2010 Y6 Mathematics HL Mock Paper 1 (Solutions)

Qn	Solution	Marks
1		[6 marks]
(a)	$\det A = -2 - 6k$, $\det B = 7h + 9$ Since $\det A = \det B$ and $\det AB = 256h$ $\mathbf{det}\,\boldsymbol{AB} = \mathbf{det}\,\boldsymbol{A} \cdot \mathbf{det}\,\boldsymbol{B} = (\det B)^2 = (7h+9)^2$, i.e., $(7h+9)^2 = 256h$. Expand and simplify, we have $49h^2 - 130h + 81 = 0$.	
(b)	By solving the equation in part (a), $(h-1)(49h-81) = 0$, $h = 1$ or $h = 81/49$ (N.A., since h is an integer) Note that $\det A = \det B$, so $-2 - 6k = 7h + 9$, which implies $k = -3$.	

REMARK: Don't waste your time to calculate the matrix $\boldsymbol{AB}$.

Qn	Solution	Marks
2		[6 marks]

The intersections of the two curves are given by $y = 2 - 3x + x^2 = 2 + x - x^2$, i.e., $2x^2 - 4x = 0$. Thus, $x = 0$ or 2.

$$\text{Area} = \int_0^2 (2 + x - x^2 - 2 + 3x - x^2)dx = \int_0^2 (4x - 2x^2)dx = \left[2x^2 - \frac{2}{3}x^3\right]_0^2 = \frac{8}{3}$$

REMARK: To find out which curve is higher, substitute x = 1 to see which y-coordinate is greater.

Qn	Solution	Marks
3		[6 marks]

(a)

$$\int_{-a}^{a} f(t)dt = 1 \Rightarrow \int_{-a}^{a} \frac{\pi}{4}\cos\left(\frac{\pi t}{2}\right)dt = \left[\frac{1}{2}\sin\left(\frac{\pi t}{2}\right)\right]_{-a}^{a}$$
$$= \frac{1}{2}\sin\left(\frac{a\pi}{2}\right) - \frac{1}{2}\sin\left(-\frac{a\pi}{2}\right) = \sin\left(\frac{a\pi}{2}\right) = 1$$

Thus,

$$\sin\left(\frac{a\pi}{2}\right) = 1 \Rightarrow \frac{a\pi}{2} = \frac{\pi}{2}$$

Thus, $= 1$.

(b)

$$F(x) = \int_{-1}^{x} f(t)dt = \int_{-1}^{x} \frac{\pi}{4}\cos\left(\frac{\pi t}{2}\right)dt = \left[\frac{1}{2}\sin\left(\frac{\pi t}{2}\right)\right]_{-1}^{x}$$
$$= \frac{1}{2}\sin\left(\frac{x\pi}{2}\right) - \frac{1}{2}\sin\left(-\frac{\pi}{2}\right) = \frac{1}{2}\left(\sin\left(\frac{x\pi}{2}\right) + 1\right)$$

REMARK: You can also find c.d.f. in the following way:

$$F(x) = \int f(x)dx = \int \frac{\pi}{4}\cos\left(\frac{\pi x}{2}\right)dx = \frac{1}{2}\sin\left(\frac{x\pi}{2}\right) + C, \text{then use } F(-1) = 0 \text{ } or \text{ } F(1) = 1 \text{ to find C.}$$

(c)

$$F(x) = \frac{1}{2}\left(\sin\left(\frac{x\pi}{2}\right) + 1\right)$$

So,

$$F(x)=\frac{1}{4} \Rightarrow \frac{1}{2}\left(\sin\left(\frac{x\pi}{2}\right)+1\right)=\frac{1}{4} \Rightarrow \sin\left(\frac{x\pi}{2}\right)=-\frac{1}{2} \Rightarrow \frac{x\pi}{2}=-\frac{\pi}{6} \Rightarrow x=-\frac{1}{3}$$
$$F(x)=\frac{3}{4} \Rightarrow \frac{1}{2}\left(\sin\left(\frac{x\pi}{2}\right)+1\right)=\frac{3}{4} \Rightarrow \sin\left(\frac{x\pi}{2}\right)=\frac{1}{2} \Rightarrow \frac{x\pi}{2}=\frac{\pi}{6} \Rightarrow x=\frac{1}{3}$$

Thus, the interquartile range is $\frac{1}{3}-\left(-\frac{1}{3}\right)=\frac{2}{3}$.

Qn	Solution	Marks
4		[6 marks]

Let X be the marks, and $X \sim N(\mu, \sigma^2)$. We are given
$$P(X \geq 80)=0.1 \quad \text{and} \quad P(X \geq 65)=0.3$$
Then **by inverse normal distribution table (NOT GDC)**
$$\frac{80-\mu}{\sigma}=invnorm(0.9)=1.2816$$
$$\frac{65-\mu}{\sigma}=invnorm(0.7)=0.5244$$
i.e.,
$$\begin{cases}\mu-1.2816\sigma=80\\ \mu-0.5244\sigma=65\end{cases}$$
Solve the simultaneous equations (2 s.f.):
$$\begin{cases}\sigma=-19.81\approx-20\\ \mu=54.61\approx 55\end{cases}$$

REMARK: You also need to know how to use table to find other values, for example invnorm(0.1).

Qn	Solution	Marks
5		[6 marks]

- The range of arcsin function is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$:
  $4>\pi$, so $-\frac{\pi}{2}\leq \pi-4\leq\frac{\pi}{2}$, then
  $$\frac{\arcsin(\sin 4)}{4}=\frac{\arcsin(\sin(\pi-4))}{4}=\frac{\pi-4}{4}=\frac{\pi}{4}-1$$
- The range of arccos function is $[0,\pi]$:
  $0<3<\pi$, then
  $$\frac{\arccos(\cos 3)}{3}=\frac{3}{3}=1$$
- The range of arctan function is $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ (open interval):
  $2>\frac{\pi}{2}$, so $-\frac{\pi}{2}\leq 2-\pi\leq\frac{\pi}{2}$, then
  $$\frac{\arctan(\tan 2)}{2}=\frac{\arctan(\tan(2-\pi))}{2}=\frac{2-\pi}{2}=1-\frac{\pi}{2}$$
- The range of arccot function is $]0,\pi[$ (open interval):
  $0<1<\pi$, then
  $$\frac{\text{arccot}(\cot 1)}{1}=\frac{1}{1}=1$$

Thus,
$$\frac{\arcsin(\sin 4)}{4}+\frac{\arccos(\cos 3)}{3}+\frac{\arctan(\tan 2)}{2}+\frac{\text{arccot}(\cot 1)}{1}=2-\frac{\pi}{4}.$$

4 is out of the range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

4 is in the 3rd quadrant so
$\sin 4=-\sin(4-\pi)=\sin(\pi-4)$

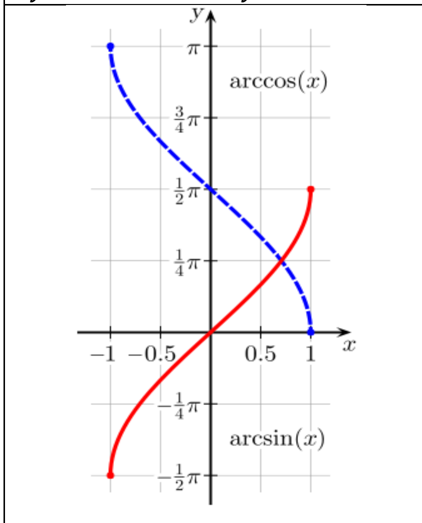
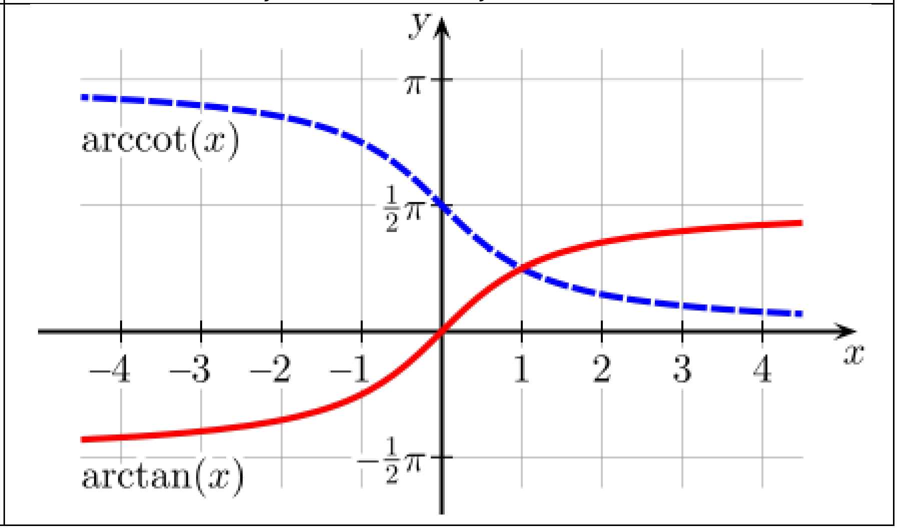
3 is within the range $[0,\pi]$

2 is out of the range $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$

2 is in the 2nd quadrant so
$\tan 2=-\tan(\pi-2)=\tan(2-\pi)$

1 is within the range $]0,\pi[$

REMARK: Graph of inverse trigo functions:

$y = \arcsin x$ and $y = \arccos x$	$y = \arctan x$ and $y = \arccot x$
	

Qn	Solution	Marks
6		[6 marks]

Given that b is real and positive, $b + i$ is in the first quadrant $\Rightarrow 0 < \arg(b+i) < \dfrac{\pi}{2}$.

Thus, $0 < arg(b+i)^2 < \pi$, so $arg(b+i)^2 = \arg z = \frac{\pi}{3}$.
$\arg(b+i) = \frac{\pi}{6}$ implies $b = 1 \times \cot\frac{\pi}{6} = \sqrt{3}$ since $\frac{1}{b} = tan\frac{\pi}{6}$.

Qn	Solution	Marks
7		[6 marks]

$$
\begin{aligned}
E\left[\left(X - E(X)\right)^2\right] &= E\left[X^2 - 2XE(X) + \left(E(X)\right)^2\right] \\
&= E[X^2] + E[-2XE(X)] + E\left[\left(E(X)\right)^2\right] \\
&= E(X^2) - 2E(X)E(X) + \left(E(X)\right)^2 \\
&= E(X^2) - \left(E(X)\right)^2
\end{aligned}
$$

Note that $\left(X - E(X)\right)^2 \geq 0$ for any random variable X, so we have $E\left[\left(X - E(X)\right)^2\right] \geq 0$.
Thus, $E(X^2) - \left(E(X)\right)^2 \geq 0$, i.e., $E(X^2) \geq \left(E(X)\right)^2$

Qn	Solution	Marks
8		[6 marks]

First, change the equations of lines to the standard form:

$$\frac{x-1}{1} = \frac{y-1}{-2} = \frac{z-2}{1} \quad \text{and} \quad \frac{x+1}{3} = \frac{y-2}{-3} = \frac{z+2}{5}.$$

The normal of the plane is given by

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix}$$

So the equation of the plane is of the form

$$-7x - 2y + 3z = d$$

Substitute the point $(1,1,2)$, we have $d = -3$, i.e., the equation of the plane is

$$-7x - 2y + 3z = -3.$$

OR The equation of the plane is given by

$$\boldsymbol{r} = \begin{pmatrix}1\\1\\2\end{pmatrix} + \mu\begin{pmatrix}1\\-2\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\-3\\5\end{pmatrix}.$$

REMARK: The second solution is much easier than the first one. But when the question is in section B, and the later parts need the Cartesian equation, you still need to use the first method.

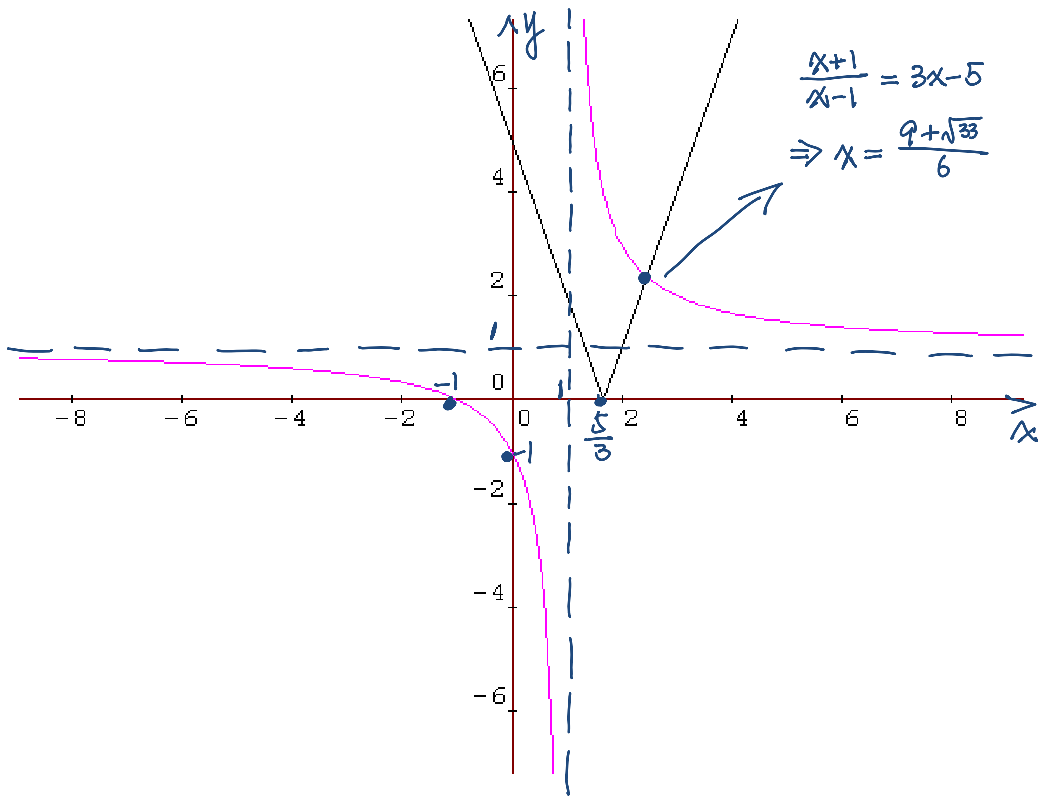
Qn	Solution	Marks
9		[6 marks]
(a)	From the graph the amplitude is 3, and max is 2, i.e., $a = 3,\ c = -1$. You can consider the transformation from $y = \sin x$ to $y = a\sin(x+b)+c$ first. So the transformation is given by • Translate the graph along negative y-axis by c; • Scale the graph along y-axis by $\frac{1}{a}$ • Translate the graph along positive x-axis by b;	
(b)	From the graph the amplitude is 3, and max is 2, i.e., $a = 3,\ c = -1$. The point $\left(\frac{3\pi}{4}, 2\right)$ will be shifted to the left to become $\left(\frac{\pi}{2}, 2\right)$ So $b = \frac{\pi}{2} - \frac{3\pi}{4} = -\frac{\pi}{4}$ $y = 3\sin\left(x - \frac{\pi}{4}\right) - 1$ becomes $y = 3\sin\left(x - \frac{\pi}{4}\right)$, then $y = \sin\left(x - \frac{\pi}{4}\right)$ and then $y = \sin x$	

Qn	Solution	Marks
10		[6 marks]
	$(1-2x)^5(1+3x)^4 = (1 - 10x + 40x^2 - \cdots)(1 + 12x + 54x^2 + \cdots) = 1 + 2x - 26x^2$ So, $a = 1$, $b = 2$, $c = -26$.	

Qn	Solution	Marks
11		[6 marks]
(a)	$f(g(x)) = (ax+b+2)^2 - 3$ $= a^2x^2 + abx + 2ax + abx + b^2 + 2b + 2ax + 2b + 4 - 3$ $= a^2x^2 + x(2ab + 4a) + (b^2 + 4b + 1)$ $(\text{Given that}) = 4x^2 + 6x - \frac{3}{4}$ Equating coefficients of x^2 gives $a^2 = 4$. Thus, $a = 2$, since $a > 0$. Equating coefficients of x gives $2ab + 4a = 6$, so $b = -\frac{1}{2}$.	
(b)	$h(k(x)) = 5(cx^2 - x + 2) + 2 = 0$ $5cx^2 - 5x + 12 = 0$ Condition for real equal roots is $b^2 - 4ac = 0$, thus, $25 - 240c = 0$, i.e., $c = \frac{5}{48}$.	

Qn	Solution	Marks
12		[14 marks]

(a)



(b) By the graph of part (a), there is only one intersection between the two graphs.
For the intersection, it satisfies

$$\frac{x+1}{x-1} = 3x-5 \Rightarrow x = \frac{9+\sqrt{33}}{6} \quad \text{or} \quad \frac{9-\sqrt{33}}{6}\left(\text{N.A., since } x < \frac{5}{3}\right)$$

Therefore, the solution to the inequality is given by

$$x < 1 \quad or \quad x > \frac{9+\sqrt{33}}{6}.$$

Qn	Solution	Marks
13		[16 marks]

(a) Let P_n be the statement: $\frac{d^n}{dx^n}(\cos x) = \cos\left(x + \frac{n\pi}{2}\right)$ for $n \in Z^+$.

P_1: $\frac{d}{dx}(\cos x) = -\sin x = \cos\left(x + \frac{\pi}{2}\right) \quad \Rightarrow P_1$ *is* true.

Assume P_k is true i.e. $\frac{d^k}{dx^k}(\cos x) = \cos\left(x + \frac{k\pi}{2}\right)$

Then for $n = k+1$

$$\frac{d^{k+1}}{dx^{k+1}}(\cos x) = \frac{d}{dx}\left[\frac{d^k}{dx^k}(\cos x)\right]$$
$$= \frac{d}{dx}\left[\cos\left(x + \frac{k\pi}{2}\right)\right]$$

$$= -\sin\left(x + \frac{k\pi}{2}\right)$$
$$= \sin\left(-x - \frac{k\pi}{2}\right)$$
$$= \cos\left[\frac{\pi}{2} - \left(-x - \frac{k\pi}{2}\right)\right]$$
$$= \cos\left(x + \frac{(k+1)\pi}{2}\right)$$

P_{k+1} is true whenever P_1 and P_k are true.
Hence P_n is true for all $n \in \mathbb{Z}^+$ by Mathematical Induction.

(b) We know that $\sin 4x = \cos\left(\frac{\pi}{2} - 4x\right)$,
$\cos(x) = \cos(x + 2k\pi)$ *and* $\cos x = \cos(-x) = \cos(-x + 2k\pi)$
Thus

$$\sin 4x = \cos x \Rightarrow \cos\left(\frac{\pi}{2} - 4x\right) = \cos(x + 2k\pi) \text{ and } \cos\left(\frac{\pi}{2} - 4x\right) = \cos(-x + 2k\pi)$$

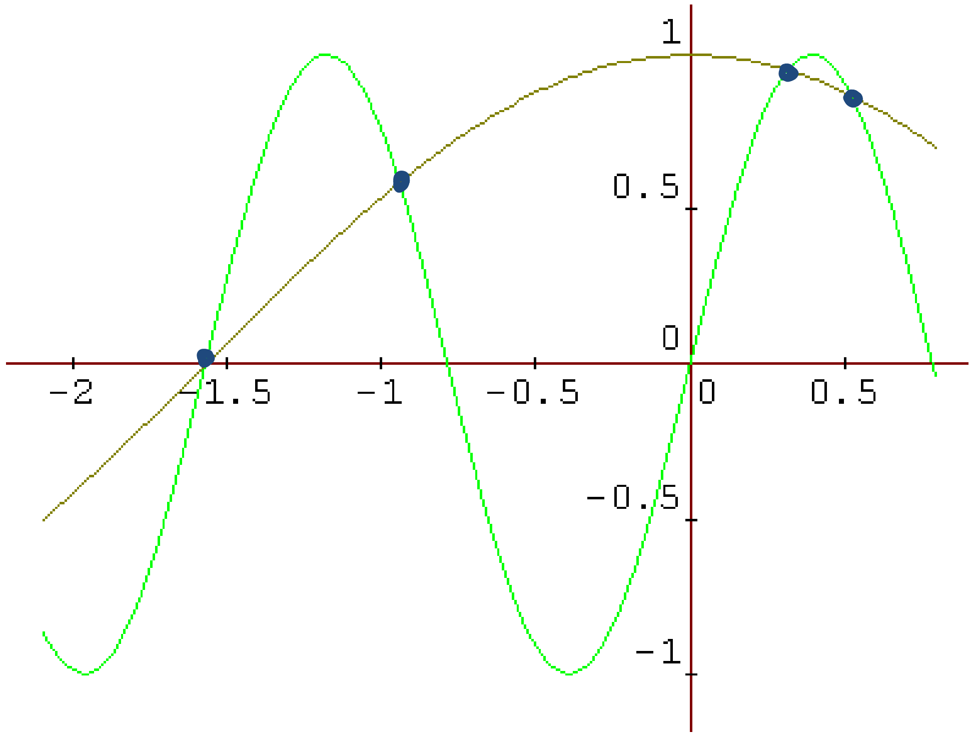
Hence the following cases arise:

- $\frac{\pi}{2} - 4x = x \Rightarrow x = \frac{\pi}{10}$
- $\frac{\pi}{2} - 4x = x + 2\pi \Rightarrow x = -\frac{3\pi}{10}$
- $\frac{\pi}{2} - 4x = x - 2\pi \Rightarrow x = \frac{\pi}{2}$ $\left(\text{N.A. since } x > \frac{\pi}{4}\right)$
- $\frac{\pi}{2} - 4x = x + 4\pi \Rightarrow x = -\frac{7\pi}{10}$ $\left(\text{N.A. since } x < -\frac{2\pi}{3}\right)$
- $\frac{\pi}{2} - 4x = -x \Rightarrow x = \frac{\pi}{6}$
- $\frac{\pi}{2} - 4x = -x + 2\pi \Rightarrow x = -\frac{\pi}{2}$
- $\frac{\pi}{2} - 4x = -x - 2\pi \Rightarrow x = \frac{5\pi}{6}$ $\left(\text{N.A. since } x > \frac{\pi}{4}\right)$
- $\frac{\pi}{2} - 4x = -x + 4\pi \Rightarrow x = -\frac{7\pi}{6}$ $\left(\text{N.A. since } x < -\frac{2\pi}{3}\right)$

Thus, the answers are

$$x = -\frac{\pi}{2}, -\frac{3\pi}{10}, \frac{\pi}{10}, \frac{\pi}{6}.$$

We can use GDC to check (when practising)



Qn	Solution	Marks
14		[17 marks]

(a) (i) $\overrightarrow{BA} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} -2 \\ m \\ 1 \end{pmatrix}$, so $\overrightarrow{BA} \cdot \overrightarrow{BC} = 1 - m$.

(ii)

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = \left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right| \cos A\hat{B}C = \sqrt{3}\sqrt{5 + m^2} \times \frac{\sqrt{2}}{3}$$

Square both sides,

$$(1 - m)^2 = \frac{2}{3}(5 + m^2) \;\Rightarrow\; m^2 - 6m - 7 = 0 \;\Rightarrow\; m = -1 \;\text{ or }\; m = 7\ (\text{N.A.})$$

(b) Using vector product:

$$\boldsymbol{n} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & -1 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$$

Substituting coordinates of a point (e.g. $A(2, -1, 0)$)

$$-2(x - 2) + 3(y + 1) - z = 0 \quad (-2x + 3y - z = -7)$$

(c)

$$\text{Area of } ABC = \frac{1}{2}\left|\overrightarrow{BA} \cdot \overrightarrow{BC}\right| = \frac{1}{2}\left|\begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}\right| = \frac{\sqrt{14}}{2}$$

(d) (i) Line perpendicular to plane ABC $\Rightarrow$ line parallel to $\boldsymbol{n}$.

Equation of line is $r = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$

(ii) $\overrightarrow{AD} = \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix}$, so volume of pyramid $= \frac{1}{3} \times \text{area}_{ABC} \times \left|\overrightarrow{AD}\right| = \frac{1}{3}\frac{\sqrt{14}}{2}\sqrt{56} = \frac{14}{3}$.