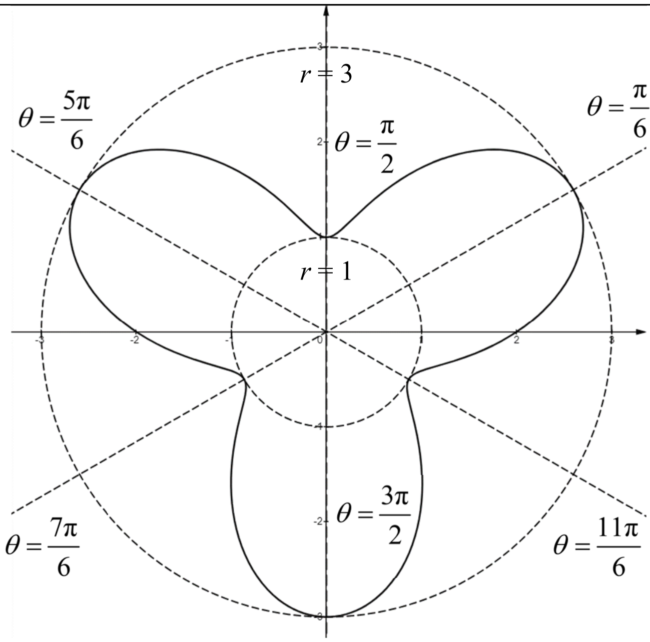
| Q1 | Solution |
|---|---|
| (a) |  |
| (b) | Since $r$ is minimum at $\theta = \frac{11}{6}\pi$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{\tan\left(\frac{11}{6}\pi\right)} = \sqrt{3}$<br><br>At $\theta = \frac{11}{6}\pi$, $x = \frac{\sqrt{3}}{2}$ and $y = -\frac{1}{2}$.<br><br>Hence the equation is $y - \left(-\frac{1}{2}\right) = \sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right) \Rightarrow y = \sqrt{3}x - 2$ |

| Q2 | Solution |
| --- | --- |
| (a) | $\frac{\mathrm{d}}{\mathrm{d}x}(xy)=x\frac{\mathrm{d}y}{\mathrm{d}x}+y$<br><br>$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(xy)=x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}$<br><br>$\frac{\mathrm{d}^3}{\mathrm{d}x^3}(xy)=x\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+3\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$<br><br>Conjecture: $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(xy)=x\frac{\mathrm{d}^ny}{\mathrm{d}x^n}+n\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}$, $n\in\mathbb{Z}^+$ |
| (b) | Let $P_n$ be the proposition $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(xy)=x\frac{\mathrm{d}^ny}{\mathrm{d}x^n}+n\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}$ where $n\in\mathbb{Z}^+$ .<br><br>When $n = 1$:<br><br>$\text{RHS}=x\frac{\mathrm{d}y}{\mathrm{d}x}+y$<br>$=\frac{\mathrm{d}}{\mathrm{d}x}(xy)=\text{LHS}$<br><br>$\therefore P_1$ is true.<br><br>Assume $P_k$ is true for some $k\in\mathbb{Z}^+$: $\frac{\mathrm{d}^k}{\mathrm{d}x^k}(xy)=x\frac{\mathrm{d}^ky}{\mathrm{d}x^k}+k\frac{\mathrm{d}^{k-1}y}{\mathrm{d}x^{k-1}}$<br><br>To prove $P_{k+1}$ is true:<br><br>$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(xy)=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}^k}{\mathrm{d}x^k}(xy)\right]$<br>$=\frac{\mathrm{d}}{\mathrm{d}x}\left[x\frac{\mathrm{d}^ky}{\mathrm{d}x^k}+k\frac{\mathrm{d}^{k-1}y}{\mathrm{d}x^{k-1}}\right]$<br>$=x\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}}+\frac{\mathrm{d}^ky}{\mathrm{d}x^k}+k\frac{\mathrm{d}^ky}{\mathrm{d}x^k}$<br>$=x\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}}+(k+1)\frac{\mathrm{d}^ky}{\mathrm{d}x^k}$<br><br>Since $P_1$ is true and $P_k$ is true $\Rightarrow$ $P_{k+1}$ is true, by Mathematical Induction, $P_n$ is true for all $n\in\mathbb{Z}^+$. |
| (c) | Let $y=\mathrm{e}^{-x}$, $\frac{\mathrm{d}^ny}{\mathrm{d}x^n}=(-1)^n\mathrm{e}^{-x}$<br><br>$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x\mathrm{e}^{-x})=x\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x})+n\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(\mathrm{e}^{-x})$<br>$=x(-1)^n\mathrm{e}^{-x}+n(-1)^{n-1}\mathrm{e}^{-x}$<br>$=(-1)^n\mathrm{e}^{-x}(x-n)$ |

| Q3 | Solution |
| --- | --- |
| (a) | $\begin{pmatrix} 1 & 5 & 1 & 1 \\ 2 & 11 & a & 4 \\ 1 & 2a & 16 & -11 \end{pmatrix} \xrightarrow[R_3-R_1]{R_2-2R_1} \begin{pmatrix} 1 & 5 & 1 & 1 \\ 0 & 1 & a-2 & 2 \\ 0 & 2a-5 & 15 & -12 \end{pmatrix}$ <br> $\xrightarrow{R_3-(2a-5)R_2} \begin{pmatrix} 1 & 5 & 1 & 1 \\ 0 & 1 & a-2 & 2 \\ 0 & 0 & -2a^2+9a+5 & -4a-2 \end{pmatrix}$ <br> $\xrightarrow{-R_3} \begin{pmatrix} 1 & 5 & 1 & 1 \\ 0 & 1 & a-2 & 2 \\ 0 & 0 & (2a+1)(a-5) & 2(2a+1) \end{pmatrix}$ <br> For system to be consistent, $(2a+1)(a-5) \neq 0$ OR $(2a+1)(a-5) = 0 = 2(2a+1)$ <br> $a \neq -\frac{1}{2}, 5$ OR $a = -\frac{1}{2}$ <br> $\therefore a \neq 5$ |
| (b) | When $a = -\frac{1}{2}$ , <br> $\begin{pmatrix} 1 & 5 & 1 & 1 \\ 0 & 1 & -\frac{5}{2} & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ <br> $x+5y+z=1$ <br> $y-\frac{5}{2}z=2$ <br> Let $z=\lambda$ , <br> $y=2+\frac{5}{2}\lambda$ <br> $x=-5\left(2+\frac{5}{2}\lambda\right)-\lambda+1$ <br> $=-9-\frac{27}{2}\lambda$ <br> $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -9 \\ 2 \\ 0 \end{pmatrix} + \frac{\lambda}{2}\begin{pmatrix} -27 \\ 5 \\ 2 \end{pmatrix}$ , $\lambda \in \mathbb{R}$ |
| (c) | $x+5y+(z-1)=1$ <br> $2x+11y+a(z-1)=4$ <br> $x+2ay+16(z-1)=-11$ <br> Substituting $z-1$ into solution above, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -9 \\ 2 \\ 1 \end{pmatrix} + \frac{\lambda}{2}\begin{pmatrix} -27 \\ 5 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$ |

| Q4 | Solution |
|---|---|
| (a) | Since $\mathrm{f}(1)\,\mathrm{f}(2)=\left(1^3-3-1\right)\left(2^3-6-1\right)=-3<0$, there is at least one root in the interval.<br><br>Since $\mathrm{f}'(x)=3x^2-3$, which is positive over the interval (1, 2), there will be exactly one root in the interval. |
| (b) | $\beta=\dfrac{(1)\,\mathrm{f}(2)-(2)\,\mathrm{f}(1)}{\mathrm{f}(2)-\mathrm{f}(1)}=\dfrac{(1)(1)-(2)(-3)}{(1)-(-3)}=\dfrac{7}{4}=1.75$<br><br>Since $\mathrm{f}'(x)=3x^2-3$ and $\mathrm{f}''(x)=6x$, are both positive over the interval (1, 2), the curve is upward sloping and concave upwards, hence the chord used for linear interpolation will lie above the actual curve, resulting in an underestimate. |
| (c) (i) | $x_{n+1}=x_n-\dfrac{x_n^{\ 3}-3x_n-1}{3x_n^{\ 2}-3}\quad\left(\text{or } \dfrac{2x_n^{\ 3}+1}{3x_n^{\ 2}-3}\right)$ |
| (c) (ii) | For $x_1=1$, $\mathrm{f}'(1)=0$ and so $\mathrm{g}(1)$ is undefined, causing the iterative process to fail.<br><br>From the GC, using $x_1=2$,<br><br>we have $\alpha=1.879$ (3 d.p.) |
| (d) | $u_{n+1}=\dfrac{u_n(1)-2\left(u_n^{\ 3}-3u_n-1\right)}{1-\left(u_n^{\ 3}-3u_n-1\right)}=\dfrac{2+7u_n-2u_n^{\ 3}}{2+3u_n-u_n^{\ 3}}$<br><br>Using this formula, $u_4=1.879303, u_5=1.879378$. This suggests that the linear interpolation converges almost as fast as the Newton-Raphson method in this particular context. |

| $n$ | $x_n$ |
|---|---|
| 1 | 2 |
| 2 | 17/9 |
| 3 | 10555/5616 |
| 4 | 1.879385 |
| 5 | 1.879385 |

| Q5 | Solution |
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| (a) | $S=\left\{\begin{pmatrix}a\\b\\c\end{pmatrix}\in\mathbb{R}^3 : 2a-b+4c=0\right\}$<br><br>Note that $\mathbf{0}\in S$ , hence $S$ is non-empty.<br><br>Let $\mathbf{u},\mathbf{v}\in S$ such that $\mathbf{u}=\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix},\mathbf{v}=\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}$, and $\lambda\in\mathbb{R}$<br><br>Then $\mathbf{u}+\lambda\mathbf{v}=\begin{pmatrix}a_1+\lambda a_2\\b_1+\lambda b_2\\c_1+\lambda c_2\end{pmatrix}$<br><br>$2(a_1+\lambda a_2)-(b_1+\lambda b_2)+4(c_1+\lambda c_2)$<br>$=2a_1-b_1+4c_1+\lambda(2a_2-b_2+4c_2)$<br>$=0$<br><br>$\therefore \mathbf{u}+\lambda\mathbf{v}\in S$<br><br>As $S\subseteq\mathbb{R}^3$ is closed under addition and scalar multiplication, $S$ is a linear space. |
| (b) | Let $\mathbf{u},\mathbf{v}\in S$ such that $\mathbf{u}=\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix},\mathbf{v}=\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}$, and $\lambda\in\mathbb{R}$<br><br>$\mathbf{u}+\lambda\mathbf{v}=\begin{pmatrix}a_1+\lambda a_2\\b_1+\lambda b_2\\c_1+\lambda c_2\end{pmatrix}$<br><br>$\mathrm{T}(\mathbf{u}+\lambda\mathbf{v})=\mathrm{T}\begin{pmatrix}a_1+\lambda a_2\\b_1+\lambda b_2\\c_1+\lambda c_2\end{pmatrix}$<br>$=\begin{pmatrix}a_1+\lambda a_2+3(c_1+\lambda c_2)\\6(a_1+\lambda a_2)+4(b_1+\lambda b_2)+2(c_1+\lambda c_2)\\a_1+\lambda a_2+b_1+\lambda b_2-(c_1+\lambda c_2)\end{pmatrix}$<br>$=\begin{pmatrix}a_1+3c_1\\6a_1+4b_1+2c_1\\a_1+b_1-c_1\end{pmatrix}+\lambda\begin{pmatrix}a_2+3c_2\\6a_2+4b_2+2c_2\\a_2+b_2-c_2\end{pmatrix}$<br>$=\mathrm{T}(\mathbf{u})+\lambda\mathrm{T}(\mathbf{v})$<br><br>Therefore T is a linear transformation. |

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| **(c)** | $\mathbf{A}=\begin{pmatrix}1&0&3\\6&4&2\\1&1&-1\end{pmatrix}$<br><br>$\begin{pmatrix}1&0&3\\6&4&2\\1&1&-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$<br><br>Using GC,<br><br>$\text{rref}(\mathbf{A})=\begin{pmatrix}1&0&3\\0&1&-4\\0&0&0\end{pmatrix}$<br><br>Let $z=\lambda$,<br><br>$x=-3\lambda,\ y=4\lambda$<br><br>Basis for the null space of $\mathbf{A}$ = $\left\{\begin{pmatrix}-3\\4\\1\end{pmatrix}\right\}$ |
| **(d)** | As $\text{nullity}(\mathbf{A})=1$,<br><br>$\text{rank}(\mathbf{A})=\dim(\mathbb{R}^3)-\text{nullity}(\mathbf{A})$<br>$=3-1=2$ |
| **(e)** | as $\text{T}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}a+3c\\6a+4b+2c\\a+b-c\end{pmatrix}$, and<br><br>$2(a+3c)-(6a+4b+2c)+4(a+b-c)$<br>$=(2-6+4)a+(-4+4)b+(6-2-4)c$<br>$=0$<br><br>$\text{Range(T)}\subseteq S$.<br><br>As $\dim(\text{Range(T)})=\text{rank}(\mathbf{A})=2$, and $\dim(S)=2$,<br><br>Then $\text{Range(T)}=S$.<br><br>**Alternatively,**<br><br>$\text{Range(T)}=\text{Column Space of }\mathbf{A}$<br>$=\text{Span}\left\{\begin{pmatrix}1\\6\\1\end{pmatrix},\begin{pmatrix}0\\4\\1\end{pmatrix}\right\}$<br><br>As $2(1)-1(6)+4(1)=0$ and $2(0)-1(4)+4(1)=0$, |

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| | Range(T) $\subseteq S$ .<br>As $\dim(\text{Range(T)}) = \text{rank}(\mathbf{A}) = 2$ , and $\dim(S) = 2$ ,<br>Then $\text{Range(T)} = S$ . |
| **(f)** | $\begin{pmatrix}-2\\0\\1\end{pmatrix} = \begin{pmatrix}1\\6\\1\end{pmatrix} - \begin{pmatrix}0\\4\\1\end{pmatrix} - \begin{pmatrix}3\\2\\-1\end{pmatrix}$<br>$\therefore \text{T}\begin{pmatrix}1\\-1\\-1\end{pmatrix} = \begin{pmatrix}-2\\0\\1\end{pmatrix}$<br>Subset of $\mathbb{R}^3$ required is $\left\{\begin{pmatrix}1\\-1\\-1\end{pmatrix} + \lambda\begin{pmatrix}-3\\4\\1\end{pmatrix},\ \lambda \in \mathbb{R}\right\}$ |

| Q6 | Solution |
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| (a) | $u_n - 4u_{n-1} - u_{n-2} = 0$<br><br>Auxiliary equation: $\lambda^2 - 4\lambda - 1 = 0$<br><br>$\lambda = \dfrac{4 \pm \sqrt{20}}{2}$<br>$= 2 \pm \sqrt{5}$<br><br>General solution: $u_n = A\left(2+\sqrt{5}\right)^n + B\left(2-\sqrt{5}\right)^n$, where $A$, $B$ are constants<br><br>When $n=0$, $u_0 = 1 = A + B$ ----- (1)<br><br>When $n=1$, $u_1 = \alpha = A\left(2+\sqrt{5}\right) + B\left(2-\sqrt{5}\right)$ ----- (2)<br><br>$\left(2+\sqrt{5}\right)$ (1) – (2): $\left(2+\sqrt{5}\right) - \alpha = 2\sqrt{5}B$<br><br>$B = \dfrac{\left(2+\sqrt{5}\right)-\alpha}{2\sqrt{5}}$<br><br>$A = 1 - B$<br>$= \dfrac{\sqrt{5}-2+\alpha}{2\sqrt{5}}$<br><br>$\therefore \quad u_n = \left(\dfrac{\sqrt{5}-2+\alpha}{2\sqrt{5}}\right)\left(2+\sqrt{5}\right)^n + \left(\dfrac{\left(2+\sqrt{5}\right)-\alpha}{2\sqrt{5}}\right)\left(2-\sqrt{5}\right)^n,\ n \geq 0$<br><br><u>Alternatively:</u><br><br>$(1)+(2)\left(2+\sqrt{5}\right)$:<br>$\left(2+\sqrt{5}\right)\alpha + 1 = \left(10+4\sqrt{5}\right)A$<br>$A = \dfrac{1+\left(2+\sqrt{5}\right)\alpha}{10+4\sqrt{5}}, \qquad B = \dfrac{9+4\sqrt{5}-\left(2+\sqrt{5}\right)\alpha}{10+4\sqrt{5}}$ |
| (b) | As $\left\|2+\sqrt{5}\right\| > 1$, $\left(2+\sqrt{5}\right)^n \to \infty$ as $n \to \infty$ .<br><br>As $\left\|2-\sqrt{5}\right\| < 1$ , $\left(2-\sqrt{5}\right)^n \to 0$ as $n \to \infty$.<br><br>Therefore $u_n$ converges if $\dfrac{\sqrt{5}-2+\alpha}{2\sqrt{5}} = 0$<br><br>$\therefore \alpha = 2 - \sqrt{5}$ . |

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| **(c)(i)** | $\text{RHS} = 1 + \dfrac{1}{4(1+s_n)}$<br>$= 1 + \dfrac{1}{4\left(1 + \dfrac{2u_n + u_{n-1}}{2u_n}\right)}$<br>$= 1 + \dfrac{2u_n}{4(4u_n + u_{n-1})}$<br>$= 1 + \dfrac{2u_n}{4u_{n+1}}$<br>$= \dfrac{2u_{n+1} + u_n}{2u_{n+1}}$<br>$= s_{n+1} = \text{LHS}$ |
| **(c)(ii)** | $s_{n+1} = 1 + \dfrac{1}{4(1+s_n)}$<br>Let the limit of the sequence be $l$.<br>Then as $n \to \infty$, $l = 1 + \dfrac{1}{4(1+l)} \quad \Rightarrow \quad l^2 = \dfrac{5}{4}$<br>$l \geq 0$, since $u_0 = 1$, $u_1 = 0 \Rightarrow u_n \geq 0$ for $n \geq 0 \Rightarrow s_n \geq 0$ for $n \geq 2$<br>$\therefore l = \dfrac{\sqrt{5}}{2}$ |
| **(c)(iii)** | $s_8 = \dfrac{2u_8 + u_7}{2u_8}$<br>Using GC, $u_8 = 5473$, $u_7 = 1292$<br>$s_8 = \dfrac{12238}{10946}$<br>$s_8 \approx \dfrac{\sqrt{5}}{2}$<br>$\therefore \sqrt{5} \approx \dfrac{12238}{5473}$ |

<table>
<tr><th>Q7</th><th>Solution</th></tr>
<tr><td>(a)</td><td>Consider the triangle PQR, with $\angle RPQ = \theta$, $PR = r$, $PQ = 2c$ and $RQ = r - 2a$.<br>Using cosine rule, $(RQ)^2 = (PQ)^2 + (PR)^2 - 2(PQ)(PR)\cos(\angle RPQ)$<br>$\Rightarrow \quad (r-2a)^2 = (2c)^2 + (r)^2 - 2(2c)(r)\cos\theta$<br>$\Rightarrow \quad r^2 - 4ar + 4a^2 = 4c^2 + r^2 - 4cr\cos\theta$<br>$\Rightarrow \quad r\left(1 - \frac{c}{a}\cos\theta\right) = \frac{a^2 - c^2}{a}$<br>$\Rightarrow \quad u = \frac{c}{a}, \quad v = \frac{a^2 - c^2}{a}$</td></tr>
<tr><td>(b)</td><td>$e = \frac{c}{a}$</td></tr>
<tr><td>(c)</td><td>$r\cos\theta = x + c, \quad r\sin\theta = y, \quad r^2 = (x+c)^2 + y^2$<br>$r\left(1 - \frac{c}{a}\cos\theta\right) = \frac{a^2 - c^2}{a} \quad \Rightarrow \quad ar - c(r\cos\theta) = a^2 - c^2$ (from (a))<br>Substituting $r\cos\theta$, $ar - c(x+c) = a^2 - c^2 \quad \Rightarrow \quad a^2 r^2 = (a^2 + cx)^2$<br>Substituting $r^2$, $a^2\left[(x+c)^2 + y^2\right] = a^4 + 2a^2cx + c^2x^2$<br>$\Rightarrow \quad a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2 = a^4 + 2a^2cx + c^2x^2$<br>$\Rightarrow \quad (a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$</td></tr>
<tr><td>(d)</td><td>$(3 - 2^2)x^2 + 3y^2 = 3(3 - 2^2) \quad \Rightarrow \quad \frac{x^2}{3} - y^2 = 1$<br>At intersection with $y = mx + k$,<br>$\frac{x^2}{3} - (mx+k)^2 = 1 \quad \Rightarrow \quad x^2 - 3(m^2x^2 + 2mkx + k^2) = 3$<br>$\Rightarrow \quad (3m^2 - 1)x^2 + (6mk)x + (3k^2 + 3) = 0$<br>Since $y = mx + k$ is a tangent, there is only 1 solution, i.e.<br>$\Rightarrow \quad (6mk)^2 - 4(3m^2 - 1)(3k^2 + 3) = 0$<br>$\Rightarrow \quad 36m^2k^2 - 4(9m^2k^2 + 9m^2 - 3k^2 - 3) = 0$<br>$\Rightarrow \quad 36m^2k^2 - 36m^2k^2 - 36m^2 + 12k^2 + 12 = 0$<br>$\Rightarrow \quad k^2 = 3m^2 - 1$<br><u>Alternatively:</u><br>$\frac{x^2}{3} - y^2 = 1 \quad \Rightarrow \quad \frac{2}{3}x - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{3y}$<br>Equation of tangent at $(x_1, y_1)$ is $y - y_1 = \frac{x_1}{3y_1}(x - x_1)$</td></tr>
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| | $\Rightarrow \quad y=\dfrac{x_1}{3y_1}x+\dfrac{3y_1^{\,2}-x_1^{\,2}}{3y_1}=\dfrac{x_1}{3y_1}x-\dfrac{1}{y_1}$, where $y_1<0$.<br><br>Then $m=\dfrac{x_1}{3y_1}$, $\quad k=-\dfrac{1}{y_1}\Rightarrow k^2=\dfrac{1}{y_1^{\,2}}$.<br><br>$\text{RHS}=3m^2-1=3\left(\dfrac{x_1}{3y_1}\right)^2-1=\dfrac{x_1^{\,2}-3y_1^{\,2}}{3y_1^{\,2}}=\dfrac{3}{3y_1^{\,2}}=\dfrac{1}{y_1^{\,2}}=k^2$ (shown) |
| **(e)** | $\text{f}(m)=3\left(-\dfrac{1}{m}\right)^2-1$<br>$=\dfrac{3}{m^2}-1$ |
| **(f)** | At the intersection,<br><br>$mx+\sqrt{3m^2-1}=-\dfrac{1}{m}x+\sqrt{\dfrac{3}{m^2}-1}$<br>$\Rightarrow m^2x+m\sqrt{3m^2-1}=-x+\sqrt{3-m^2}$<br>$\Rightarrow x=\dfrac{\sqrt{3-m^2}-m\sqrt{3m^2-1}}{m^2+1}$<br>$y=m\left(\dfrac{\sqrt{3-m^2}-m\sqrt{3m^2-1}}{m^2+1}\right)+\sqrt{3m^2-1}=\dfrac{m\sqrt{3-m^2}+\sqrt{3m^2-1}}{m^2+1}$<br><br>Substituting,<br><br>$x^2+y^2=\left(\dfrac{\sqrt{3-m^2}-m\sqrt{3m^2-1}}{m^2+1}\right)^2+\left(\dfrac{m\sqrt{3-m^2}+\sqrt{3m^2-1}}{m^2+1}\right)^2$<br>$=\dfrac{1}{\left(m^2+1\right)^2}\left[\left(3-m^2\right)+m^2\left(3m^2-1\right)+m^2\left(3-m^2\right)+\left(3m^2-1\right)\right]$<br>$=\dfrac{2+4m^2+2m^4}{1+2m^2+m^4}=2$ |

| Q8 | Solution |
|---|---|
| (a) | $x = r\cos\theta, \quad y = r\sin\theta$<br>$\left(x^2+y^2\right)^2 = 7y^3 + 2x^2y$<br>$\Rightarrow \left(r^2\right)^2 = 7r^3\sin^3\theta + 2r^3\cos^2\theta\sin\theta$<br>$\Rightarrow r = 7\sin^3\theta + 2\left(1-\sin^2\theta\right)\sin\theta$<br>$\Rightarrow r = 5\sin^3\theta + 2\sin\theta$ |
| (b) | $A = \int_0^{\frac{1}{2}\pi}\left[\mathrm{f}(\theta)\right]^2 \,\mathrm{d}\theta = \int_0^{\frac{1}{2}\pi}\left(5\sin^3\theta + 2\sin\theta\right)^2 \,\mathrm{d}\theta$<br>$= \int_0^{\frac{1}{2}\pi} 25\sin^6\theta + 20\sin^4\theta + 4\sin^2\theta \,\mathrm{d}\theta$<br><br>Let $\mathrm{g}(\theta) = 25\sin^6\theta + 20\sin^4\theta + 4\sin^2\theta$, then $\mathrm{g}(0)=0$, $\mathrm{g}\left(\frac{\pi}{2}\right) = 49$, and<br>$\mathrm{g}\left(\frac{\pi}{4}\right) = 25\left(\frac{1}{\sqrt{2}}\right)^6 + 20\left(\frac{1}{\sqrt{2}}\right)^4 + 4\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{81}{8}$.<br>Using Simpson's Rule,<br>$A \approx \frac{1}{6}\left(\frac{\pi}{2}-0\right)\left[\mathrm{g}(0) + 4\mathrm{g}\left(\frac{\pi}{4}\right) + \mathrm{g}\left(\frac{\pi}{2}\right)\right]$<br>$= \frac{\pi}{12}\left[0 + \frac{81}{2} + 49\right] = \frac{179}{24}\pi$ |
| (c) | $\int_0^{\frac{1}{2}\pi} \sin^n\theta \,\mathrm{d}\theta = \int_0^{\frac{1}{2}\pi} \sin\theta\sin^{n-1}\theta \,\mathrm{d}\theta$<br>$= \left[-\cos\theta\sin^{n-1}\theta\right]_0^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi}(-\cos\theta)(n-1)\left(\sin^{n-2}\theta\right)(\cos\theta)\,\mathrm{d}\theta$<br>$= \left[-(0)(1)+(1)(0)\right] + (n-1)\int_0^{\frac{1}{2}\pi}\left(\cos^2\theta\right)\left(\sin^{n-2}\theta\right)\mathrm{d}\theta$<br>$= (n-1)\int_0^{\frac{1}{2}\pi}\left(1-\sin^2\theta\right)\left(\sin^{n-2}\theta\right)\mathrm{d}\theta$<br>$\int_0^{\frac{1}{2}\pi} \sin^n\theta \,\mathrm{d}\theta = (n-1)\int_0^{\frac{1}{2}\pi}\left(\sin^{n-2}\theta\right)\mathrm{d}\theta - (n-1)\int_0^{\frac{1}{2}\pi}\sin^n\theta\,\mathrm{d}\theta$<br>$n\int_0^{\frac{1}{2}\pi} \sin^n\theta \,\mathrm{d}\theta = (n-1)\int_0^{\frac{1}{2}\pi}\left(\sin^{n-2}\theta\right)\mathrm{d}\theta$<br>$\int_0^{\frac{1}{2}\pi} \sin^n\theta \,\mathrm{d}\theta = \left(\frac{n-1}{n}\right)\int_0^{\frac{1}{2}\pi}\left(\sin^{n-2}\theta\right)\mathrm{d}\theta$ |

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| **(d)** | $A=\int_0^{\frac{1}{2}\pi} 25\sin^6\theta+20\sin^4\theta+4\sin^2\theta\,\mathrm{d}\theta$<br>$=\int_0^{\frac{1}{2}\pi} 25\left(\frac{6-1}{6}\right)\sin^4\theta+20\left(\frac{4-1}{4}\right)\sin^2\theta+4\left(\frac{2-1}{2}\right)\mathrm{d}\theta$<br>$=\int_0^{\frac{1}{2}\pi} \frac{125}{6}\left(\frac{4-1}{4}\right)\sin^2\theta+15\left(\frac{2-1}{2}\right)+2\,\mathrm{d}\theta$<br>$=\int_0^{\frac{1}{2}\pi} \frac{125}{8}\left(\frac{2-1}{2}\right)+\frac{19}{2}\,\mathrm{d}\theta$<br>$=\int_0^{\frac{1}{2}\pi} \frac{277}{16}\,\mathrm{d}\theta=\frac{277}{32}\pi$ |
| **(e)** | Percentage error $=\left\|\frac{179}{24}\pi-\frac{277}{32}\pi\right\|\div\frac{277}{32}\pi\times 100\%=13.8\%$ (3 s.f.).<br><br>Hence Brian's claim that the estimate is unreliable is valid. |

| Q9 | Solution |
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| (a) | $\mathbf{A}=\begin{pmatrix}0.98 & 0.1\\ 0.02 & 0.9\end{pmatrix}$<br><br>$\begin{pmatrix}B_n\\ E_n\end{pmatrix}=\begin{pmatrix}0.98 & 0.1\\ 0.02 & 0.9\end{pmatrix}\begin{pmatrix}B_{n-1}\\ E_{n-1}\end{pmatrix}$<br>$=\begin{pmatrix}0.98B_{n-1}+0.1E_{n-1}\\ 0.02B_{n-1}+0.9E_{n-1}\end{pmatrix}$<br><br>$B_n+E_n=\left(0.98B_{n-1}+0.1E_{n-1}\right)+\left(0.02B_{n-1}+0.9E_{n-1}\right)$<br>$=B_{n-1}+E_{n-1}$<br><br>As $B_n+E_n=B_{n-1}+E_{n-1}$ for all $n\geq 1$, $B_n+E_n=B_0+E_0$ for all $n\geq 0$.<br><br>Hence $B_n+E_n$ is a constant for all $n\geq 0$. |
| (b) | $\lvert\mathbf{A}-\lambda\mathbf{I}\rvert=\begin{vmatrix}0.98-\lambda & 0.1\\ 0.02 & 0.9-\lambda\end{vmatrix}=0$<br><br>$\Rightarrow(0.98-\lambda)(0.9-\lambda)-(0.1)(0.02)=0$<br><br>$\lambda^2-1.88\lambda+0.88=0$<br><br>$\lambda=\dfrac{1.88\pm\sqrt{1.88^2-4(0.88)}}{2}$<br>$=\dfrac{1.88\pm 0.12}{2}$<br><br>Eigenvalues are $\lambda=1, 0.88$<br><br>Let $\lambda_1=1$, $\left(\begin{array}{cc\|c}-0.02 & 0.1 & 0\\ 0.02 & -0.1 & 0\end{array}\right)\xrightarrow[-\frac{1}{0.02}R_1]{R_2+R_1}\left(\begin{array}{cc\|c}1 & -5 & 0\\ 0 & 0 & 0\end{array}\right)$<br><br>Corresponding eigenvector $\mathbf{e}_1=\begin{pmatrix}5\\ 1\end{pmatrix}$<br><br>Let $\lambda_2=0.88$, $\left(\begin{array}{cc\|c}0.1 & 0.1 & 0\\ 0.02 & 0.02 & 0\end{array}\right)\xrightarrow[10R_1]{R_2-0.2R_1}\left(\begin{array}{cc\|c}1 & 1 & 0\\ 0 & 0 & 0\end{array}\right)$<br><br>Corresponding eigenvector $\mathbf{e}_2=\begin{pmatrix}-1\\ 1\end{pmatrix}$ |
| (c) | $\mathbf{A}=\begin{pmatrix}5 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0.88\end{pmatrix}\begin{pmatrix}5 & -1\\ 1 & 1\end{pmatrix}^{-1}$<br><br>$\begin{pmatrix}5 & -1\\ 1 & 1\end{pmatrix}^{-1}=\dfrac{1}{6}\begin{pmatrix}1 & 1\\ -1 & 5\end{pmatrix}$ |

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| | $\mathbf{A}^n=\frac{1}{6}\begin{pmatrix}5 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0.88^n\end{pmatrix}\begin{pmatrix}1 & 1\\-1 & 5\end{pmatrix}$<br>$=\frac{1}{6}\begin{pmatrix}5+(0.88)^n & 5-5(0.88)^n\\1-(0.88)^n & 1+5(0.88)^n\end{pmatrix}$<br>$\begin{pmatrix}\mathrm{B}_n\\\mathrm{E}_n\end{pmatrix}=\mathbf{A}^n\begin{pmatrix}50\\80\end{pmatrix}$<br>$=\frac{1}{6}\begin{pmatrix}5+(0.88)^n & 5-5(0.88)^n\\1-(0.88)^n & 1+5(0.88)^n\end{pmatrix}\begin{pmatrix}50\\80\end{pmatrix}$<br>$=\frac{1}{6}\begin{pmatrix}650-350(0.88)^n\\130+350(0.88)^n\end{pmatrix}$<br>$=\frac{5}{3}\begin{pmatrix}65-35(0.88)^n\\13+35(0.88)^n\end{pmatrix}$ |
| **(d)** | The model does not support this position.<br>As $n\to\infty$ ,<br>$\begin{pmatrix}\mathrm{B}_n\\\mathrm{E}_n\end{pmatrix}=\frac{5}{3}\begin{pmatrix}65-35(0.88)^n\\13+35(0.88)^n\end{pmatrix}\to\frac{5}{3}\begin{pmatrix}65\\13\end{pmatrix}=\frac{65}{3}\begin{pmatrix}5\\1\end{pmatrix}$<br>Hence we would expect the populations of city B and E to stabilise at a ratio of 5:1, and city E will not be abandoned. |
| **(e)** | The population of city B and E at the end of 2025 can be modelled as $\begin{pmatrix}B_3\\E_3\end{pmatrix}+\begin{pmatrix}-5\\5\end{pmatrix}$, where $\mathrm{B}_3$ and $\mathrm{E}_3$ are the populations under the old simulation.<br>As $\begin{pmatrix}-5\\5\end{pmatrix}=5\begin{pmatrix}-1\\1\end{pmatrix}$ is an eigenvector of **A** with eigenvalue 0.88, the difference between the new model and the original model *k* years after 2025 can be represented as<br>$\mathbf{A}^k\begin{pmatrix}-5\\5\end{pmatrix}=0.88^k\begin{pmatrix}-5\\5\end{pmatrix}$<br>$0.88^k(5)<0.1$<br>$k>\frac{\ln 0.02}{\ln 0.88}$<br>$k>30.603$<br>Hence the difference will first fall below 100 in the year 2056. |