Question 1	
(a) **[2]**	$T_0(x)=1$ $T_1(x)=x$ $T_2(\cos\theta)=\cos(2\theta)=2\cos^2\theta-1$ and thus $T_2(x)=2x^2-1$.
(b) **[3]**	$T_{n+1}(x)+T_{n-1}(x)$ $=\cos\left((n+1)\cos^{-1}x\right)+\cos\left((n-1)\cos^{-1}x\right)$ $=2\cos\left(n\cos^{-1}x\right)\cos\left(\cos^{-1}x\right)$ $=2xT_n(x)$ Alternate Solution $T_{n+1}(\cos\theta)=\cos\left((n+1)\theta\right)$ $=\cos(n\theta)\cos\theta-\sin(n\theta)\sin\theta$ $=(\cos\theta)T_n(\cos\theta)+\frac{1}{2}\left[\cos\left((n+1)\theta\right)-\cos\left((n-1)\theta\right)\right]$ $=(\cos\theta)T_n(\cos\theta)+\frac{1}{2}\left[T_{n+1}(\cos\theta)-T_{n-1}(\cos\theta)\right]$ $T_{n+1}(\cos\theta)=2(\cos\theta)T_n(\cos\theta)-T_{n-1}(\cos\theta)$ $T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)$
(c) **[3]**	We have $T_0(0)=1$ and $T_1(0)=0$. Substituting $x=0$ into **(b)** we get $T_{n+1}(0)=-T_{n-1}(0)$ and thus $T_n(0)=\begin{cases}0 & \text{if } n \text{ is odd}\\(-1)^{\frac{n}{2}} & \text{if } n \text{ is even}\end{cases}$ Alternatively, $T_n(0)=\cos\left(n\cos^{-1}0\right)=\cos\left(\frac{n\pi}{2}\right)=\begin{cases}0 & \text{if } n \text{ is odd}\\(-1)^{\frac{n}{2}} & \text{if } n \text{ is even}\end{cases}$

(d) **[2]**	$T_n(x)=0$ $\Rightarrow 0=T_n(\cos\theta)=\cos(n\theta)$ $\Rightarrow n\theta=\dfrac{\pi}{2}+k\pi, k\in\mathbb{Z}$ $\Rightarrow \theta=\dfrac{\pi}{2n}+\dfrac{k\pi}{n}, k\in\mathbb{Z}$ Since the polynomial is of degree *n* (a simple recursion using the relation in **(b)**) we must have *n* roots to the equation. The real numbers $\cos\theta_k=\cos\left(\dfrac{\pi}{2n}+\dfrac{k\pi}{n}\right), k\in\mathbb{Z}$ are therefore the roots of $T_n(x)=0$. If we restrict *k* to be 0, 1, …, *n* – 1, we have *n* distinct roots as the function cosine is a bijection from $(0,\pi)$ to $(-1,1)$.
(e) **[4]**	From **(d)**, we know that $T_n(x)=a(x-x_0)(x-x_1)...(x-x_{n-1}),\ x_k=\cos\left(\dfrac{2k+1}{2n}\pi\right)$ Hence the desired product can be obtained by substituting 0 into the above relation and obtaining $T_n(0)=a(-x_0)(-x_1)...(-x_{n-1})$ $\quad\quad=a(-1)^n x_0x_1...x_{n-1}$ We need to find *a*. From **(b)**, we can see recursively that $a=2^{n-1}$. Therefore, $\displaystyle\prod_{k=0}^{n-1}\cos\left(\frac{2k+1}{2n}\pi\right)=x_0x_1...x_{n-1}$ $=\dfrac{T_n(0)}{a(-1)^n}$ $=\begin{cases}0 & \text{if } n \text{ is odd}\\ \dfrac{(-1)^{\frac{n}{2}}}{2^{n-1}(-1)^n}=\dfrac{(-1)^{\frac{n}{2}}}{2^{n-1}} & \text{if } n \text{ is even}\end{cases}$

Question 2	
(a) **[2]**	$S_1 = 2$, $S_2 = 3$, $S_3 = 4$
(b) **[4]**	The *well-spaced* subsets of $\{1, 2, 3, ..., n, n+1, n+2, n+3\}$ contains either the element $n+3$ or not. Case 1: $n+3$ is an element in the subset. Then elements $n+1$ and $n+2$ are not in the subset and the number of such *well-spaced* subsets is the number of *well-spaced* subsets, including the empty set, of the set $\{1, 2, 3, ..., n\} = S_n$ Case 2: $n+3$ is not an element in the subset. Thus the number of such *well-spaced* subsets is the number of *well-spaced* subsets, including the empty set, of the set $\{1, 2, 3, ..., n, n+1, n+2\} = S_{n+2}$ Hence $S_{n+3} = S_{n+2} + S_n$. $S_4 = S_3 + S_1 = 6$ $S_5 = S_4 + S_2 = 9$ $S_6 = S_5 + S_3 = 13$ $S_7 = S_6 + S_4 = 19$ $S_8 = S_7 + S_5 = 28$
(c) **[4]**	Let *A* be the set of *k*-combinations of $\{1, 2, 3, ..., n\}$ that are well-spaced and *B* be the set of *k*-combinations of *Y*, where $Y = \{1, 2, 3, ..., n-2k+2\}$. For any $a \in A$, $a = \{a_1, a_2, ..., a_k\}$ where we assume WLOG $a_1 < a_2 < ... < a_k$. We define a mapping f: $A \to B$ such that $\text{f}(a) = \{a_1, a_2 - 2, a_3 - 4, ..., a_k - 2(k-1)\}$. Note that f(*a*) $\in B$ since $a_k \le n \Rightarrow a_k - 2(k-1) \le n - 2(k-1)$. Clearly, f is injective since if *a*, *b* $\in$ *A* and *a* $\neq$ *b*, then $a_i \neq b_i$ for at least one *i*, so therefore $a_i - 2(i-1) \neq b_i - 2(i-1)$, so f(*a*) $\neq$ f(*b*). For each $p = \{p_1, p_2, ..., p_r\} \in B$, consider $q = \{p_1, p_2 + 2, ..., p_k + 2(k-1)\}$.

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We need to show that q is well-spaced, which is clear since the difference between any 2 consecutive terms is at least 3 (2 + 1).

Hence $q \in A$ and f(q) = p which implies that f is surjective.

Thus, f is a bijection and $|A| = |B| = \binom{n-2k+2}{k}$. (shown)

<u>Alternative Solution</u>

From the set $\{1,2,3,...,n\}$, if the number is selected, it is represented by ‘0’, otherwise, it is represented by ‘1’. That is, if the subset formed is $\{1,4,8\}$, its corresponding binary string representation of length n is 011011101111….111.

For the subset to be well-spaced, the binary string representation for such a subset must have at least two ‘1’s between any two consecutive ‘0’s.

Thus, to obtain a well-spaced subset of size k, there are k ‘0’s and $(n-k)$ ‘1’s. For the $(n-k)$ ‘1’s, $2(k-1)$ of them are placed as a pair between the ‘0’s as shown below.

0 1 1 0 1 1 0 1 1 0 …0 1 1 0

The remaining $(n-k-2(k-1)) = n-3k+2$ of the ‘1’s can be placed in any of the $(k+1)$ positions (i.e. before/after/in between the ‘0’s).
0 1 1 0 1 1 0 1 1 0 …0 1 1 0
↑ ↑ ↑ ↑ ↑ ↑ ↑

The number of ways this can be done is the same as the number of ways to distribute $(n-3k+2)$ identical objects into $(k+1)$ distinct boxes.
Hence $T_{n,k} = \binom{n-3k+2+(k+1)-1}{(k+1)-1} = \binom{n-2k+2}{k}$.
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<tr><td>(d)
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To have a well-spaced subset, we need

$n-2k+2 \geq k \Rightarrow k \leq \frac{n+2}{3}$

$S_n = \sum_{k=0}^{\left\lfloor \frac{n+2}{3} \right\rfloor} T_{n,k} = \sum_{k=0}^{\left\lfloor \frac{n+2}{3} \right\rfloor} \binom{n-2k+2}{k}$ (or equivalently $\sum_{k=0}^{\left\lceil \frac{n}{3} \right\rceil} \binom{n-2k+2}{k}$).
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Question 3	
(a) **[3]**	Let $y = -x$. Then $\int_{-a}^{a} \mathrm{g}(x)\mathrm{h}(x)\ \mathrm{d}x$ $= \int_{a}^{-a} \mathrm{g}(-y)\mathrm{h}(-y)\ (-\mathrm{d}y)$ $= \int_{-a}^{a} \mathrm{g}(-y)\mathrm{h}(-y)\ \mathrm{d}y$ Therefore, $\int_{-a}^{a} \mathrm{g}(x)\mathrm{h}(x)\ \mathrm{d}x$ $= \frac{1}{2}\left(\int_{-a}^{a} \mathrm{g}(x)\mathrm{h}(x)\ \mathrm{d}x + \int_{-a}^{a} \mathrm{g}(-x)\mathrm{h}(-x)\ \mathrm{d}x\right)$ $= \frac{1}{2}\left(\int_{-a}^{a} \mathrm{g}(x)\mathrm{h}(x)\ \mathrm{d}x + \int_{-a}^{a} \mathrm{g}(-x)\mathrm{h}(x)\ \mathrm{d}x\right)$ $= \frac{1}{2}\left(\int_{-a}^{a} \mathrm{h}(x)\left(\mathrm{g}(x)+\mathrm{g}(-x)\right)\ \mathrm{d}x\right)$ $= \frac{1}{2}\int_{-a}^{a} \mathrm{h}(x)\ \mathrm{d}x$ $= \int_{0}^{a} \mathrm{h}(x)\ \mathrm{d}x$ since h is even
(b) **[4]**	Let $\mathrm{h}(x) = \sqrt{1-x^2}$ which is even and $\mathrm{g}(x) = \frac{1}{1+2^x}$. Then $\mathrm{g}(x) + \mathrm{g}(-x) = \frac{1}{1+2^x} + \frac{1}{1+2^{-x}} = \frac{1}{1+2^x} + \frac{2^x}{2^x+1} = 1$. Hence $\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+2^x}\ \mathrm{d}x = \int_{0}^{1} \sqrt{1-x^2}\ \mathrm{d}x = \frac{\pi}{4}$ since the integral represents the area of the quadrant of the circle with radius 1 centered at the origin.

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Let us first evaluate

$$\int_0^2 \mathrm{f}(x)\mathrm{e}^{-x}\ \mathrm{d}x$$

$$= \int_0^1 x\mathrm{e}^{-x}\ \mathrm{d}x + \int_1^2 (2-x)\mathrm{e}^{-x}\ \mathrm{d}x$$

$$= \left[-x\mathrm{e}^{-x}\right]_0^1 + \int_0^1 \mathrm{e}^{-x}\ \mathrm{d}x + \left[-(2-x)\mathrm{e}^{-x}\right]_1^2 - \int_1^2 \mathrm{e}^{-x}\ \mathrm{d}x$$

$$= -\mathrm{e}^{-1} + \left[-\mathrm{e}^{-x}\right]_0^1 + \mathrm{e}^{-1} - \left[-\mathrm{e}^{-x}\right]_1^2$$

$$= -\mathrm{e}^{-1} + 1 + \mathrm{e}^{-2} - \mathrm{e}^{-1}$$

$$= 1 - \frac{2}{\mathrm{e}} + \frac{1}{\mathrm{e}^2} = \left(1 - \frac{1}{\mathrm{e}}\right)^2$$

Then

$$\int_0^\infty \mathrm{f}(x)\mathrm{e}^{-x}\ \mathrm{d}x = \lim_{n\to\infty} \int_0^{2n} \mathrm{f}(x)\mathrm{e}^{-x}\ \mathrm{d}x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int_{2(k-1)}^{2k} \mathrm{f}(x)\mathrm{e}^{-x}\ \mathrm{d}x$$

To evaluate each integral we perform a substitution $x = 2(k-1)+t$ to bring each integral back to the interval [0, 2]. We thus obtain

$$\lim_{n\to\infty} \sum_{k=1}^{n} \int_0^2 \mathrm{f}(2(k-1)+t)\mathrm{e}^{-2(k-1)+t}\ \mathrm{d}t$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int_0^2 \mathrm{f}(t)\mathrm{e}^{-2(k-1)-t}\ \mathrm{d}t \quad \text{since f is 2-periodic}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \mathrm{e}^{-2(k-1)} \int_0^2 \mathrm{f}(t)\mathrm{e}^{-t}\ \mathrm{d}t$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{\mathrm{e}^2}\right)^{k-1} \left(1 - \frac{1}{\mathrm{e}}\right)^2$$

$$= \left(1 - \frac{1}{\mathrm{e}}\right)^2 \frac{1}{1 - \frac{1}{\mathrm{e}^2}} = \frac{1 - \frac{1}{\mathrm{e}}}{1 + \frac{1}{\mathrm{e}}} = \frac{\mathrm{e}-1}{\mathrm{e}+1}$$

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Student Solution:

$$\int_0^\infty f(x)e^{-x}\,dx = \lim_{n\to\infty}\sum_{i=0}^{n}\int_{2i}^{2i+2} f(x)e^{-x}\,dx$$

$$= \lim_{n\to\infty}\sum_{i=0}^{n}\left[\int_{2i}^{2i+1}(x-2i)e^{-x}\,dx + \int_{2i+1}^{2i+2}(2i+2-x)e^{-x}\,dx\right] \quad -(*)$$

Notice $\int xe^{-x}\,dx = -e^{-x}x - \int(-e^{-x})(1)\,dx$

$= -xe^{-x} - e^{-x}$

D	I
x	e^{-x}
1	$-e^{-x}$

Hence, $\int_{2i}^{2i+1}(x-2i)e^{-x}\,dx + \int_{2i+1}^{2i+2}(2i+2-x)e^{-x}\,dx$

$$= \left[-xe^{-x}-e^{-x}\right]_{2i}^{2i+1} + \left[2ie^{-x}\right]_{2i}^{2i+1} + \left[-(2i+2)e^{-x}\right]_{2i+1}^{2i+2} - \left[-xe^{-x}-e^{-x}\right]_{2i+1}^{2i+2}$$

$$= \left[(2i+1)e^{-2i} - (2i+2)e^{-2i-1}\right] + 2i\left[e^{-2i-1}-e^{-2i}\right] - (2i+2)\left[e^{-2i-2}-e^{-2i-1}\right] + \left[(2i+3)e^{-2i-2}-(2i+2)e^{-2i-1}\right]$$

$$= e^{-2i} + e^{-2i-1}(-2) + e^{-2i-2}$$

Continuing from (*), $\int_0^\infty f(x)e^{-x}\,dx = \lim_{n\to\infty}\sum_{i=0}^{n}\left(e^{-2i}-2e^{-2i-1}+e^{-2i-2}\right) \quad -(**)$

Notice $\lim_{n\to\infty}\sum_{i=0}^{n} e^{-2i} = \sum_{i=0}^{\infty}\left(\frac{1}{e^2}\right)^i = -\frac{1}{\left(\frac{1}{e^2}\right)-1} = \frac{e^2}{e^2-1}$.

So continuing from (**), $\int_0^\infty f(x)e^{-x}\,dx = \left(\frac{e^2}{e^2-1}\right) - \frac{2}{e}\left(\frac{e^2}{e^2-1}\right) + \frac{1}{e^2}\left(\frac{e^2}{e^2-1}\right)$

$$= \frac{e^2-2e+1}{e^2-1}$$

$$= \frac{(e-1)^2}{(e+1)(e-1)}$$

$$= \frac{e-1}{e+1}$$

(*) is true since $f(x) = \begin{cases} x-2i & \text{if } 2i\le x\le 2i+1 \text{ for some } i\in\mathbb{Z} \\ 2i+2-x & \text{if } 2i+1< x\le 2i \text{ for some } i\in\mathbb{Z}. \end{cases}$

Question 4	
(a) **[6]**	$u_1 = a$ $u_2 = b = \frac{b}{a}(a) = \frac{b}{a}u_1$ $u_3 = \frac{u_2}{u_1}(2u_1 - u_2) = \frac{b}{a}(2a-b) = \left(2-\frac{b}{a}\right)u_2$ $\therefore c = 2 - \frac{b}{a}$ Hence if we let P_n be the statements $u_{2n} = \frac{b}{a}u_{2n-1}$ and $u_{2n+1} = \left(2-\frac{b}{a}\right)u_{2n}$ for all $n \in \mathbb{Z}^+$, the above shows that P_1 . Assuming P_k for some $k \in \mathbb{Z}^+$, $u_{2k} = \frac{b}{a}u_{2k-1}$ and $u_{2k+1} = \left(2-\frac{b}{a}\right)u_{2k}$. Then $u_{2(k+1)} = u_{2k+2}$ $= \frac{u_{2k+1}}{u_{2k}}(2u_{2k} - u_{2k+1})$ $= u_{2k+1}\left(2 - \frac{u_{2k+1}}{u_{2k}}\right)$ $= u_{2k+1}(2-c)$ Note: $2 - c = 2 - \left(2 - \frac{b}{a}\right) = \frac{b}{a}$ $= \frac{b}{a}u_{2k+1} = \frac{b}{a}u_{2(k+1)-1}$ and similarly $u_{2(k+1)+1} = u_{2k+3}$ $= \frac{u_{2k+2}}{u_{2k+1}}(2u_{2k+1} - u_{2k+2})$ $= u_{2k+2}\left(2 - \frac{u_{2k+2}}{u_{2k+1}}\right)$ $= u_{2k+2}\left(2 - \frac{b}{a}\right)$ Note: From induction hypothesis $= cu_{2k+2} = cu_{2(k+1)}$ Thus $P_k \Rightarrow P_{k+1}$. Since P_1 and $P_k \Rightarrow P_{k+1}$, by Mathematical Induction, P_n for all $n \in \mathbb{Z}^+$.

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Student Solution:

(a) $U_{n+2} = \frac{U_{n+1}}{U_n}(2U_n - U_{n+1})$

$\frac{U_{n+2}}{U_{n+1}} = \frac{1}{U_n}(2U_n - U_{n+1})$

$= 2 - \frac{U_{n+1}}{U_n}$

$\frac{U_{n+3}}{U_{n+2}} = 2 - \frac{U_{n+2}}{U_{n+1}}$

$= 2 - \left(2 - \frac{U_{n+1}}{U_n}\right) = \frac{U_{n+1}}{U_n}$

Let $V_n = \frac{U_{n+1}}{U_n}$, $V_{n+2} = V_n$ so V_n is periodic with period 2.

$V_{2n-1} = V_{(2n-1) \bmod 2} = V_1 = \frac{U_2}{U_1} = \frac{b}{a}$

$\Rightarrow \frac{U_{2n}}{U_{2n-1}} = \frac{b}{a} \Rightarrow U_{2n} = \frac{b}{a} U_{2n-1}$ (proven)

$U_3 = \frac{U_2}{U_1}(2U_1 - U_2) = \frac{b}{a}(2a-b)$

$V_{2n} = V_{(2n) \bmod 2 + 2} = V_2 = \frac{U_3}{U_2} = \frac{\frac{b}{a}(2a-b)}{b} = \frac{1}{a}(2a-b) = 2 - \frac{b}{a}$

$\Rightarrow \frac{U_{2n+1}}{U_{2n}} = 2 - \frac{b}{a} \Rightarrow U_{2n+1} = \left(2 - \frac{b}{a}\right) U_{2n}$ (proven)
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(b)

[5]
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From the above, we know that

$$u_{2n} = \frac{b}{a}u_{2n-1} = \frac{bc}{a}u_{2n-2} = \ldots = \left(\frac{bc}{a}\right)^{n-1} u_2 = b\left(\frac{bc}{a}\right)^{n-1}$$

and

$$u_{2n-1} = cu_{2n-2} = \frac{bc}{a}u_{2n-3} = \ldots = \left(\frac{bc}{a}\right)^{n-1} u_1 = a\left(\frac{bc}{a}\right)^{n-1}.$$

Thus the series is made up of 2 geometric series essentially (with same common ratio), and we know that the series converges if

$$\left|\frac{bc}{a}\right| < 1 \Leftrightarrow -1 < 2\left(\frac{b}{a}\right) - \left(\frac{b}{a}\right)^2 < 1.$$

Solving this inequality in terms of b/a, we have from GC,

$\frac{b}{a} \in (-0.414, 2.41) \setminus \{0,1\}$, or equivalently $1-\sqrt{2} < \frac{b}{a} < 1+\sqrt{2}$, $\frac{b}{a} \neq 0,1$ (exact)
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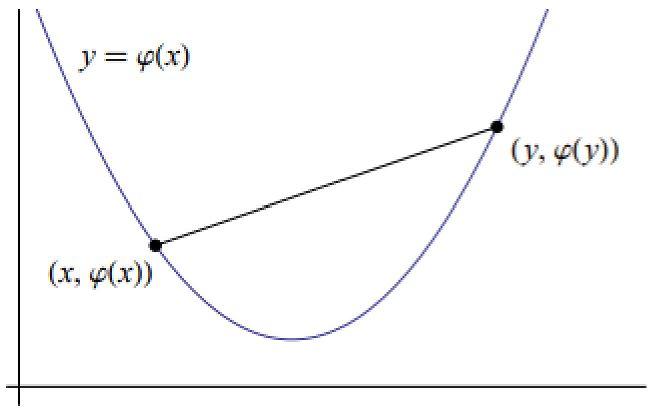
(c) **[3]**	For u_n to have period 4, $u_{2n+3} = u_{2n-1}$ and $u_{2n+4} = u_{2n}$. From above, we know that $u_{2n+3} = \left(\frac{bc}{a}\right)^2 u_{2n-1}$ and $u_{2n+4} = \left(\frac{bc}{a}\right)^2 u_{2n}$. Thus we require $\left(\frac{bc}{a}\right)^2 = 1$. However if $\frac{bc}{a} = 1$, the sequence would be periodic with period 2, and thus we need $\frac{bc}{a} = -1$. This gives us $\frac{1}{2}\left(\frac{b}{a}\right) - \left(\frac{b}{a}\right)^2 = -1$. From (b), we see that this gives us $\frac{b}{a} = 1-\sqrt{2}$ or $1+\sqrt{2}$.

Question 5

(a) **[3]**



The line segment joining the two points is given by

$$\mathbf{r} = \begin{pmatrix} x \\ \varphi(x) \end{pmatrix} + \lambda \begin{pmatrix} y - x \\ \varphi(y) - \varphi(x) \end{pmatrix} = \begin{pmatrix} (1-\lambda)x + \lambda y \\ (1-\lambda)\varphi(x) + \lambda\varphi(y) \end{pmatrix}$$

where $\lambda \in [0,1]$.

From the graph we see that since the line segment is above the curve, we must have

$$\varphi\left((1-\lambda)x + \lambda y\right) \le (1-\lambda)\varphi\left(x\right) + \lambda\varphi\left(y\right).$$

(b) **[4]**

Consider the function $\varphi(z) = z^n$. Since $n \ge 1$,

$\varphi''(z) = n(n-1)z^{n-2} \ge 0$ for nonnegative z, and thus the function is convex. Let $\lambda = \frac{1}{2}$.

Then $\left(\frac{x+y}{2}\right)^n \le \frac{x^n + y^n}{2}$.

From **(a)**, we see that equality holds when either $x = y$ or when the graph is linear ($n = 1$), in which case, any x and y that are nonnegative gives equality.

Note: The equality case boils down to if the line segment coincides with the graph, or is reduced to a single point ($x = y$).

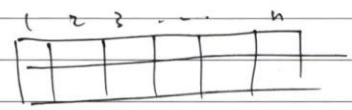
(c) **[4]**	When a or b is 0, the LHS of the inequality is 0 and the inequality holds trivially. So suppose a and b are positive. Consider the function $\varphi(z)=\mathrm{e}^z$, which is convex since $\varphi''(z)=\mathrm{e}^z>0$. Let $\lambda=\frac{1}{q}$, then $\mathrm{e}^{\frac{x}{p}+\frac{y}{q}}\le\frac{1}{p}\mathrm{e}^x+\frac{1}{q}\mathrm{e}^y.$ Now let $x=\ln a^p, y=\ln b^q$ (since a and b are positive reals, x and y are well defined). This gives us $\mathrm{e}^{\frac{\ln a^p}{p}+\frac{\ln b^q}{q}}\le\frac{1}{p}\mathrm{e}^{\ln a^p}+\frac{1}{q}\mathrm{e}^{\ln b^q}$ $\Rightarrow ab=\mathrm{e}^{\ln a+\ln b}\le\frac{a^p}{p}+\frac{b^q}{q}.$

Question 6	
(a) **[1]**	Write down the numbers 1 to n in the 1st row in the order. For each subsequent row, cyclically permute the previous row, i.e. $\begin{array}{\|c\|c\|c\|c\|} \hline 1 & 2 & \ldots & N \\ \hline 2 & 3 & \ldots & 1 \\ \hline & & & \\ \hline N & 1 & \ldots & N-1 \\ \hline \end{array}$
(b) **[3]**	For a Latin square of order 2, there are just 2 of them, since the first row is either 1, 2 or 2, 1 and the second row is the other. For a Latin square of order 3, assume WLOG that the first row is 1, 2, 3. The second row is either 2, 3, 1 or 3, 1, 2. After inserting the 2nd row, there is only one way to complete the 3rd row. Hence 2 such Latin squares. Taking into account permutations, there are thus $2\times 3!=12$ such Latin squares.
(c) **[3]**	Look at the last row n. For each cell (n, i) in row n, look at column i. There are n – 1 distinct numbers in column i, and thus just one number, say k not present in the column. Write the number k in the cell. We claim that this algorithm creates a Latin square. To check this, it suffices to check whether any numbers are repeated in any row or column. By construction, we know that our choice of number does not cause any repetition of numbers in any column; as well, we know that no number is repeated in any row other than possibly the n-th row, because we started with a partial Latin square. Therefore, it suffices to check the n-th row for any repeated numbers. To do this, proceed by contradiction: i.e. suppose not, that there are two cells in the bottom row such that we've placed the same numbers in those two cells. This means that there is some number that we've never written in our last row. But this means that this number is used somewhere in all n columns within the first $n-1$ rows, which forces some row in those first $n-1$ to contain two copies of k by the Pigeonhole Principle. This is a contradiction. Alternatively, consider the number that appears twice in the last row. Taking away the two columns they are in, the remaining n – 2 columns have thus n – 1 occurrences of this number (since each row must have 1 of the number). By the Pigeonhole Principle, one column must have 2 of this number, a contradiction.

(d) **[5]**	There are $n!$ ways to arrange the first row. Let us consider the case where the first row is 1, 2, …, n. For each $i = 1, 2, \ldots, n$, let A_i be the set of arrangements of the 2 rows such that i appears twice in the same column. The problem is thus to enumerate $\left\|\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}\right\|$. We have clearly $\|A_i\| = (n-1)!$, $\|A_i \cap A_j\| = (n-2)!$, and so on, since fixing one, two (respectively) numbers leaves the remaining $n - 1$, $n - 2$ (respectively) numbers to be in the remaining positions with no restrictions. By the Principle of Inclusion and Exclusion, $\left\|\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}\right\|$ $= \|S\| - \|A_1 \cup A_2 \cup \ldots \cup A_n\|$ $= \|S\| - \sum_{i=1}^{n} \|A_i\| + \sum_{i<j} \|A_i \cap A_j\| - \sum_{i<j<k} \|A_i \cap A_j \cap A_k\| + \ldots + (-1)^n \|A_1 \cap A_2 \cap \ldots \cap A_n\|$ $n! - \binom{n}{1}(n-1)! + \ldots + (-1)^k \binom{n}{k}(n-k)! + \ldots + (-1)^n \binom{n}{n}(n-n)!$ $= \sum_{k=0}^{n} (-1)^k \binom{n}{k}(n-k)!$ $= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!}(n-k)!$ $= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ For each of the $n!$ permutations, it suffices to rearrange the 2nd row in the same way the first row is rearranged to obtain a $2 \times n$ Latin rectangle. Hence the total number of $2 \times n$ Latin rectangles is $n! \times n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = (n!)^2 \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$

Student Solution:

d)



Let A_i denote the event that there are two same entries in the same column i

Note $\left|\bigcap A_i\right| = \binom{n}{k}\cdot k!\cdot(n-k)!\cdot(n-k)!$

(for k such events; first row, second row)

By PIE $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_1 \cap A_j \ldots A_m| \times \binom{n}{i} \times (-1)^{i-1}$

(i terms)

$= \sum_{i=1}^{n} \binom{n}{i}\cdot i!\cdot(n-i)!^2\cdot\binom{n}{i}\times(-1)^{i-1}$

$= \sum_{i=1}^{n} \frac{n!}{(n-i)!\,i!}\cdot i!\cdot(n-i)!\cdot\frac{n!}{(n-i)!\,(i)!}\times(-1)^{i-1}$

$= \sum_{i=1}^{n} \frac{(n!)^2}{(i!)}\times(-1)^{i-1}$

Total number of rectangles:

($n!$ to arrange top row, $n!$ to arrange bottom row)

$(n!)^2 - \sum_{i=1}^{n} \frac{(n!)^2}{(i!)}\times(-1)^{i-1}$

$= (n!)^2 \sum_{i=0}^{n} \frac{(-1)^i}{i!}$ (shown)

Question 7

(a) [1]

From the first house, there are (n – 1) choices for the subsequent house the postman goes to, then (n – 2) for the next, etc. Hence the desired number is (n – 1)!.

(b) [3]

Let $h_0, h_1, \ldots, h_n$ be the sequence of houses the postman goes to on his route, with $h_0 = h_n = 1$. Suppose $h_i = n$.

Then the length of the route is

$$\begin{aligned}
l &= |h_1 - h_0| + |h_2 - h_1| + \ldots + |h_n - h_{n-1}| \\
&= \left(|h_1 - h_0| + \ldots + |h_i - h_{i-1}|\right) + \left(|h_{i+1} - h_i| + \ldots + |h_n - h_{n-1}|\right) \\
&\geq \left(|h_1 - h_0 + \ldots + h_i - h_{i-1}|\right) + \left(|h_{i+1} - h_i + \ldots + h_n - h_{n-1}|\right) \\
&= |h_i - h_0| + |h_n - h_i| = n - 1 + n - 1 = 2(n-1).
\end{aligned}$$

The route of minimum length is 2(n – 1) since we can attain it from a route 1, 2, 3, …, n, 1.

Student Solution:

Let the route be $1, a_1, a_2, \cdots, a_{n-1}, 1$.
Let the number of times the postman traverses the part of the street between house i and $i+1$ be N_i, for each $i = 1, 2, \cdots, n-1$.
We claim $N_i \geq 2 \ \forall \ i = 1, 2, \cdots, n-1$.
Suppose otherwise $N_i = 1$ for some $i \in \{1, 2, \cdots, n-1\}$.
Since the postman was initially at house 1, ~~which is at the side of the~~ the first time the postman traverses the part of street between i and $i+1$ is in the direction $i \to i+1$ since $1 \leq i$. Afterwards, he never crosses this part of street again, so he always remains at the same side of house $i+1$, which is now the side opposite house 1. Therefore he never returns to house 1, a contradiction.
Total length $= \sum_{i=1}^{n-1} N_i \geq \sum_{i=1}^{n-1} 2 = 2(n-1)$.
Minimum length $\geq 2(n-1)$.
A route of length $2(n-1)$ is possible: $1, 2, 3, 4, \cdots, n-1, n, 1$.
Then length $= |1-2| + |2-3| + \cdots + |n-1-n| + |n-1|$
$= \underbrace{1 + 1 + \cdots + 1}_{n-1 \text{ 1's}} + n - 1$
$= 2(n-1)$.

(c) **[2]**	Suppose $h_0, h_1, ..., h_n$ is a route of minimum length, 2(n – 1). Then we know from (b) that $\lvert h_1 - h_0 \rvert + ... + \lvert h_i - h_{i-1} \rvert = \lvert h_i - h_0 \rvert = n - 1$ and $\lvert h_{i+1} - h_i \rvert + ... + \lvert h_n - h_{n-1} \rvert = \lvert h_n - h_i \rvert = n - 1$. This means that the sequence $h_0, h_1, ..., h_n$ is strictly increasing up to $h_i = n$ and then strictly decreasing to $h_n = 1$. Hence it suffices to choose amongst {2, …, n – 1} which terms should be in the increasing portion and which terms are in the decreasing portion. Since there are 2 choices for each, there are altogether 2^{n-2} such routes of minimum length.
(d) **[5]**	For a route $h_0, h_1, ..., h_n$, consider 2 sequences of indices $0 = b_1, b_2, ..., b_k, b_{k+1} = n$ and $m_1, ..., m_k$ such that the sequence $h_0, h_1, ..., h_n$ is increasing between h_{b_1} and h_{m_1}, decreasing between h_{m_1} and h_{b_2}, and in general, increasing between h_{b_i} and h_{m_i}, decreasing between h_{m_i} and $h_{b_{i+1}}$. Then the length of the route satisfies $l = h_{m_1} - h_{b_1} + h_{m_1} - h_{b_2} + ... + h_{m_k} - h_{b_k} + h_{m_k} - h_{b_{k+1}}$ $= 2\left(\left(h_{m_1} + ... + h_{m_k}\right) - \left(h_{b_1} + ... + h_{b_{k+1}}\right)\right)$ We may assume WLOG that the two sequences $\left(h_{m_i}\right), \left(h_{b_i}\right)$ are strictly increasing and decreasing respectively (suffices to relabel the terms if necessary). Then we have for all $1 \le i \le k$, $h_{m_i} \le n + 1 - i$, $h_{b_i} \ge i$. So $l = 2\left(h_{m_1} - h_{b_1} + h_{m_2} - h_{b_2} + ... + h_{m_k} - h_{b_k}\right)$ $\le 2\left((n-1) + (n-3) + ... + (n - 2k + 1)\right)$ $= 2k(n-k)$

The quadratic function $f(k) = k(n-k)$ for integers k, attains its maximum at $k = \left\lfloor \frac{n}{2} \right\rfloor$, and thus the desired maximum length is $\left\lfloor \frac{n^2}{2} \right\rfloor$. This maximum length is attained by considering the route 1, n, 2, $n-1$, 3, …, $\left\lfloor \frac{n+1}{2} \right\rfloor$, 1.

Student Solution:

Claim: $N_i \le 2\min(i, n-i)$ $\forall\ i = 1, 2, \cdots, n$.

Each time the postman traverses the part of street between i and $i+1$, he is on his way to deliver a letter to a different house on the other side of this portion of the street.

When he is going $i \to i+1$, he is delivering a letter to one of the $n-i$ homes $i+1, i+2, \cdots, n$

Else when he is going $i \leftarrow i+1$, he is heading to one of the i homes $1, 2, \cdots, i$.

The postman starts and ends at home 1, so he goes $i \to i+1$ as many times as he goes $i \leftarrow i+1$. Therefore, since he can at most go $i \to i+1$ $(n-i)$ times to go to each one of the $(n-i)$ houses at most once, and he can at most go $i \leftarrow i+1$ (i) times to go to each one of the i homes at most once, he crosses this portion of the street at most $2\min(i, n-i)$ times.

$$\therefore \text{length} = N_1 + N_2 + \cdots + N_{n-1}$$
$$\le \sum_{i=1}^{n-1} 2\min(i, n-i)$$
$$= \begin{cases} 2\left(1+2+\cdots+\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \\ (1+2+\cdots+\frac{n}{2}) + \left(\frac{n-2}{2} + \frac{n-4}{2} + \cdots + 1\right) & \text{if } n \text{ is even} \end{cases}$$
$$= \begin{cases} (n+1)(n-1)\left(\frac{1}{4}\right) & \text{if } 2 \nmid n \\ \frac{n}{2} + (n)(n-1)\left(\frac{1}{2}\right) & \text{if } 2 \mid n \end{cases}$$
$$= \begin{cases} \frac{n^2-1}{2} & \text{if } 2 \nmid n \\ \frac{n^2}{2} & \text{if } 2 \mid n \end{cases}$$

If $2 \nmid n$, a length of $\frac{n^2-1}{2}$ is attainable: go on the path $1, n, 2, n-1, \cdots, \frac{n-1}{2}, \frac{n+3}{2}, \frac{n+1}{2}, 1$.

$$\text{length} = |1-n| + |n-2| + |2-(n-1)| + \cdots + \left|\frac{n+3}{2} - \frac{n+1}{2}\right| + \left|\frac{n+1}{2} - 1\right|$$
$$= (n-1) + (n-2) + (n-3) + \cdots + 2 + 1 + \frac{n-1}{2}$$
$$= \frac{n(n-1)}{2} + \frac{n-1}{2}$$
$$= \frac{n^2-1}{2}.$$

If $2 \mid n$, a length of $\frac{n^2}{2}$ is attainable, go on the path $1, n, 2, n-1, \cdots, \frac{n-2}{2}, \frac{n+4}{2}, \frac{n}{2}, \frac{n+2}{2}, 1$

$$\text{length} = |1-n| + |n-2| + |2-(n-1)| + \cdots + \left|\frac{n}{2} - \frac{n+2}{2}\right| + \left|\frac{n+2}{2} - 1\right|$$
$$= (n-1) + (n-2) + (n-3) + \cdots + 2 + 1 + \frac{n}{2}$$
$$= \frac{(n-1)(n)}{2} + \frac{n}{2}$$
$$= n^2/2.$$

Question 8	
(a) [1]	$\sigma(n)=\sigma(2^m)=1+2+...+2^m=\dfrac{2^{m+1}-1}{2-1}=2(2^m)-1=2n-1.$
(b) [3]	It suffices to show the result is true if $a=p^m, b=q^n$, where p and q are distinct primes. The result will then follow by the Fundamental Theorem of Arithmetic. If $a=p^m, b=q^n$, then $\sigma(p^m)=1+p+...+p^m, \sigma(q^n)=1+q+...+q^n$. But $\sigma(ab)=\sigma(p^m q^n)$ $=1+p+....+p^m$ $+q(1+p+....+p^m)$ $+q^2(1+p+....+p^m)$ ... $+q^n(1+p+....+p^m)$ $=\left(1+p+....+p^m\right)\left(1+q+....+q^n\right)$ $=\sigma(a)\sigma(b)$
(c) [3]	Let $n=p_1^{r_1}p_2^{r_2}...p_k^{r_k}$, where the p's are distinct odd primes (FTA). Then $2n-1=\sigma(n)=\sigma\left(p_1^{r_1}p_2^{r_2}...p_k^{r_k}\right)$ $=\sigma\left(p_1^{r_1}\right)\sigma\left(p_2^{r_2}\right)...\sigma\left(p_k^{r_k}\right)$ $=\left(1+p_1+p_1^2+...+p_1^{r_1}\right)...\left(1+p_k+p_k^2+...+p_k^{r_1}\right)$ Since 2n – 1 is odd, the product must be odd too. This implies that each term must be odd. Since the primes are odd, this means that we must have an odd number of terms in each bracket, and thus the exponents are even; this means that n is a perfect square.

	Student Solution: If n is odd, all its factors are odd. If $\sigma(n) = 2n-1$ which is odd, it has an odd number of factors, since a sum of $2k$ odd numbers is even. If n is not a perfect squares, n has factors in pairs, i.e. a and $\frac{n}{a}$, which are all distinct. since $a \neq \frac{n}{a}$. Then n will have an even number of factors. Hence n must be a
(d) **[1]**	We have $1+\frac{1}{p}+\ldots+\frac{1}{p^a}<\sum_{k=0}^{\infty}\frac{1}{p^k}=\frac{1}{1-\frac{1}{p}}=\frac{p}{p-1}.$
(e) **[3]**	Suppose n is a prime power. Then $\sigma(n)=\sigma(p^m)=\dfrac{p^{m+1}-1}{p-1}=2(p^m)-1$. This implies that $p^{m+1}-1=2p^{m+1}-2p^m-p+1$ $\Rightarrow p^{m+1}-2p^m=p-2$ $\Rightarrow p^m(p-2)=p-2$ $\Rightarrow p=2$ or $p^m=1$ which are both not possible since n is odd and thus n is not perfect. Suppose n is only a product of two distinct primes. Then we have $2n-1=\sigma(n)=\sigma\left(p^aq^b\right)$ $=\sigma\left(p^a\right)\sigma\left(q^b\right)$ $=\left(1+p+p^2+\ldots+p^a\right)\left(1+q+q^2+\ldots+q^b\right)$ Dividing by n throughout we get $2=\left(1+\frac{1}{p}+\ldots+\frac{1}{p^a}\right)\left(1+\frac{1}{q}+\ldots+\frac{1}{q^b}\right)+\frac{1}{p^aq^b}.$

	Assume WLOG that $3 \le p < q$. Then from **(d)**, $1+\frac{1}{p}+...+\frac{1}{p^a} \le 1+\frac{1}{3}+...+\frac{1}{3^a} < \frac{3}{2}$. Likewise, $1+\frac{1}{q}+...+\frac{1}{q^a} \le 1+\frac{1}{5}+...+\frac{1}{5^a} < \frac{5}{4}$. Therefore $2 = \left(1+\frac{1}{p}+...+\frac{1}{p^a}\right)\left(1+\frac{1}{q}+...+\frac{1}{q^b}\right)+\frac{1}{p^a q^b}$ $< \frac{3}{2}\times\frac{5}{4}+\frac{1}{3^2\times 5^2} < \frac{15}{8}+\frac{1}{8} = 2$ a contradiction.