Name:_____ **Class:**_____

JURONG PIONEER JUNIOR COLLEGE

JC2 Preliminary Examination 2023

FURTHER MATHEMATICS 9649/02
Higher 2

Paper 2

18 September 2023
3 hours

Additional materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

An answer booklet will be provided with this question paper. You should follow the instructions on the front cover of the booklets. If you need any additional answer paper ask the invigilator for a continuation booklet.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **8** printed pages.

[Turn over

Section A: Pure Mathematics [50 marks]

1 The sequence $\{X_n\}$ is given by

$$\sqrt{X_{n+2}} = \frac{X_{n+1}}{(X_n)^2}, \qquad \text{for} \quad n \geq 1.$$

By applying the natural logarithm to the recurrence relation, use a suitable substitution to find the general solution of the sequence, expressing your answer in trigonometric form. [5]

2 Let $I_n = \int_{-1}^{1} (1-x^2)^n \, \mathrm{d}x$, for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

(a) Use integration by parts to show that

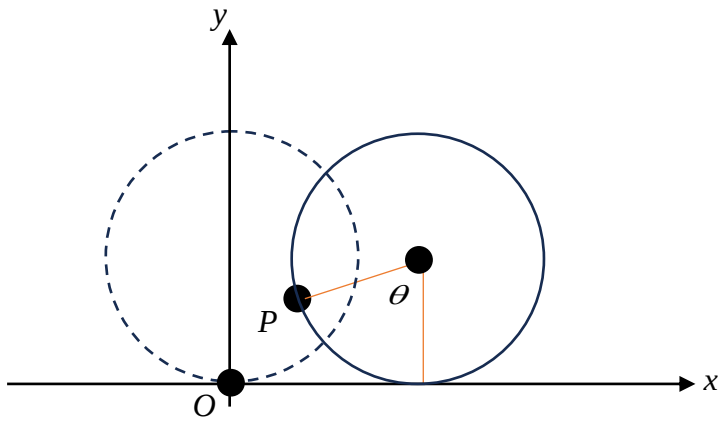
$$(2n+1) I_n = 2n \, I_{n-1}, \qquad \text{for } n > 0. \qquad [3]$$

(b) Hence, or otherwise, evaluate $I_{\frac{3}{2}}$, giving your answer as a multiple of π. [3]

(c) Use Simpson's rule with five ordinates to find an approximation to $I_{\frac{3}{2}}$, giving your answer in exact form. [2]

(d) Deduce an approximation to π, giving your answer in exact form. [1]

3



The diagram above shows a circle with radius r cm. Initially, the lowest point P on the circumference of the circle is at the origin. As the circle rolls along the x-axis in the positive direction, P traces a path and this locus is known as a *cycloid*.

At any time, the angle through which the rolling circle has rotated is θ radians. Show that the locus of P has parametric equations

$$x = r(\theta - \sin\theta), \qquad y = r(1 - \cos\theta). \qquad [2]$$

The part of this locus for which $0 \le \theta \le 2\pi$ is denoted by C.

(a) Sketch C and find the coordinates of its maximum point in terms of r. [3]

(b) Show that the surface area formed when C is completely rotated about the x-axis can be expressed as the integral $8\pi r^2 \int_0^{2\pi} \sin^3 \frac{\theta}{2} \mathrm{d}\theta$, and hence, find its exact area in terms of r. [5]

4 For $t \ge 1$, the function $y = \mathrm{f}(t)$ satisfies

$$t\frac{\mathrm{d}y}{\mathrm{d}t} + ky = t,$$

where k is a constant and $y = 0$ when $t = 1$.

(a) When $k \ne -1$, find the solution for y in terms of t and k. [4]

It is given that $k = 2$.

(b) Use Euler's method with step size 0.1 to estimate the value of y when $t = 1.2$. [2]

(c) Sketch the curve for the solution found in **(a)**, indicating the equation of any asymptote(s) and the coordinates of any axial intercept(s). Hence, explain whether your answer in **(b)** is an overestimate or underestimate of the exact value of y when $t = 1.2$. [4]

5 **(a)** Given that the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\sqrt{3}\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0$$

is satisfied by $y = \mathrm{f}(x)$, where $\mathrm{f}(0) = 0$ and $\mathrm{f}'(0) = 1$, determine $\mathrm{f}(x)$. [4]

(b) Prove by induction that

$$\mathrm{f}^{(n)}(x) = 2^n \mathrm{e}^{x\sqrt{3}} \sin\left(x + \frac{1}{6}n\pi\right)$$

for all positive integers *n*, where $\mathrm{f}^{(n)}(x)$ denotes $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{f}(x)$. [4]

(c) Sketch the graph of $y = \mathrm{f}(x)$ for $0 \le x \le 3\pi$, giving the coordinates of the stationary points and the points where the graph meets the *x*-axis. [3]

(d) The positive roots of the equation $\mathrm{f}(x) = \frac{1}{x}$ are denoted by $\alpha_1, \alpha_2, ..., \alpha_n, ...$, in increasing order.

(i) Use the Newton-Raphson method once, with first approximation 0.6, to estimate α_1. [3]

(ii) State, with working, a first approximation to α_n when *n* is large. [2]

Section B: Probability and Statistics [50 marks]

6 The average number of bacteria in 1 ml of drugs is being investigated.

(a) State, in context, an assumption that needs to be made for the number of bacteria in 1 ml of drug to be well modelled by a Poisson distribution. [1]

Assume that the number of bacteria in 1 ml of drug *A* has a Poisson distribution with mean 0.5.

(b) Given that the probability that there are at most 3 bacteria in *k* ml of drug *A* is less than 0.02, find the least possible integer value of *k*. [2]

The number of the same bacteria in 1 ml of drug *B* has a Poisson distribution with mean 0.8. A mixture of these drugs used to treat a particular disease consists of 1.4 ml of drug *A* and 1.1 ml of drug *B*. Bacteria in the drugs will cause infection in a patient if 5 or more bacteria are injected.

(c) Assuming that there are no chemical reactions between drugs *A* and *B* when they are mixed, find the probability that a patient will get infection after being injected with the drug mixture. [4]

7 In a particular country, large number of ducks live on lakes *A* and *B*. The mass, in kg, of a duck on lake *A* is denoted by x and the mass, in kg, of a duck on lake *B* is denoted by y. A random sample of 8 ducks is taken from lake *A* and a random sample of 10 ducks is taken from lake *B*. Their masses are summarised as follows.

$$n = 8 \quad \sum x = 10.56 \quad \sum x^2 = 14.1775$$
$$n = 10 \quad \sum y = 12.39 \quad \sum y^2 = 15.894$$

A scientist claims that ducks on lake *A* are heavier on average than ducks on lake *B*.

Test, at the 10% level of significance, whether the scientist's claim is justified. You should assume that both distributions are normal. State another assumption necessary for the method to be valid. [7]

8 Two suppliers, *P* and *Q* are under examination for the quality of their items, which are rated poor, fair or good. A random sample of items is taken from each supplier, and the numbers in each category are recorded. A test is conducted at the 5% level of significance to test whether the quality rating is independent of the supplier and a report is generated. Unfortunately, parts of the report are dirtied by blots of ink, as shown below.

Report on supplier *P* and *Q*

H_0 : Quality rating is independent of the supplier.

H_1 : Quality rating is dependent of the supplier.

Data is shown as follows:

Supplier	Poor	Fair	Good	Total
P			120	180
Q			30	60
Total	30	60	150	240

Using a chi-squared test at the 5% level of significance,

$\chi^2_{CALC} = 6.4$

there is [redacted] evidence to conclude that the quality rating is dependent of the supplier.

(a) Calculate the expected frequencies for each category in the table found in the report. [2]

(b) It is known that there are less than 20 items that are rated poor for both suppliers, find the number of items that are rated poor and fair from Supplier *P* and *Q* respectively. [4]

(c) Hence, carry out the chi-squared test to verify if there is sufficient evidence to conclude that the quality rating is dependent of the supplier. [2]

9 A greengrocer has a large number of apples to sell. He claims that the weight, X kg, of a randomly chosen apple has the probability density function given by

$$\mathrm{f}(x) = \begin{cases} 200(x - 0.1), & 0.1 \leq x \leq 0.2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the cumulative distribution function F, of X. [2]

When a customer comes to buy an apple, the greengrocer takes two apples at random, and sells the customer the heavier of the two. The weight of the heavier apple is Y kg.

(b) Show that the cumulative distribution function G, of Y is given by $\left[\mathrm{F}(y)\right]^2$. [1]

(c) Find the probability density function of Y and hence find the numerical value for $\mathrm{E}(Y)$. [4]

The greengrocer would like to test whether his claim on the distribution of X is valid. The weight, in kg, of 200 apples was recorded and the table below shows the frequency distribution.

Weight of apple (x)	0.1 – 0.12	0.12 – 0.14	0.14 – 0.16	0.16 – 0.18	0.18 – 0.2
Frequency	14	18	45	58	65

(d) Using your answer found in **(a)**, complete the following expected frequency table. [2]

Weight of apple (x)	0.1 – 0.12	0.12 – 0.14	0.14 – 0.16	0.16 – 0.18	0.18 – 0.2
Frequency	8	24		56	

(e) Hence, test at the 5% level of significance, whether the greengrocer's claim is valid. [3]

10 Carpal Tunnel syndrome is a condition which affects a person's ability to grip with their hands. Researchers tested a treatment for this syndrome which was applied to 8 randomly chosen patients. A pre-treatment and a post treatment test of grip, in kg, is given to each patient, with the following results.

Patient	1	2	3	4	5	6	7	8
Pre-treatment grip	24.3	29.5	28.0	28.5	21.5	28.7	25.1	26.3
Post-treatment grip	28.3	34.6	30.3	31.6	21.5	29.8	26.0	27.5

(a) Stating the distributional assumption, test, at the 1% significant level, whether the mean grip of people with the syndrome increases after undergoing the treatment. [7]

(b) It is given that there is evidence at the 10% significant level that the mean grip increases by more than w kg after undergoing the treatment. Find the range of values of w. [2]

(c) The researchers then claim that, at the 1% level of significance, the average grip increases by more than 1 kg after undergoing the treatment. Carry out a sign test to test the researchers' claim. [4]

Another random sample of 20 patients who went through the treatment are selected and a pre-treatment and a post treatment test of grip is given to each patient. A sign test is carried out at the 1% level of significance. Find the least number of patients who should have their mean grip increases by more than 1 kg after undergoing the treatment, such that the researchers' claim in **(c)** is valid. You may assume that none of the patient's grip increases by exactly 1 kg. [3]