.



Let h be the height of the cylinder.



From the diagram $y = \sqrt{R^2 - x^2}$ and $h = 2y$.

$$A = 2\pi x h = 2\pi x\left(2\sqrt{R^2 - x^2}\right)$$

$$A = 4\pi x\sqrt{R^2 - x^2}$$

$$A^2 = 16\pi^2 x^2\left(R^2 - x^2\right) = 16\pi^2\left(R^2x^2 - x^4\right)$$

$$2A\frac{\mathrm{d}A}{\mathrm{d}x} = 16\pi^2\left(2xR^2 - 4x^3\right) \text{---(1)}$$

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{16\pi^2}{A}\left(xR^2 - 2x^3\right)$$

When $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$,

$$xR^2 - 2x^3 = 0$$

$$x\left(R^2 - 2x^2\right) = 0$$

$$x^2 = \frac{R^2}{2} \text{ or } x = 0 \text{ (rej)}$$

$$x = \frac{R}{\sqrt{2}} \text{ or } x = -\frac{R}{\sqrt{2}} \text{ (rej)}$$

Differentiation (1) w.r.t x

$$2A\frac{\mathrm{d}^2A}{\mathrm{d}x^2} + 2\left(\frac{\mathrm{d}A}{\mathrm{d}x}\right)^2 = 16\pi^2\left(2R^2 - 12x^2\right)$$

When $x = \frac{R}{\sqrt{2}}$,

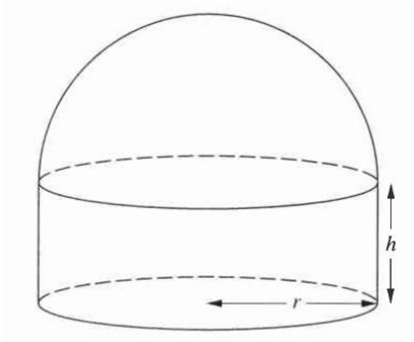
$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = \frac{16\pi^2}{2A}\left(2R^2 - 12\left(\frac{R^2}{2}\right)\right) = \frac{8\pi^2}{A}\left(-4R^2\right) < 0$$

$$h = 2\sqrt{R^2 - \frac{R^2}{2}} = 2\sqrt{\frac{R^2}{2}} = 2\frac{R}{\sqrt{2}} = 2x$$

Therefore, maximum value of A is obtained when the height of the cylinder is equal to its diameter.

23 N12/I/10

[It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]



A model of a concert hall is made up of three parts.

- The roof is modelled by the curved surface of a hemisphere of radius r cm.
- The walls are modelled by the curved surface of a cylinder of radius r cm and height h cm.
- The floor is modelled by a circular disc of radius r cm.

The three parts are joined together as shown in the diagram. The model is made of material of negligible thickness.

(i) It is given that the volume of the model is a fixed value $k\,\text{cm}^3$, and the external surface area is a minimum. Use differentiation to find the values of r and h in terms of k. Simplify your answers. $r = h = \left(\frac{3k}{5\pi}\right)^{\frac{1}{3}}$ [7]

(ii) It is given instead that the volume of the model is $200\,\text{cm}^3$ and its external surface area is $180\,\text{cm}^2$. Show that there are two possible values of r. Given also that $r < h$, find the value of r and the value of h. $r = 3.04$, $h = 4.88$ [5]

(i)	volume of model= $\dfrac{2\pi r^3}{3}+\pi r^2 h=k$ $h=\dfrac{k}{\pi r^2}-\dfrac{2r}{3}$ total surface area,S $=2\pi r^2+\pi r^2+2\pi rh$ $=3\pi r^2+2\pi r\left(\dfrac{k}{\pi r^2}-\dfrac{2r}{3}\right)$ $=3\pi r^2+\dfrac{2k}{r}-\dfrac{4\pi r^2}{3}$ $=\dfrac{5\pi r^2}{3}+\dfrac{2k}{r}$ $\dfrac{dS}{dr}=\dfrac{10\pi r}{3}-\dfrac{2k}{r^2}$ $\dfrac{dS}{dr}=0$ $\dfrac{10\pi r}{3}-\dfrac{2k}{r^2}=0$ $\dfrac{10\pi r}{3}=\dfrac{2k}{r^2}$ $r^3=\dfrac{3k}{5\pi}$ $r=\left(\dfrac{3k}{5\pi}\right)^{\frac{1}{3}}$ $\dfrac{d^2S}{dr^2}=\dfrac{10\pi}{3}+\left(\dfrac{4k}{r^3}\right)=\dfrac{10\pi}{3}+\dfrac{4k}{\left(\dfrac{3k}{5\pi}\right)}$ $=\dfrac{10\pi}{3}+\dfrac{20\pi}{3}=10\pi>0$ Thus, surface area is minimum.	

	$r=\left(\frac{3k}{5\pi}\right)^{\frac{1}{3}}$ $h=\frac{k}{\pi r^2}-\frac{2r}{3}=r\left(\frac{k}{\pi r^3}-\frac{2}{3}\right)$ $=r\left(\frac{5}{3}-\frac{2}{3}\right)$ $=r=\left(\frac{3k}{5\pi}\right)^{\frac{1}{3}}$	
(ii)	$k=200, S=180$ $180=3\pi r^2+\frac{400}{r}-\frac{4\pi r^2}{3}$ $\frac{5}{3}\pi r^3-180r+400=0$ by GC, $r=-6.758\text{(NA)}, 3.715, \text{or } 3.037$ when $r=3.715,\ h=2.1156$ when $r=3.037, h=4.8775$ $\therefore r=3.04, h=4.88$	