Jurong Pioneer Junior College
H2 Mathematics **JC2-2023**
Prelim Paper 2 Solution

1

(i)

Method 1

$$\ln\left(1+e^{-x}\right) \approx \ln\left[1+\left(1-x+\frac{x^2}{2}\right)\right]$$

$$= \ln\left(2-x+\frac{x^2}{2}\right)$$

$$= \ln\left[2\left(1-\frac{x}{2}+\frac{x^2}{4}\right)\right]$$

$$= \ln 2 + \ln\left[1+\left(-\frac{x}{2}+\frac{x^2}{4}\right)\right]$$

$$= \ln 2 + \left(-\frac{x}{2}+\frac{x^2}{4}\right) - \frac{\left(-\frac{x}{2}+\frac{x^2}{4}\right)^2}{2} + \ldots$$

$$= \ln 2 - \frac{x}{2} + \frac{x^2}{4} - \left(\frac{x^2}{8}\right) + \ldots$$

$$= \ln 2 - \frac{x}{2} + \frac{1}{8}x^2 + \ldots$$

Method 2

$$f(x) = \ln\left(1+e^{-x}\right)$$

$$f'(x) = \frac{1}{1+e^{-x}}\left(-e^{-x}\right) = -\frac{1}{e^x+1}$$

$$f''(x) = \left(e^x+1\right)^{-2} e^x$$

When $x=0$, $f(0)=\ln 2$, $f'(0) = -\frac{1}{2}$, $f''(0) = \frac{1}{4}$

$$f(x) = \ln 2 - \frac{1}{2}x + \left(\frac{1}{4}\right)\frac{x^2}{2!} + \ldots\ldots$$

$$f(x) = \ln 2 - \frac{1}{2}x + \frac{x^2}{8} + \ldots\ldots$$

(ii)

$$\frac{d}{dx}\ln\left(1+e^{-x}\right)=\frac{-e^{-x}}{1+e^{-x}}=-\frac{1}{1+e^{x}}$$

Using the series from part **(i)**, $\frac{1}{1+e^x}\approx-\frac{d}{dx}\left(\ln 2-\frac{x}{2}+\frac{1}{8}x^2\right)=-\left(-\frac{1}{2}+\frac{2}{8}x\right)=\frac{1}{2}-\frac{1}{4}x$.

2

$$\frac{dv}{dt}=9.8-R$$
$$\frac{dv}{dt}=9.8-kv \qquad \text{(Since } R=kv,\ \text{where } k>0\text{)}$$
$$\int\frac{1}{9.8-kv}\,dv=\int 1\,dt$$
$$-\frac{1}{k}\int\frac{-k}{9.8-kv}\,dv=\int 1\,dt$$
$$-\frac{1}{k}\ln\left|9.8-kv\right|=t+C$$
$$\ln\left(9.8-kv\right)=-kt-kC \qquad (9.8-kv>0)$$
$$9.8-kv=Ae^{-kt},\text{where } A=e^{-kC}$$
When $t=0$, $v=0$ (helicopter is stationary)
$$9.8-k(0)=Ae^0\Rightarrow A=9.8$$
$$\therefore 9.8-kv=9.8e^{-kt}$$
$$v=\frac{9.8(1-e^{-kt})}{k}$$
As $t\to\infty$, $e^{-kt}\to 0, v\to\frac{9.8}{k}$

$\therefore$ The terminal velocity of the parachutist is $\frac{9.8}{k}$ ms^{-1}

3

(a)

Method 1

$y=e^x\to y=-e^x\to y=-e^x+2$
Reflect about the x-axis.
Translate by 2 units in the positive y-direction.

Method 2

$y=e^x\to y=e^x-2\to y=-\left(e^x-2\right)=-e^x+2$
Translate by 2 units in the negative y-direction.
Reflect about the x-axis.
(b)

After A : $y = 3x + 6$

After B : $y = 3(x-2) + 6 = 3x$

After C : $y = \frac{1}{3}(3x) = x$

(c)

(i)

Asymptotes : $x = a$, $y = \frac{1}{d}$

Axial intercepts : $(c, 0)$, $\left(0, \frac{1}{b}\right)$

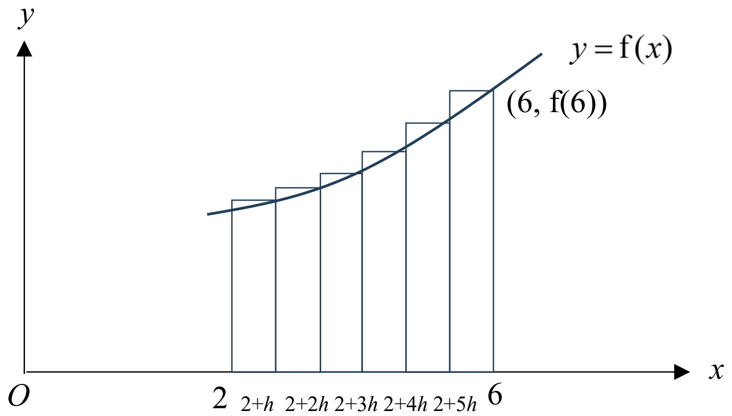
(ii)

Asymptotes : $x = d$, $y = c$

Axial intercepts : $(0, a)$, $(b, 0)$

4

(i)



$h = \frac{2}{3}$

$\sum_{n=1}^{6}\left(\mathrm{f}(2+nh)\right)h = \mathrm{f}(2+h)h + \mathrm{f}(2+2h)h + \mathrm{f}(2+3h)h + \mathrm{f}(2+4h)h + \mathrm{f}(2+5h)h + \mathrm{f}(2+6h)h$ represents the sum of areas of six rectangles drawn and the rectangles are above the curve, so it is greater than area of A.

(ii)

$\sum_{n=0}^{5}\left(\mathrm{f}(2+nh)\right)h$

(iii)

Upper bound of area of A = $\sum_{n=1}^{6}\left(\mathrm{f}(2+nh)\right)h=\frac{2}{3}\sum_{n=1}^{6}\left[\left(2+\frac{2}{3}n\right)\mathrm{e}^{\left(2+\frac{2}{3}n\right)}\right]\approx 2914$

Lower bound of area of A = $\sum_{n=0}^{5}\left(\mathrm{f}(2+nh)\right)h=\frac{2}{3}\sum_{n=0}^{5}\left[\left(2+\frac{2}{3}n\right)\mathrm{e}^{\left(2+\frac{2}{3}n\right)}\right]\approx 1311$

(iv)

Exact area of A

$\int_2^6 x\mathrm{e}^x \, \mathrm{d}x$

$=\left[x\mathrm{e}^x\right]_2^6-\int_2^6 \mathrm{e}^x \mathrm{d}x$

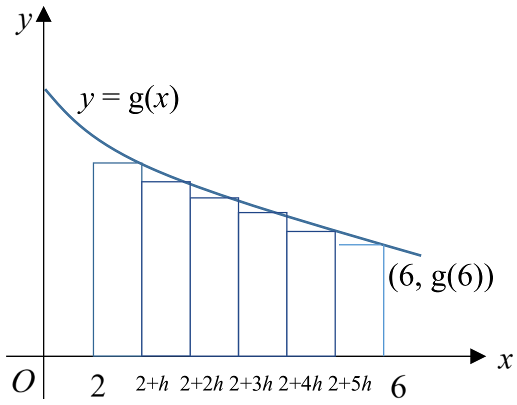
$=6\mathrm{e}^6-2\mathrm{e}^2-\left[\mathrm{e}^x\right]_2^6$

$=5\mathrm{e}^6-\mathrm{e}^2$ units ^{2}

Let $u=x$ and $\frac{\mathrm{d}v}{\mathrm{d}x}=\mathrm{e}^x$.

Then $\frac{\mathrm{d}u}{\mathrm{d}x}=1$ and $v=\mathrm{e}^x$

(v)



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(a)
(i)

Given $\mathbf{r}\bullet\mathbf{q}=\mathbf{p}\bullet\mathbf{q}$.
R represents any point on the plane that is perpendicular to $\mathbf{q}$ and containing the point P.

(ii)

Given $\mathbf{r}\times\mathbf{q}=\mathbf{p}\times\mathbf{q}$

$\mathbf{r}\times\mathbf{q}-\mathbf{p}\times\mathbf{q}=\mathbf{0}$

$(\mathbf{r}-\mathbf{p})\times\mathbf{q}=\mathbf{0}$

$(\mathbf{r}-\mathbf{p})//\mathbf{q}$

$(\mathbf{r}-\mathbf{p})=k\mathbf{q},\ k\in\mathbb{R}$

$\therefore \mathbf{r}=\mathbf{p}+k\mathbf{q},\ k\in\mathbb{R}$

R represents any point on the line containing the point P and parallel to $\mathbf{q}$.

(b)

Given that $AC:CB = 3:2$. By Ratio Theorem, $\mathbf{b} = \dfrac{\mathbf{c}+\mathbf{2a}}{3} \Rightarrow \mathbf{c} = 3\mathbf{b} - \mathbf{2a}$.

Consider $|2\mathbf{a}-\mathbf{c}|^2 = (2\mathbf{a}-\mathbf{c})\bullet(2\mathbf{a}-\mathbf{c})$

$|2\mathbf{a}-\mathbf{c}|^2 = 4\mathbf{a}\bullet\mathbf{a} - \mathbf{2a}\bullet\mathbf{c} - \mathbf{c}\bullet\mathbf{2a} + \mathbf{c}\bullet\mathbf{c}$

$|2\mathbf{a}-\mathbf{c}|^2 = 4|\mathbf{a}|^2 - 4\mathbf{a}\bullet\mathbf{c} + |\mathbf{c}|^2$ (Since $\mathbf{a}\bullet\mathbf{c} = \mathbf{c}\bullet\mathbf{a}$, $\mathbf{a}\bullet\mathbf{a} = |\mathbf{a}|^2$, $\mathbf{c}\bullet\mathbf{c} = |\mathbf{c}|^2$)

$|2\mathbf{a}-\mathbf{c}|^2 = 4|\mathbf{a}|^2 - 4\mathbf{a}\bullet(3\mathbf{b} - \mathbf{2a}) + |\mathbf{c}|^2$

$|2\mathbf{a}-\mathbf{c}|^2 = 4|\mathbf{a}|^2 - 12\mathbf{a}\bullet\mathbf{b} + 8|\mathbf{a}|^2 + |\mathbf{c}|^2$

$|2\mathbf{a}-\mathbf{c}|^2 = 12|\mathbf{a}|^2 - 12|\mathbf{a}||\mathbf{b}|\cos 60^\circ + |\mathbf{c}|^2$

$|2\mathbf{a}-\mathbf{c}|^2 = 12(1)^2 - 12(1)(1)\left(\dfrac{1}{2}\right) + \left(\sqrt{7}\right)^2$

$|2\mathbf{a}-\mathbf{c}|^2 = 13$

$\therefore |2\mathbf{a}-\mathbf{c}| = \sqrt{13}$ since $|2\mathbf{a}-\mathbf{c}| \geq 0$

6

(i)

Let X be the mass of the badminton racket.
Let μ be the mean mass of a badminton racket.

H_0: $\mu = 800$
H1: $\mu \neq 800$

Under H_0, since n is large, By Central Limit Theorem, $\overline{X} \sim N\left(800, \dfrac{20^2}{n}\right)$ approximately.

Test statistic, $z = \dfrac{\overline{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \dfrac{807-800}{\frac{20}{\sqrt{n}}} = \dfrac{7\sqrt{n}}{20}$

At 2% level of significance, for a two-tail test, critical region is $z < -2.3263$ or $z > 2.3263$
Since H_0 is rejected, z lies inside the critical region.

$\dfrac{7\sqrt{n}}{20} < -2.3263$ or $\dfrac{7\sqrt{n}}{20} > 2.3263$

(no real solution)

$\sqrt{n} > 6.6467$

$n > 44.179$

Least $n = 45$

(ii)

There is no need to know anything about the distribution of the population since the sample size is large, by Central Limit Theorem, the sample mean distribution is approximately normal.

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(i)

Number of ways = $^4C_2 = 6$

(ii)

<u>Case 1: All same colour</u>
$^3C_1 = 3$ (excludes green bricks)

<u>Case 2: 2 different colours</u>
$^4C_2 = 6$ (to choose 2 colours)
$^2C_1 = 2$ (to choose the colour which will be repeated)
Number of ways = $6 \times 2 = 12$

Or
$^4C_1 \times {}^3C_1 = 12$

<u>Case 3: All different colours</u>
$^4C_3 = 4$

Total number of ways = 3 + 12 + 4 = 19

(iii)

<u>Case 1: 2R 1B 1G</u>
$^5C_2 \times {}^4C_1 \times {}^2C_1 = 80$

<u>Case 2: 2R 2Y</u>
$^5C_2 \times {}^4C_2 = 60$

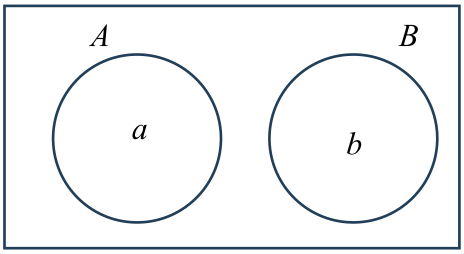
<u>Case 3: 1R 2B 1Y</u>
$^5C_1 \times {}^4C_2 \times {}^4C_1 = 120$

<u>Case 4: 4B</u>
$^4C_4 = 1$

Total number of ways = 80 + 1 + 120 + 60 = 261

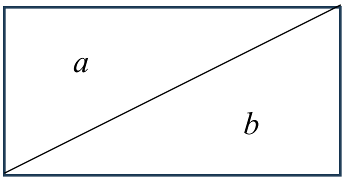
8

(i)

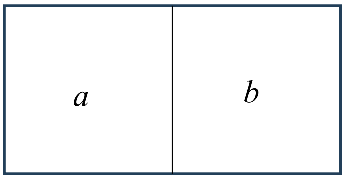


$\mathrm{P}(A' \cap B') = 1 - \mathrm{P}(A) - \mathrm{P}(B) = 1 - a - b$

(ii)



or



If A' and B' are mutually exclusive events, then

$\mathrm{P}(A' \cap B') = 1 - a - b = 0$

$a + b = 1$

(iii)

Since A and C are independent events, $\mathrm{P}(A \cap C) = ac$

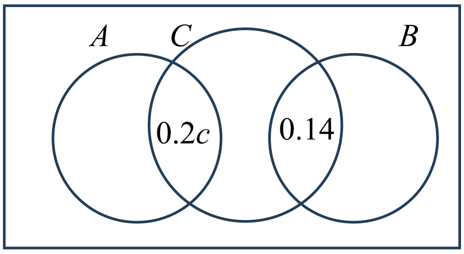
$\mathrm{P}(A \cup B \cup C)$

$= \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap C) - \mathrm{P}(B \cap C)$

$= a + b + c - ac - 0.14$

$= 0.2 + 0.3 + c - 0.2c - 0.14$

$= 0.36 + 0.8c$



(iv)

$\mathrm{P}(A' \cap B' \cap C)$

$= \mathrm{P}(A \cup B \cup C) - \mathrm{P}(A) - \mathrm{P}(B)$

$= 0.36 + 0.8c - 0.2 - 0.3$

$= 0.8c - 0.14$

Or

$\mathrm{P}(A' \cap B' \cap C)$

$= \mathrm{P}(C) - \mathrm{P}(A \cap C) - \mathrm{P}(B \cap C)$

$= c - ac - 0.14$

$= c - 0.2c - 0.14$

$= 0.8c - 0.14$

(v)

$\mathrm{P}(A \cup B \cup C) = 0.36 + 0.8c \le 1$ and $\mathrm{P}(A' \cap B' \cap C) = 0.8c - 0.14 \ge 0$

$0.8c \le 0.64$ $\qquad$ $0.8c \ge 0.14$

$c \le 0.8$ $\qquad$ $c \ge 0.175$

$0.175 \le c \le 0.8$

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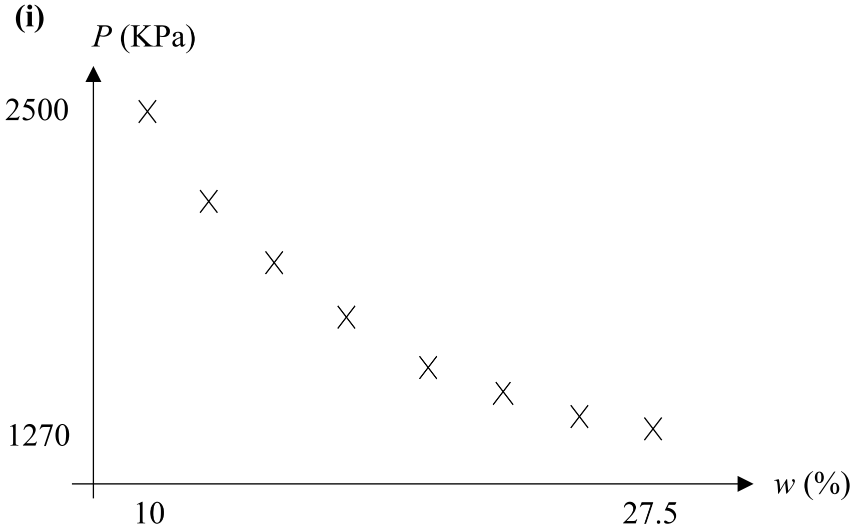
(i)



From the scatter plot,

the points seem to lie close on a curve rather than a straight line.

Or

P decreases at a decreasing rate as w increases.

Therefore, the relationship between P and w is unlikely to be well modelled by a linear equation of the form $P = aw + b$, where a and b are constants.

(ii)

Between P and w: $r = -0.949$ (3 sf)

Between P and $\sqrt{w}$: $r = -0.969$ (3 sf)

Since the product moment correlation coefficient between P and $\sqrt{w}$ is closer to -1 as compared with that between P and w, there is a stronger negative linear relationship between P and $\sqrt{w}$. Hence, the model given by $P = a\sqrt{w} + b$ is better.

From GC, $P = -569.66\sqrt{w} + 4130.0$

The equation of the line is $P = -570\sqrt{w} + 4130$.

(iii)

When $w = 40$, $P = -569.66\sqrt{40} + 4130.0 = 527$

As $w = 40$ is outside the range of the given data, an extrapolation is being done. The linear relationship between P and $\sqrt{w}$ may not hold at this extrapolated range. Estimation may not be reliable.

(iv)

For the regression line to remain unchanged, $\left(\sqrt{w_9}, P_9\right)$ which resulted from the missing data point $\left(w_9, P_9\right)$ must be equivalent to $\left(\overline{\sqrt{w}}, \overline{P}\right)$ for the 8 points.

From GC, $\overline{P} = 1693.75 = P_9$

Shear strength for the data point is 1694 KPa (nearest integer).

(v)

$$1000P = -569.66\sqrt{w} + 4130.0$$
$$P = -0.56966\sqrt{w} + 4.130$$
$$P = -0.570\sqrt{w} + 4.13$$

The product moment correlation coefficient would not differ as product moment correlation coefficient is not affected by a change in unit of the variables (or scaling of the graph).

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(i)

$X =$ BMI of 18-year-old boys $\quad X \sim \mathrm{N}\left(22.7, \sigma^2\right)$

$$\mathrm{P}\left(X \leq 16.7\right) = 0.05$$
$$\mathrm{P}\left(Z \leq \frac{16.7 - 22.7}{\sigma}\right) = 0.05$$
$$\frac{-6}{\sigma} = -1.6449$$
$$\sigma = 3.6476 \approx 3.65 \text{ (shown)}$$

(ii)

$\mathrm{P}\left(X \geq m\right) \leq 0.1$
Using GC, $m \geq 27.4$
Minimum value of BMI is 27.4

(iii)

$Y =$ BMI of 15-year-old boys $\quad Y \sim \mathrm{N}\left(21.6, 3.49^2\right)$

$$\frac{X+Y}{2} \sim \mathrm{N}\left(\frac{22.7 + 21.6}{2}, \frac{3.6476^2 + 3.49^2}{4}\right)$$
$$\frac{X+Y}{2} \sim \mathrm{N}(22.15,\ 6.3713)$$

$$\mathrm{P}\left(\frac{X+Y}{2} > 20.1\right) \approx 0.792$$

(iv)

The BMI of a randomly chosen 15-year-old boy and a randomly chosen 18-year-old boy are independent of each other.

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Since $\mathrm{P}(T=1)=\mathrm{P}(T=4)=\frac{1}{6}$, then $\mathrm{P}(T=2)+\mathrm{P}(T=3)=\frac{4}{6}$.

If $\mathrm{P}(T=2)=\mathrm{P}(T=3)=\frac{2}{6}$, then the modes of T are 2 and 3, which contradicts the question.

Hence, $\mathrm{P}(T=2)=\frac{3}{6}=\frac{1}{2}$ and $\mathrm{P}(T=3)=\frac{1}{6}$.

t	1	2	3	4
$\mathrm{P}(T=t)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$

(i)

$$\mathrm{P}(Y=0)=\mathrm{P}(1\text{ or }3)+\mathrm{P}(2)\times\mathrm{P}(1\text{ or }3)+\mathrm{P}(2)\times\mathrm{P}(2)\times\mathrm{P}(1\text{ or }3)+\left[\mathrm{P}(2)\right]^3\times\mathrm{P}(1\text{ or }3)+\dots$$

$$=\frac{2}{6}+\left(\frac{1}{2}\right)\left(\frac{2}{6}\right)+\left(\frac{1}{2}\right)^2\left(\frac{2}{6}\right)+\left(\frac{1}{2}\right)^3\left(\frac{2}{6}\right)+\dots$$

$$=\frac{\frac{2}{6}}{1-\frac{1}{2}}$$

$$=\frac{2}{3} \quad \text{(shown)}$$

(ii)

y	0	1	2	3	…
$\mathrm{P}(Y=y)$	$\frac{2}{3}$	$\frac{1}{6}$	$\left(\frac{1}{2}\right)\frac{1}{6}$	$\left(\frac{1}{2}\right)^2\frac{1}{6}$	

$$\mathrm{E}\left(Y^2\right)=0+1^2\left(\frac{1}{6}\right)+2^2\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)+3^2\left(\frac{1}{2}\right)^2\left(\frac{1}{6}\right)+\ldots$$

$$=\frac{1}{6}\left(1^2+2^2\left(\frac{1}{2}\right)+3^2\left(\frac{1}{2}\right)^2+\ldots\right)$$

$$=\frac{1}{6}\times\frac{1+\frac{1}{2}}{\left(1-\frac{1}{2}\right)^3}=2$$

$$\mathrm{Var}\left(Y\right)=\mathrm{E}\left(Y^2\right)-\left(\mathrm{E}\left(Y\right)\right)^2=2-\left(\frac{2}{3}\right)^2=\frac{14}{9}$$

12

(i)

The probability that a screen protector is cracked is constant at 0.04.
The event that a screen protector is cracked is independent of any other screen protector.

(ii)

Let X = number of cracked screen protectors out of n
$X \sim \mathrm{B}(n, 0.04)$

Given that mode is 2,

$\mathrm{P}(X=1)<\mathrm{P}(X=2)$ and $\mathrm{P}(X=3)<\mathrm{P}(X=2)$

$$\binom{n}{1}(0.04)^1(0.96)^{n-1}<\binom{n}{2}(0.04)^2(0.96)^{n-2}$$

$$\frac{n!}{1!(n-1)!}(0.04)(0.96)^{n-1}<\frac{n!}{2!(n-2)!}(0.04)^2(0.96)^{n-2}$$

$$\frac{2!}{1!}\frac{(0.96)^{n-1}}{(0.96)^{n-2}}<\frac{(n-1)!}{(n-2)!}\frac{(0.04)^2}{0.04}$$

$$2(0.96)<(n-1)(0.04)$$

$$n>49$$

$$\binom{n}{3}(0.04)^3(0.96)^{n-3}<\binom{n}{2}(0.04)^2(0.96)^{n-2}$$

$$\frac{n!}{3!(n-3)!}(0.04)^3(0.96)^{n-3}<\frac{n!}{2!(n-2)!}(0.04)^2(0.96)^{n-2}$$

$$\frac{(n-2)!}{(n-3)!}\frac{(0.04)^3}{(0.04)^2}<\frac{3!}{2!}\frac{(0.96)^{n-2}}{(0.96)^{n-3}}$$

$$(n-2)(0.04)<3(0.96)$$

$$n<74$$

$$\therefore 49<n<74 \qquad n\in\mathbb{Z}$$

(iii)

Y = number of cracked screen protectors out of 50
$Y \sim \mathrm{B}(50, 0.04)$

Required Probability $=\mathrm{P}(Y\le 3)$

$=0.86087\approx 0.861 \quad (3 \text{ s.f.})$

(iv)

Required Probability

$= \mathrm{P}(Y > 2 \mid Y \leq 6)$

$= \dfrac{\mathrm{P}(Y > 2 \ \cap \ Y \leq 6)}{\mathrm{P}(Y \leq 6)}$

$= \dfrac{\mathrm{P}(2 < Y \leq 6)}{\mathrm{P}(Y \leq 6)}$

$= \dfrac{\mathrm{P}(Y \leq 6) - \mathrm{P}(Y \leq 2)}{\mathrm{P}(Y \leq 6)}$

$= \dfrac{0.99639 - 0.67671}{0.99639}$

$\approx 0.321 \quad (3 \text{ s.f.})$

(v)

Required Probability

$= {}^{6}C_{3} \times (0.86087)^{3} \times (1 - 0.86087)^{3} \times (0.86087)$

$\approx 0.0296 \quad (3 \text{ s.f.})$