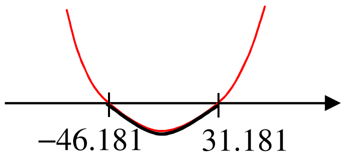
H2 Math Prelim Paper 2 Solutions and Markers' Comments

1(i)	$\int \mathrm{f}(x)\ \mathrm{d}x = 6\int \dfrac{(x^2-3)}{(x-3)(x^2+9)}\ \mathrm{d}x$ $= \dfrac{6}{3}\int \left(\dfrac{1}{x-3}+\dfrac{2x+6}{x^2+9}\right)\mathrm{d}x$ $= 2\int \left(\dfrac{1}{x-3}+\dfrac{2x}{x^2+9}+\dfrac{6}{x^2+3^2}\right)\mathrm{d}x$ $= 2\left[\ln\lvert x-3\rvert+\ln\left(x^2+9\right)+\dfrac{6}{3}\tan^{-1}\left(\dfrac{x}{3}\right)\right]+C$ $= 2\left[\ln\lvert x-3\rvert+\ln\left(x^2+9\right)+2\tan^{-1}\left(\dfrac{x}{3}\right)\right]+C$
1 (ii)	Area $= \int_{-\sqrt{3}}^{0} \mathrm{f}(x)\ \mathrm{d}x$ $= \int_{-\sqrt{3}}^{0} \dfrac{6(x^2-3)}{(x-3)(x^2+9)}\ \mathrm{d}x$ $= 2\left[\ln\lvert x-3\rvert+\ln\left(x^2+9\right)+2\tan^{-1}\left(\dfrac{x}{3}\right)\right]_{-\sqrt{3}}^{0}$ $= 2\left[\ln 3+\ln 9-\ln\left(\sqrt{3}+3\right)-\ln 12-2\left(-\dfrac{\pi}{6}\right)\right]$ $= 2\left[\ln\left(\dfrac{9}{4\left(\sqrt{3}+3\right)}\right)+\dfrac{\pi}{3}\right]$ units^2
2 (a)	1^{st} Lesson - 40 minutes $\xrightarrow{+5}$ 2^{nd} Lesson - 45 minutes $\xrightarrow{+5}$ 3^{rd} Lesson - 50 minutes

	$\vdots$ $\vdots$ This is an arithmetic progression: $u_1 = 40$ and common difference, $d = 5$ 60 hours = (60×60) minutes = 3600 minutes Total time (in mins) after n lessons, S_n $= \frac{n}{2}\left[2(40)+(n-1)(5)\right]$ $= \frac{n}{2}\left[75+5n\right]$ $\therefore$ For Adrian to attend at least 60 hours of lessons, $\frac{n}{2}\left[75+5n\right] \geq 3600$ **Method 1:** $\therefore$ For Adrian to attend ***at least*** 60 hours of lessons,

$$\frac{n}{2}[75+5n] \geq 3600$$

$$\Rightarrow n^2 + 15n - 1440 \geq 0$$



$\Rightarrow n \leq -46.181$ or $n \geq 31.181$

(rejected $\because n > 0$)

$\therefore$ Adrian has to attend a minimum of 32 lessons before he is qualified to take the test.

Method 2: Using GC to set up the table of values

Using GC to tabulate a table of values of S_n for various values of n,

n	$S_n = \frac{n}{2}[75+5n]$	Comparison with 3600
31	3565	< 3600
32	3760	> 3600

$\therefore$ Adrian has to attend a minimum of 32 lessons before he is qualified to take the test.

2 (b)	$u_{n+1} = S_n$, $n \in \mathbb{Z}^+$ For $n \geq 2$, $\frac{u_{n+1}}{u_n} = \frac{S_n}{u_n}$ $= \frac{u_n + S_{n-1}}{u_n}$ $= \frac{u_n + u_n}{u_n} \quad \because u_n = S_{n-1}$ $= 2$ $\because \frac{u_{n+1}}{u_n} =$ constant, $\{u_2, u_3, u_4, \ldots\}$ follows a geometric progression with common ratio 2. $\sum_{r=1}^{N+1} u_r = u_1 + \sum_{r=2}^{N+1} u_r$ $= u_1 + \frac{u_2\left(2^N - 1\right)}{2-1}$ $= u_1 + u_1\left(2^N - 1\right) \quad \because u_1 = u_2$ $= u_1 2^N$
3(i)	Since N lies on l_1,

$$\overrightarrow{ON} = \begin{pmatrix} 1-3\lambda \\ -1+4\lambda \\ 2\lambda \end{pmatrix} \text{ for some } \lambda \in \mathbb{R}.$$

$$\begin{aligned} \overrightarrow{AN} &= \overrightarrow{ON} - \overrightarrow{OA} \\ &= \begin{pmatrix} 1-3\lambda \\ -1+4\lambda \\ 2\lambda \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -3-3\lambda \\ -6+4\lambda \\ -3+2\lambda \end{pmatrix} \end{aligned}$$

Since $\overrightarrow{AN}$ is perpendicular to l_1,

	$\overrightarrow{AN} \bullet \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} = 0$ $\begin{pmatrix} -3-3\lambda \\ -6+4\lambda \\ -3+2\lambda \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} = 0$ $9+9\lambda-24+16\lambda-6+4\lambda=0$ $29\lambda=21$ $\lambda=\dfrac{21}{29}$ Hence, $\overrightarrow{ON} = \begin{pmatrix} 1-3\left(\dfrac{21}{29}\right) \\ -1+4\left(\dfrac{21}{29}\right) \\ 0+2\left(\dfrac{21}{29}\right) \end{pmatrix} = \begin{pmatrix} -34/29 \\ 55/29 \\ 42/29 \end{pmatrix}$ The coordinates of N are $\left(-\dfrac{34}{29}, \dfrac{55}{29}, \dfrac{42}{29}\right)$
3(ii)	$l_2:\ \mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -5 \\ -1 \end{pmatrix},\ \mu \in \mathbb{R}$

Since the plane Π_1 is parallel to both lines, the normal to Π_1 can be obtained by taking the cross-product of the two director vectors.

$$\mathbf{n_1} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ -5 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ -3 \\ 15 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

Hence, $\mathbf{n_1} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$.

Since Π_1 contains l_1, Π_1 contains the point with p.v. $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

Hence, Cartesian equation of Π_1:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

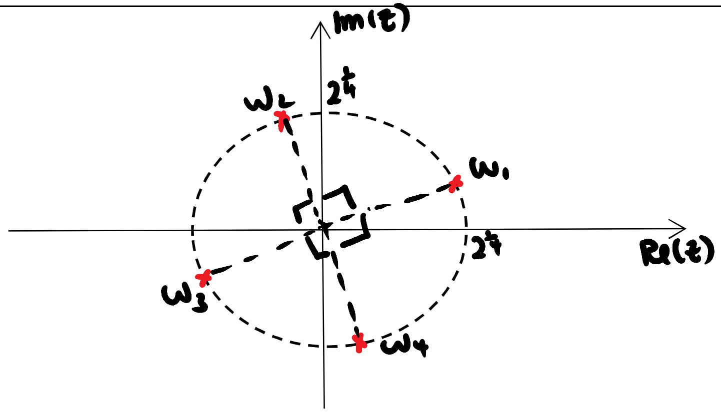
$$2x - y + 5z = 2 + 1$$

$$2x - y + 5z = 3$$

3(iii)	$\Pi_1:\ 2x-y+5z=3$ --(1) $\Pi_2:\ 3x-y+z=4$ --(2) Using G.C., $l_3:\mathbf{r}=\begin{pmatrix}1\\-1\\0\end{pmatrix}+s\begin{pmatrix}4\\13\\1\end{pmatrix},\ s\in\mathbb{R}$ **Alternative solution:** Direction vector of l_3

	$=\begin{pmatrix}2\\-1\\5\end{pmatrix}\times\begin{pmatrix}3\\-1\\1\end{pmatrix}$ $=\begin{pmatrix}-1+5\\-(2-15)\\-2+3\end{pmatrix}$ $=\begin{pmatrix}4\\13\\1\end{pmatrix}$ Since $\begin{pmatrix}1\\-1\\0\end{pmatrix}\bullet\begin{pmatrix}3\\-1\\1\end{pmatrix}$=4, and $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ lies on $\prod_1$, $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ lies on both $\prod_1$ and $\prod_2$, resulting it to be a common point on both planes and must line on l_3. Hence, $l_3:\mathbf{r}=\begin{pmatrix}1\\-1\\0\end{pmatrix}+s\begin{pmatrix}4\\13\\1\end{pmatrix},\ s\in\mathbb{R}$
(iv)	If the three planes intersect at one point, it implies that Π_3 cuts the line l_3 at exactly one point.

	$\Leftrightarrow$ it means that Π_3 and l_3 are not parallel. $\Leftrightarrow$ normal to Π_3, $\mathbf{n_1}$, is and l_3 are not perpendicular, i.e. $\begin{pmatrix} t \\ -2 \\ 6 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 13 \\ 1 \end{pmatrix} \neq 0$ $4t - 26 + 6 \neq 0$ $4t \neq 20$ $t \neq 5$ d can take any value, i.e. $d \in \mathbb{R}$
4(a)	$w^4 + 1 - \sqrt{3}i = 0,$ $w^4 = -1 + \sqrt{3}i$ $= 2e^{i(\frac{2\pi}{3} + 2kn)}, \quad k = -2, -1, 0, 1$ $w = 2^{1/4} e^{i(\frac{\pi}{6} + \frac{k\pi}{2})}, \quad k = -2, -1, 0, 1$ $\therefore w = 2^{1/4} e^{i(-\frac{5\pi}{6})}, 2^{1/4} e^{i(-\frac{\pi}{3})}, 2^{1/4} e^{i(\frac{\pi}{6})}, 2^{1/4} e^{i(\frac{2\pi}{3})}.$

	
4b(i) **(ii)**	$-y+x\mathrm{i}=iz$ (Ans) $wx^2-y^2+2xy\mathrm{i}=z^2$ (Ans) $\therefore\ vw=iz^3$ $\lvert vw\rvert=\lvert iz^3\rvert=\lvert z\rvert^3=r^3$ $\arg(vw)=\arg(v)+\arg(w)$ $=\arg(\mathrm{i}z)+\arg(z^2)$ $=\arg(i)+\arg(z)+2\arg(z)$ $=\dfrac{\pi}{2}+3\theta$

	$\therefore vw = r^3 e^{i(\frac{\pi}{2}+3\theta)}$ (Ans)
4biii)	$-4-4\sqrt{3}\mathrm{i} = 8\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}$ Thus, $r^3 e^{i(\frac{\pi}{2}+3\theta)} = 8\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}$ Since $\frac{\pi}{2} < 3\theta + \frac{\pi}{2} < 2\pi$, we need to compare $r^3 e^{i(\frac{\pi}{2}+3\theta)} = 8\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{3}\right)}$ Thus, comparing modulus and argument, $r = 2$ and $\frac{\pi}{2} + 3\theta = \frac{4\pi}{3}$ $\theta = \frac{5\pi}{18}$ Thus, $z = 2e^{\mathrm{i}\left(\frac{5\pi}{18}\right)}$ **Alternative Method** $iz^3 = -4-4\sqrt{3}\mathrm{i} = 8\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}$

	$z^3 = \dfrac{8e^{i\left(-\frac{2\pi}{3}\right)}}{i} = \dfrac{8e^{i\left(\frac{-2\pi}{3}\right)}}{e^{i\left(\frac{\pi}{2}\right)}} = 8e^{i\left(-\frac{7\pi}{6}\right)} = 8e^{i\left(\frac{5\pi}{6}\right)}$ $z^3 = 8e^{i\left(\frac{5\pi}{6}+2k\pi\right)}$ for $k \in \mathbb{Z}$ $z = 2e^{i\left(\frac{5\pi}{18}+\frac{2k\pi}{3}\right)}$ for $k = -1, 0, 1$ $z = 2e^{i\left(-\frac{7\pi}{18}\right)}, 2e^{i\left(\frac{5\pi}{18}\right)}, 2e^{i\left(\frac{17\pi}{18}\right)}$ Since it's given that $0 < \theta < \dfrac{\pi}{2}$, $z = 2e^{i\left(\frac{5\pi}{18}\right)}$
5(i)	Select the first 25 male students and 25 female students found leaving the school compounds to do the survey. A disadvantage is that it is biased towards selecting students who leave the school compounds early.
(ii)	Stratified sampling. Divide all students in school into two strata: male and female students. Within each stratum, use simple random sampling to select a sample with size proportional to the size of the stratum.
(iii)	Stratified sampling requires the sampling frame, which is the list of all customers at their retail outlets during the given month, and this is difficult to obtain.
6(i)	$P(B' \mid A) = \dfrac{1}{3}$ $\dfrac{P(B' \cap A)}{P(A)} = \dfrac{1}{3} - - - (1)$

	$P(A\|B') = \dfrac{17}{41}$ $\dfrac{P(B' \cap A)}{P(B')} = \dfrac{17}{41} --- (2)$ Taking (1)/(2), we have $\dfrac{P(B')}{P(A)} = \dfrac{1/3}{17/41} = \dfrac{41}{51}$ $\dfrac{1-P(B)}{P(A)} = \dfrac{41}{51}$ $\dfrac{0.41}{P(A)} = \dfrac{41}{51}$ $P(A) = 0.51$
(ii)	$P(A \cap B) = P(A) - P(A \cap B')$ Sub P(B)=0.59 into (2) to get $P(B' \cap A) = 0.17$ Thus $P(A \cap B) = P(A) - P(A \cap B')$ $= 0.51 - 0.17$ $= 0.34$

7(i)	Let X be the no of patients who are relieved of the symptoms, out of 15. X – B(15, 0.74) P(X> 7) = 1 – P(X ≤ 7) = 0.978
(ii)	Let U be the no of patients who are relieved of the symptoms, out of n+15. U ~ B(n+15, 0.74) $P(U \le n+13) \ge 0.99$ $1-\left(P(U=n+14)+P(U=n+15)\right) \ge 0.99$ $P(U=n+14)+P(U=n+15) \le 0.01$ $(0.74)^{n+15}+\binom{n+15}{n+14}(0.74)^{n+14}(0.26) \le 0.01$ $(0.74)^{n+15}+(n+15)(0.74)^{n+14}(0.26) \le 0.01$ From GC, least n=8 <u>Alternative Method</u> Let U' be the no of patients who are not relieved of the symptoms, out of n+15. $U' \sim B(n+15, 0.26)$

	$P(U'\geq 2)\geq 0.99$ $1-P(U'\leq 1)\geq 0.99$ $P(U'\leq 1)\leq 0.01$ $P(U'=0)+P(U'=1)\leq 0.01$ $(0.74)^{n+15}+\binom{n+15}{1}(0.26)(0.74)^{n+14}\leq 0.01$ $(0.74)^{n+15}+(n+15)(0.74)^{n+14}(0.26)\leq 0.01$ From GC, least n=8
(iii)	Let Y be the number of patients who are not relieved of the symptoms out of 36. Y ~ B(36, 0.08) Y ~ Po(2.88) approximately since n=36 large, np=2.88<5 P(Y ≤ 5) = 0.928
8(i)	apply ln to both sides to show linearization: $\ln(y)=\ln(ab^t)$ $\ln y=\ln a+\ln(b^t)$ $\ln y=\ln a+t\ln b$
(ii)	From GC, $I=0.377423t+0.26295183$ Thus, comparing with $\ln y=t\ln b+\ln a$

	$\ln a = 0.26295183 \Rightarrow a = e^{0.26295183} = 1.300764 = 1.30(\text{to 3 sf})$ $\ln b = 0.377423 \Rightarrow b = e^{0.377423} = 1.4582 = 1.46(\text{to 3 sf})$
(iii)	r = 0.996741=0.997 (to 3 sf)
(iv)	When t =15, $I = 0.377423(15) + 0.26295183$ $I = 5.92429683$ $y = 374.0153$ $\Rightarrow 374$ millions Not reliable as t =15 is out of the data range, and thus the result is obtained by extrapolation
(v)	Since t is the independent variable, use line I on t. When I = 1.5 $\Rightarrow$ y = 4.48 $1.5 = 0.377423t + 0.26295183$ $t = 3.2776 = 3.28(\text{to 3 sf})$

9(i)	Let X be the r.v. ' mass of tomatoes in grams' $\bar{x} = \frac{3500}{50} = 70,$ $s^2 = \frac{1}{49}[245220.5 - \frac{3500^2}{50}]$ $= 4.5$
(ii)	Test H_0: $\mu = 69.4$ H_1: $\mu > 69.4$ 1-tailed z-test at 5% significance level. Under H_0, $\overline{X} \sim N(69.4, \frac{4.5}{50})$ approximately by Central Limit Theorem since n is large Since p-value = 0.0228 < 0.05, we reject H_0 and conclude that at 5% significance level, there is sufficient evidence that the mean mass of the tomatoes are increased with the application of the new fertiliser. There is a 5% chance that we conclude that the mean mass of the tomatoes has increased when in fact it has not. (OR There is a 5% chance that we conclude that the mean mass of the tomatoes is more than 69.4g when in fact it is 69.4g)
iii)	For the null hypotheses H_0: $\mu = \mu_0$ is not rejected at 3% level of significance, $z = \frac{\bar{x} - \mu_0}{s / \sqrt{50}} < 1.88079$

	$\Rightarrow \dfrac{70-\mu_0}{\sqrt{4.5/50}} < 1.88079$ $\Rightarrow \mu_0 > 69.4$ $\therefore$ Range of value of μ_0 is that $\mu_0 > 69.4$ (Ans)
10(i)	<u>Method 1</u> Total number of possible codes $= 26^2 \times 10^3 = 676000$ Total number of codes with three different digits and two different letters== ${}^{26}P_2 \times {}^{10}P_3 = 46800$ Required prob=$\dfrac{46800}{676000} = \dfrac{9}{13}$ <u>Method 2</u> Required prob $= 1 \times 1 \times \dfrac{9}{10} \times \dfrac{8}{10} \times \dfrac{25}{26} = \dfrac{9}{13}$
(ii)	<u>Method 1</u> Total number of palindromes $= 26 \times 10 \times 10 \times 1 \times 1 = 2600$

	$\text{P(code is a Palindrome code)} = \frac{2600}{676000} = \frac{1}{260}$ <u>Method 2</u> $\text{P(code is a Palindrome code)} = 1\times1\times1\times\frac{1}{10}\times\frac{1}{26} = \frac{1}{260}$
(iii)	<u>Method 1</u> $P\left(\text{2 and 3 used in code} \mid \text{code is palindrome}\right)$ $= \frac{P\left(\text{code is palindrome and contains 2 and 3}\right)}{P\left(\text{code is palindrome}\right)}$ $= \frac{P(\_232\_) + P\left(\_323\_\right)}{\left(1/260\right)}$ $= \frac{2\left(1\times\frac{1}{10}\times\frac{1}{10}\times\frac{1}{10}\times\frac{1}{26}\right)}{\left(1/260\right)} = 0.02$ <u>Method 2</u>

	$P\left(2 \text{ and } 3 \text{ used in code} \middle| \text{code is palindrome}\right)$ $= \dfrac{\text{no of palindrome codes with 2 and 3}}{\text{number of palindrome codes}}$ $= \dfrac{\text{no of palindrome with } (\_\,232\,\_) + \text{no of palindrome with} (\_\,323\,\_)}{2600}$ $= \dfrac{26+26}{2600} = \dfrac{1}{50}$
(iv)	<u>Let event A be "code is palindrome".</u> <u>Let event B be "code contains 2 and 3".</u> If A, B are independent, $P\left(B \middle| A\right) = P\left(B\right)$. (This is the easiest way since $P\left(B \middle| A\right)$ is found in (iii). Thus, we need to find $P\left(B\right)$, i.e. the probability of a code containing 2 and 3. <u>Case 1: Code contains 2 and 3 only.</u> No. of codes with 2 and 3 only $= 2 \times 26^2 \times {}^3C_2 = 4056$ <u>Case 2: Code contains 2, 3 and 1 other number</u> No of codes with 2, 3 and 1 other number $= {}^8C_1 \times 3! \times 26^2 = 32448$ Prob (code contains 2 and 3) $= \dfrac{32448+4056}{676000} = 0.054 \neq 0.02$

	Thus, the events are not independent.
11(i)	Let X be number of people going to the checkout counter in the period of 15 minutes. X ~Po(4.05) P(X > 5)= 1− P(X ≤ 5) = 0.2227323=0.223
(ii)	Let Y be the number of days with more than 5 people going to the checkout counter, out of 14 days. Y ~B(14, 0.2227323) P(X ≥ k)= 1− P(X ≤ k − 1) < 0.02 $P(X \le k-1) > 0.98$ From the GC, $P(X \le 6) = 0.9791$ $P(X \le 7) = 0.9951$ Least $(k-1)$=7. Thus least $k = 8$
(iii)	Let M be number of people going the checkout counter in the period of 40 minute. M~Po(10.8) Let K be number of people leaving the checkout counter iin the period of 40 minute. K~Po(12.4) Since $\lambda = 10.8 > 10, M \sim N(10.8,10.8)$ approximately

	Since $\lambda = 12.4 > 10, \mathrm{K} \sim N(12.4, 12.4)$ approximately Let K − M be the number of people who left the queue in 40 minutes $K - \mathrm{M} \sim N(1.6, 23.2)$ P(at most 10 people at the checkout counter at 0940) = P(K – M ≥ 8) = P(K – M ≥ 7.5) = 0.110
(iv)	<u>Possible Answers</u> - The **average/mean number of people** joining the queue may be different from one time interval to the next (eg more people during lunch time) over several hours, so the Poisson model may not be appropriate. - The <u>average number of people</u> at the checkout counter at night may be less than the average number of people at the checkout counter in the mornings, so since it cannot be kept constant over several hours, the Poisson model is not appropriate. - The <u>average rate of people</u> going to the checkout counter over several hours may not be a constant as their might be more people during lunchtime hour compared with other one-hour interval.