2012 NYJC H2 Math Preliminary exam Paper 1 solutions

1	Let the equation of C be $y=ax^3+bx^2+cx+d$ Subst (0, –1) into equation, $d=-1$ Subst (–1, 1) into equation, $-a+b-c+d=1$ $\Rightarrow -a+b-c=2 - - - - - (1)$ At (–1, 1), $\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow 3a-2b+c=0 - - - - (2)$ $\int_2^3 \left(ax^3+bx^2+cx-1\right)\mathrm{d}x=\frac{31}{4}$ $\left[\frac{a}{4}x^4+\frac{b}{3}x^3+\frac{c}{2}x^2-x\right]_2^3=\frac{31}{4}$ $\left(\frac{81a}{4}+9b+\frac{9c}{2}-3\right)-\left(4a+\frac{8b}{3}+2c-2\right)=\frac{31}{4}$ $\frac{65}{4}a+\frac{19}{3}b+\frac{5}{2}c=\frac{35}{4} - - - - (3)$ Solving eqn (1), (2) and (3), $a=1,\ b=0,\ c=-3$ $\therefore y=x^3-3x-1$
2	$\mathbf{a}\times\mathbf{b}=\begin{pmatrix}1\\ \sin\theta\\ \cos\theta\end{pmatrix}\times\begin{pmatrix}1\\ \sin\phi\\ \cos\phi\end{pmatrix}$ $=\begin{pmatrix}\sin\theta\cos\phi-\cos\theta\sin\phi\\ \cos\theta-\cos\phi\\ \sin\phi-\sin\theta\end{pmatrix}$ $=\begin{pmatrix}-\sin(\phi-\theta)\\ 2\sin\frac{\theta+\phi}{2}\sin\frac{\phi-\theta}{2}\\ 2\cos\frac{\theta+\phi}{2}\sin\frac{\phi-\theta}{2}\end{pmatrix}$ $=\begin{pmatrix}-\sin 2\delta\\ 2\sin\frac{\theta+\phi}{2}\sin\delta\\ 2\cos\frac{\theta+\phi}{2}\sin\delta\end{pmatrix}$

	$\lvert\mathbf{a}\times\mathbf{b}\rvert = \sqrt{\sin^2 2\delta + 4\sin^2\delta(\sin^2\beta+\cos^2\beta)}$, where $\beta = \dfrac{\theta+\phi}{2}$ $= \sqrt{\sin^2 2\delta + 4\sin^2\delta}$ $= \sqrt{4\sin^2\delta\cos^2\delta + 4\sin^2\delta}$ $= 2\sin\delta\sqrt{1+\cos^2\delta}$ Using $\lvert\mathbf{a}\times\mathbf{b}\rvert = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\alpha$ $2\sin\delta\sqrt{1+\cos^2\delta} = \left(\sqrt{1+\sin^2\theta+\cos^2\theta}\right)\left(\sqrt{1+\sin^2\phi+\cos^2\phi}\right)\sin\alpha$ $= \left(\sqrt{2}\right)\left(\sqrt{2}\right)\sin\alpha$. $\sin\alpha = \sin\delta\sqrt{1+\cos^2\delta}$
3	(i) $4x^3+3x^2y = y^3-2$ Differentiating wr.t. x: $12x^2+6xy+3x^2\dfrac{dy}{dx} = 3y^2\dfrac{dy}{dx}$ $\left(3x^2-3y^2\right)\dfrac{dy}{dx} = -12x^2-6xy$ $\dfrac{dy}{dx} = \dfrac{4x^2+2xy}{y^2-x^2}$ $= \dfrac{2x(2x+y)}{(y-x)(y+x)}$ Curve meets $y=-x$ when: $4x^3+3x^2(-x) = -x^3-2$ $2x^3=-2$ $\Rightarrow x=-1$ and $y=1$ Thus, coordinates of P is $(-1,1)$ (ii) At $(-1,1)$, $\dfrac{dy}{dx}$ is undefined. Equation of tangent at P: $x=-1$ $OQPR$ is a square.

4

(a) $1-2\sin x>0 \Rightarrow \sin x<\frac{1}{2} \Rightarrow 0\le x<\frac{\pi}{6}$.

$$\int_0^{\frac{\pi}{2}}\left|1-2\sin x\right|\mathrm{d}x=\int_0^{\frac{\pi}{6}}\left|1-2\sin x\right|\mathrm{d}x+\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left|1-2\sin x\right|\mathrm{d}x$$

$$=\int_0^{\frac{\pi}{6}}(1-2\sin x)\,\mathrm{d}x+\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}-(1-2\sin x)\,\mathrm{d}x$$

$$=\left[x+2\cos x\right]_0^{\pi/6}-\left[x+2\cos x\right]_{\pi/6}^{\pi/2}$$

$$=2\left(\sqrt{3}-1\right)-\frac{\pi}{6}.$$

(b) $$\int\frac{\cos\theta}{\sqrt{2\cos 2\theta-1}}\,\mathrm{d}\theta=\int\frac{\cos\theta}{\sqrt{2\left(1-2\sin^2\theta\right)-1}}\,\mathrm{d}\theta$$

$$=\int\frac{1}{\sqrt{1-4\sin^2\theta}}(\cos\theta)\,\mathrm{d}\theta$$

$$=\int\frac{1}{\sqrt{1-4x^2}}\,\mathrm{d}x \quad \text{using } x=\sin\theta$$

$$\int_0^{\alpha}\frac{\cos\theta}{\sqrt{2\cos 2\theta-1}}\,\mathrm{d}\theta=\int_0^{\sin\alpha}\frac{1}{\sqrt{1-4x^2}}\,\mathrm{d}x$$

$$=\int_0^{\sin\alpha}\frac{1}{2\sqrt{\frac{1}{4}-x^2}}\,\mathrm{d}x$$

$$=\left[\frac{1}{2}\sin^{-1}(2x)\right]_0^{\sin\alpha}$$

$$=\frac{1}{2}\sin^{-1}(2\sin\alpha)$$

$$\int_0^{\alpha}\frac{\cos\theta}{\sqrt{2\cos 2\theta-1}}\,\mathrm{d}\theta=\frac{\pi}{4} \Rightarrow \frac{1}{2}\sin^{-1}(2\sin\alpha)=\frac{\pi}{4}$$

$$\Rightarrow \sin^{-1}(2\sin\alpha)=\frac{\pi}{2}$$

$$\Rightarrow 2\sin\alpha=1$$

$$\Rightarrow \sin\alpha=\frac{1}{2} \Rightarrow \alpha=\frac{\pi}{6}.$$

5

$$y=\frac{1-3a}{-x+5}$$

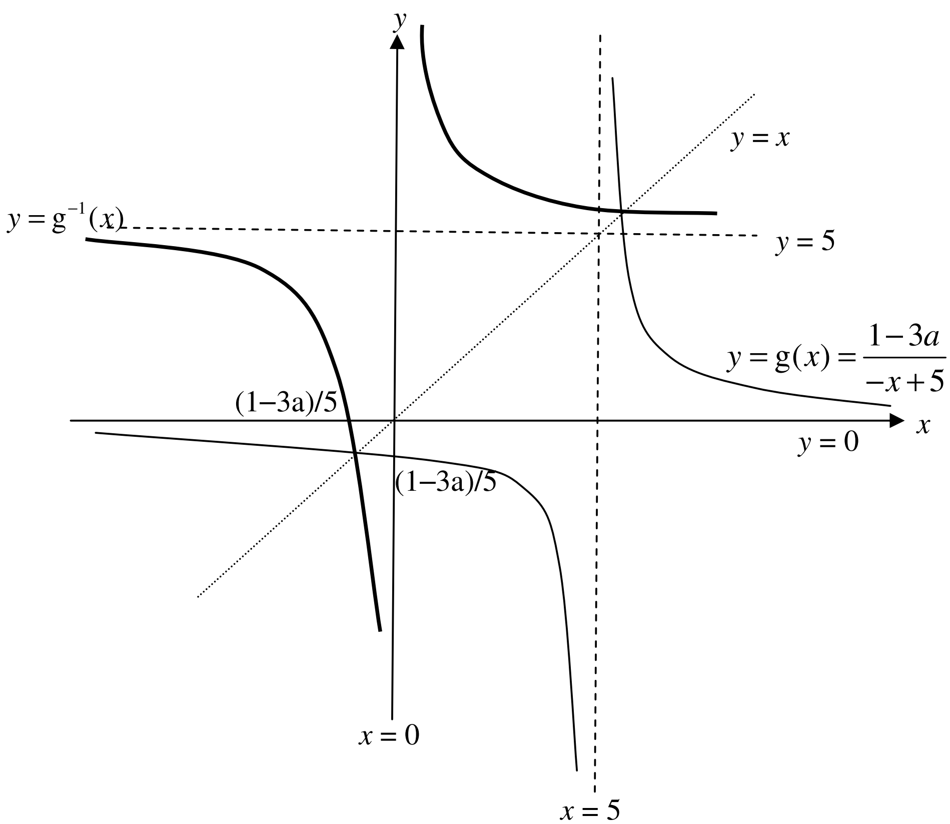
$$C^{-1}\textbf{:}\quad y=\frac{1-3a}{-x+5}+a$$

$$B^{-1}\textbf{:}\quad y=\frac{1-3a}{-(x+2)+5}+a$$

$$A^{-1}\textbf{:}\quad y=\frac{1-3a}{-(-x+2)+5}+a$$

$$=\frac{1-3a}{x+3}+a=\frac{1-3a+ax+3a}{x+3}$$

$$=\frac{1+ax}{x+3}$$



6

(a)(i) $$\frac{AC}{AB}=\tan\left(\frac{\pi}{3}-x\right)$$

$$=\frac{\tan\frac{\pi}{3}-\tan x}{1+\tan\frac{\pi}{3}\tan x}$$

$$\frac{AB}{AC}=\frac{1+\sqrt{3}\tan x}{\sqrt{3}-\tan x}$$

(ii) $\frac{AB}{AC}\approx\frac{1+\sqrt{3}x}{\sqrt{3}-x}$ when x is small

$$=\left(1+\sqrt{3}x\right)\left(\sqrt{3}-x\right)^{-1}$$

$$=\left(1+\sqrt{3}x\right)\frac{1}{\sqrt{3}}\left(1-\frac{x}{\sqrt{3}}\right)^{-1}$$

$$=\frac{1}{\sqrt{3}}\left(1+\sqrt{3}x\right)\left(1+\frac{x}{\sqrt{3}}\right)$$

$$=\frac{1}{\sqrt{3}}\left(1+\frac{x}{\sqrt{3}}+\sqrt{3}x+...\right)$$

$$=\frac{1}{\sqrt{3}}+\frac{4}{3}x+...$$

Hence, $a=\frac{1}{\sqrt{3}}$, $\quad b=\frac{4}{3}$

(b)(i) $\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}+xy=\sqrt{1+x^2}$

$$\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\frac{\mathrm{d}y}{\mathrm{d}x}+x\frac{\mathrm{d}y}{\mathrm{d}x}+y=\frac{x}{\sqrt{1+x^2}}$$

$$\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+3x\frac{\mathrm{d}y}{\mathrm{d}x}+y=\frac{x}{\sqrt{1+x^2}}$$

When $x = 0$, $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=1$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-1$

$$\therefore y=1+x-\frac{x^2}{2}+...$$

(ii) $\mathrm{e}^y\approx\mathrm{e}^{1+x-\frac{x^2}{2}}=\mathrm{e}\left(\mathrm{e}^{x-\frac{x^2}{2}}\right)$

$$\approx \mathrm{e}\left[1+\left(x-\frac{x^2}{2}\right)+\frac{1}{2}\left(x-\frac{x^2}{2}\right)^2\right]$$

$$\approx \mathrm{e}\left[1+\left(x-\frac{x^2}{2}\right)+\frac{1}{2}\left(x^2\right)\right]$$

$$\approx \mathrm{e}\left(1+x\right)$$

7

(a)(i) $r=\dfrac{a+2d}{a}=\dfrac{a+6d}{a+2d}$

$$\left(a+2d\right)^2=a\left(a+6d\right)$$

$$4d^2=2ad$$

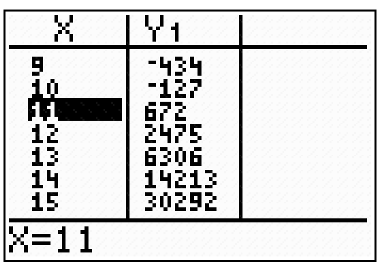
$$d=\frac{1}{2}a \quad (d\neq 0)$$

$$r=\frac{a+2\left(\frac{1}{2}a\right)}{a}=2$$

(ii) $\dfrac{0.1a\left(2^n-1\right)}{2-1}>\dfrac{2n}{2}\left[2\left(a\right)+\left(2n-1\right)\left(\dfrac{a}{2}\right)\right]$

$$2^n-1>n\left[20+5\left(2n-1\right)\right]$$

$$2^n-1-n\left[20+5\left(2n-1\right)\right]>0$$

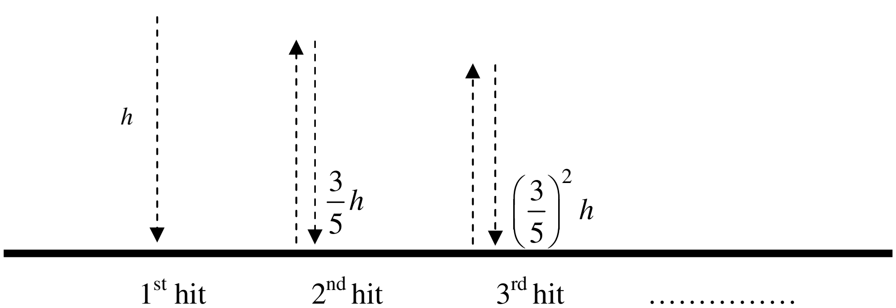


X	Y1	
9	-434	
10	-127	
11	672	
12	2475	
13	6306	
14	14213	
15	30292	

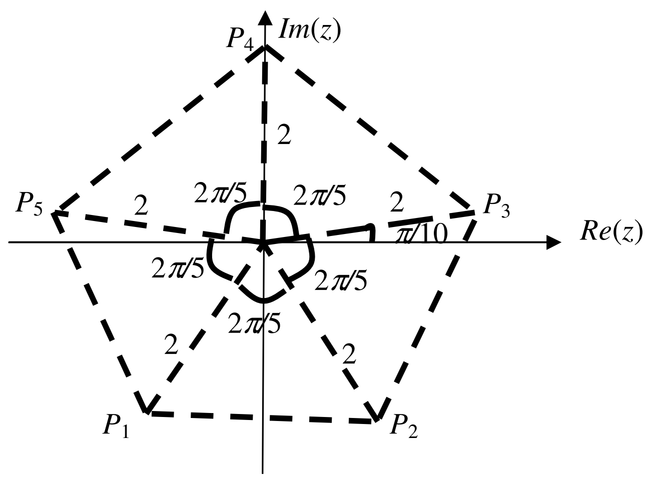
X=11

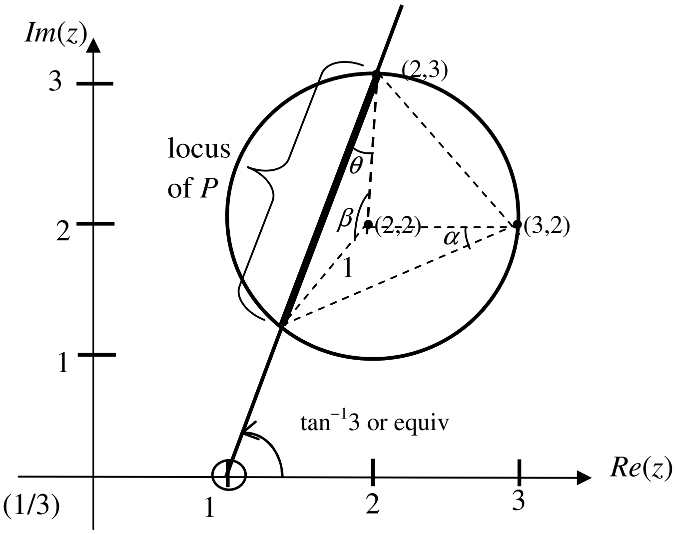
Smallest $n=11$

(b)(i)



Distance travelled by the ball when it strikes the floor for the third time

	$= h + 2\left(\frac{3}{5}\right)h + 2\left(\frac{3}{5}\right)^2 h$ $= 2.92h$ **(ii)** Total distance travelled $< S_\infty$ $= h + 2\left[(0.6)h\right] + 2\left[(0.6)^2 h\right] + 2\left[(0.6)^3 h\right] + \ldots$ $= h + 2h\left[0.6 + 0.6^2 + 0.6^3 + \ldots\right]$ $= h + 2h\left(\frac{0.6}{1-0.6}\right)$ $= 4h$ (shown)
8	(a)$z^5 = 32i = 32\ e^{i\left(\frac{\pi}{2}+2n\pi\right)}$, where $n = 0, \pm1, \pm2$. $z = 2e^{i\left(\frac{\pi}{10}+\frac{2n\pi}{5}\right)}$, where $n = 0, \pm1, \pm2$. $z = 2e^{\frac{i\pi}{10}},\ 2e^{\frac{i\pi}{2}},\ 2e^{\frac{i9\pi}{10}},\ 2e^{-\frac{i3\pi}{10}},\ 2e^{-\frac{i7\pi}{10}}$.  Area of $P_1\ P_2\ P_3\ P_4\ P_5 = 5(\frac{1}{2}(2)^2 \sin\frac{2\pi}{5})$ ≈ 9.51 units^2 (b) $\lvert z - 2 - 2i\rvert \le 1 \Rightarrow \lvert z - (2 + 2i)\rvert \le 1$



$\theta = \tan^{-1}(1/3)$

$\beta = \pi - 2\theta$

$\alpha = (\beta - (\pi/2))/2 \approx 0.46365$

$\therefore \frac{3\pi}{4} \le \arg(z - 3 - 2i) \le \pi$

or $-\pi < \arg(z - 3 - 2i) \le -(\pi - \alpha)$

$\Rightarrow \frac{3\pi}{4} \le \arg(z - 3 - 2i) \le \pi$

or $-\pi < \arg(z - 3 - 2i) \le -2.68$

9

(a) When $n \to \infty$, $x_n \to l$ and $x_{n+1} \to l$,

$$l = -\sqrt{1-2l}$$

$$l^2 = 1 - 2l$$

$$l^2 + 2l - 1 = 0$$

$$l = \frac{-2 \pm \sqrt{4 - 4(-1)}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}}{2}$$

$$= -1 \pm \sqrt{2}$$

Since $x_n < 0$, $l = -1 - \sqrt{2}$

(b)(i) $\frac{2}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}$

By cover-up rule or otherwise, $A = 1$, $B = -1$

Hence $\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+2}\right)$

$$= \frac{1}{2}\left(\begin{array}{l} \frac{1}{1} - \frac{1}{3} \\ +\frac{1}{2} - \frac{1}{4} \\ +\frac{1}{3} - \frac{1}{5} \\ + \ \vdots \\ +\frac{1}{n-2} - \frac{1}{n} \\ +\frac{1}{n-1} - \frac{1}{n+1} \\ +\frac{1}{n} - \frac{1}{n+2} \end{array}\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \frac{1}{2}\left(\frac{3}{2} - \frac{n+2+n+1}{(n+1)(n+2)}\right)$$

$$= \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$$

(ii)Let P_n denote the proposition $\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$ for $n \in \mathbb{Z}^+$

When $n = 1$, LHS = $\sum_{r=1}^{1} \frac{1}{r(r+2)} = \frac{1}{1(3)} = \frac{1}{3}$

RHS = $\frac{3}{4} - \frac{2+3}{2(1+1)(1+2)} = \frac{3}{4} - \frac{5}{12} = \frac{4}{12} = \frac{1}{3}$

$\therefore$ P_1 is true

Assume that P_k is true for some $k \in \mathbb{Z}^+$, i.e.

$\sum_{r=1}^{k} \frac{1}{r(r+2)} = \frac{3}{4} - \frac{2k+3}{2(k+1)(k+2)}$

To prove P_{k+1} is also true i.e. $\sum_{r=1}^{k+1} \frac{1}{r(r+2)} = \frac{3}{4} - \frac{2k+5}{2(k+2)(k+3)}$

LHS = $\sum_{r=1}^{k+1} \frac{1}{r(r+2)} = \sum_{r=1}^{k} \frac{1}{r(r+2)} + \frac{1}{(k+1)((k+1)+2)}$

$$=\frac{3}{4}-\frac{2k+3}{2(k+1)(k+2)}+\frac{1}{(k+1)(k+3)}$$

$$=\frac{3}{4}-\left[\frac{(2k+3)(k+3)-2(k+2)}{2(k+1)(k+2)(k+3)}\right]$$

$$=\frac{3}{4}-\frac{2k^2+9k+9-2k-4}{2(k+1)(k+2)(2k+3)}$$

$$=\frac{3}{4}-\frac{2k^2+7k+5}{2(k+1)(k+2)(k+3)}$$

$$=\frac{3}{4}-\frac{(k+1)(2k+5)}{2(k+1)(k+2)(k+3)}$$

$$=\frac{3}{4}-\frac{2k+5}{2(k+2)(2k+3)}$$

Hence P_{k+1} is true

Since P_1 is true and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$

(iii) $$\sum_{r=2}^{\infty}\frac{1}{r(r+2)}=\sum_{r=1}^{\infty}\frac{1}{r(r+2)}-\frac{1}{1(3)}$$

$$=\frac{3}{4}-\frac{1}{3}$$

$$=\frac{5}{12}$$

10 (i) Since plane $\prod$ contains point Q, therefore $\mathbf{a}=5\mathbf{i}+7\mathbf{j}+6\mathbf{k}$

$$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}2\\3\\7\end{pmatrix}$$

$$\begin{pmatrix}3\\-2\\0\end{pmatrix}\cdot\begin{pmatrix}2\\3\\7\end{pmatrix}=6-6=0$$

Since $3\mathbf{i}-2\mathbf{j}$ is perpendicular to the normal of the plane, $3\mathbf{i}-2\mathbf{j}$ can be taken as **b**.

$$\mathbf{c}=\begin{pmatrix}3\\-2\\0\end{pmatrix}\times\begin{pmatrix}2\\3\\7\end{pmatrix}=\begin{pmatrix}-14\\-21\\13\end{pmatrix}$$

	(ii) Points Q and S lie on plane $\prod$ and since vectors **b** and **c** are parallel to $\prod$, the 2 lines are coplanar. Therefore they will intersect. iii) line $PR: \mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 39 \\ 9 \end{pmatrix}$ plane $\prod : \mathbf{r} \cdot \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} = 10 + 21 + 42 = 73$ since line PR intersect plane, $\begin{pmatrix} 3+3\mu \\ 4+39\mu \\ -1+9\mu \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} = 73$ $6 + 6\mu + 12 + 117\mu - 7 + 63\mu = 73$ $186\mu = 62 \Rightarrow \mu = \frac{1}{3}$ $\overline{OS} = \begin{pmatrix} 3+1 \\ 4+13 \\ -1+3 \end{pmatrix} = \begin{pmatrix} 4 \\ 17 \\ 2 \end{pmatrix}$ $\overrightarrow{PS} = \begin{pmatrix} 4 \\ 17 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 13 \\ 3 \end{pmatrix} = \frac{1}{3}\overrightarrow{PR}$ Therefore point S does not lie on PR produced.
11	(i) $\frac{\mathrm{d}y}{\mathrm{d}t} = k(5-y)$ $\int \frac{1}{5-y}\,\mathrm{d}y = \int k\,\mathrm{d}t$ $-\ln\lvert 5-y \rvert = kt + c$ **(*)** $\lvert 5-y \rvert = e^{-(kt+c)}$ $5 - y = Ae^{-kt}$ where $A = \pm e^{-c}$ $y = 5 - Ae^{-2t}$ When $t = 0$, $y = 0$, hence $A = 5$

$y=5\left(1-e^{-kt}\right)$

When $t=3$, $y=1$,

$1=5\left(1-e^{-3k}\right)$

$e^{-3k}=\dfrac{4}{5}$

$k=-\dfrac{1}{3}\ln\dfrac{4}{5}=0.0744$ (3s.f.)

$y=5\left(1-e^{-0.0744t}\right)$

(*) Alternative working:

When $t=0$, $y=0$, hence $c=-\ln 5$

When $t=3$, $y=1$,

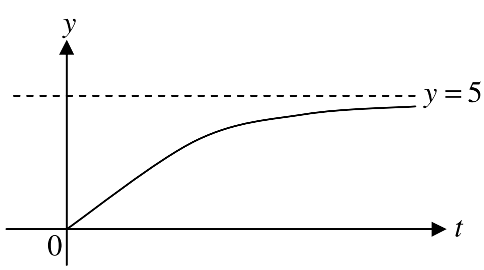
$k=-\dfrac{1}{3}\ln\dfrac{4}{5}=0.0744$ (3s.f.)

Hence $\quad |5-y|=e^{-(0.0744t-\ln 5)}$

$5-y=e^{-(0.0744t-\ln 5)}$ since $5-y>0$

$y=5-e^{-0.0744t+\ln 5}=5-5e^{-0.0744t}$

(ii)



(iii) $\dfrac{d^2 y}{dt^2}=-0.1$

$\dfrac{dy}{dt}=-0.1t+B$

$y=-0.05t^2+Bt+C$

When $t=0$, $y=1.8$, hence $C=1.8$ **(#)**

When $t=1$, $y=1.65$, hence $B=-0.1$

$y=-0.05t^2-0.1t+1.8$

Using GC, when $t=3$, $y=1.05$

when $t=4$, $y=0.6$

$6+4=10$

The company's business will become unprofitable in the 10th year.

(#) Alternative working:

When $t=6$, $y=1.8$, hence $1.8=-1.8+6B+C$

$$6B+C=3.6 \quad \text{------(1)}$$

When $t=7$, $y=1.65$, hence $1.65=-2.45+7B+C$

$$7B+C=4.1 \quad \text{------(2)}$$

Hence $B=0.5$, $C=0.6$

$y=-0.05t^2+0.5t+0.6$

Using GC, when $t=9$, $y=1.05$

when $t=10$, $y=0.6$

The company's business will become unprofitable in the 10th year.