RAFFLES INSTITUTION
2024 Year 6 H2 Mathematics Timed Practice
Questions and Solutions with comments

1 Consider the second order differential equation

$$\frac{d^2x}{dt^2}+a\frac{dx}{dt}=b,$$

where a and b are non-zero constants and $\frac{dx}{dt}\neq\frac{b}{a}$.

(a) By substituting $y=\frac{dx}{dt}$, show that the above second order differential equation can be written as $\frac{dy}{dt}=b-ay$. [2]

(b) Find y in terms of t and hence find x in terms of t. [5]

(a) **[2]**	Since $y=\frac{dx}{dt}$, then $\frac{dy}{dt}=\frac{d}{dt}\left(\frac{dx}{dt}\right)=\frac{d^2x}{dt^2}$. $\frac{d^2x}{dt^2}+a\frac{dx}{dt}=b$ becomes $\frac{dy}{dt}+ay=b$. Rearranging, we have $\frac{dy}{dt}=b-ay$.	Similar question: Tut 8C Qn 7(i)(a)
(b) **[5]**	$\frac{dy}{dt}=b-ay$ $\frac{1}{b-ay}\frac{dy}{dt}=1,$ since $b-ay=b-a\frac{dx}{dt}\neq 0$ Integrating with respect to t, $\int\frac{1}{b-ay}\,dy=\int 1\,dt$ $-\frac{1}{a}\ln\lvert b-ay\rvert=t+c,$ where $c\in\mathbb{R}$ $\ln\lvert b-ay\rvert=-at-ac$ $\lvert b-ay\rvert=e^{-at-ac}$ $b-ay=\pm e^{-ac}e^{-at}$ $b-ay=Ae^{-at}$, where $A=\pm e^{-ac}, A\neq 0$ $y=\frac{1}{a}\left(b-Ae^{-at}\right)$ Since $y=\frac{dx}{dt}$, $\frac{dx}{dt}=\frac{1}{a}\left(b-Ae^{-at}\right)$ $x=\frac{1}{a}\left(bt+\frac{A}{a}e^{-at}\right)+B$, where $B\in\mathbb{R}$ $=\frac{b}{a}t+\frac{A}{a^2}e^{-at}+B$	There are a few did not start off correctly and wrote: $y=\int b-ay\,dt$ The more common mistakes • Not having the modulus after integrating $\frac{1}{b-ay}$ • Not able to use the appropriate method to "drop" the modulus for $\lvert b-ay\rvert$ • Not writing down the constant c and d after completing the integration. Some wrote the answer as $x=Kt+Ce^{-at}+B$. Note that B and C are **arbitrary constants** (in this case, B can be any real value and C can be any non-zero value). However, K is not an arbitrary constant as $K=\frac{b}{a}$. So, in this instance, it is better to write the answer as $x=\frac{b}{a}t+Ce^{-at}+B$ Similar question: Tut 8C Qn 3, 7(i)(b), 10(i)

2 Referred to an origin O, the position vectors of three points A, B and C are $3\mathbf{i}-2\mathbf{j}+5\mathbf{k}$, $\mathbf{i}+4\mathbf{j}-2\mathbf{k}$ and $\alpha\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ respectively.

(a) Determine the value of α for which points O, A, B and C are coplanar. [2]

(b) Given that $\overrightarrow{OD}=\overrightarrow{OA}+\lambda\overrightarrow{OB}$, find the value of λ for which $\overrightarrow{OD}$ is perpendicular to the $y-$axis. [2]

(c) Hence, find the exact area of triangle OAD. [2]

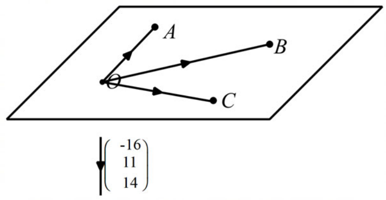
<table>
<tr>
<td>(a)
[2]</td>
<td>

<u>Method 1</u>

A vector normal to plane $OABC$

$=\overrightarrow{OA}\times\overrightarrow{OB}$

$=\begin{pmatrix}3\\-2\\5\end{pmatrix}\times\begin{pmatrix}1\\4\\-2\end{pmatrix}$



$=\begin{pmatrix}-16\\11\\14\end{pmatrix}$

So, $\overrightarrow{OC}\bullet\begin{pmatrix}-16\\11\\14\end{pmatrix}=\begin{pmatrix}\alpha\\2\\4\end{pmatrix}\bullet\begin{pmatrix}-16\\11\\14\end{pmatrix}=0$

$-16\alpha+22+56=0$

$\alpha=\frac{39}{8}$

<u>Method 2</u>

Given the points are coplanar,

$\begin{pmatrix}\alpha\\2\\4\end{pmatrix}=s\begin{pmatrix}3\\-2\\5\end{pmatrix}+t\begin{pmatrix}1\\4\\-2\end{pmatrix}$, where s and t are constants.

$\alpha=3s+t$ --- (1)
$2=-2s+4t$ --- (2)
$4=5s-2t$ --- (3)

Using GC to solve the above 3 eqns, we get

$\alpha=\frac{39}{8},\ s=\frac{5}{4},\ t=\frac{9}{8}$.

</td>
<td>This part is well done.</td>
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<tr>
<td>(b)
[2]</td>
<td>

$\overrightarrow{OD}=\overrightarrow{OA}+\lambda\overrightarrow{OB}=\begin{pmatrix}3\\-2\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\4\\-2\end{pmatrix}$

A direction vector of the y-axis is $\begin{pmatrix}0\\1\\0\end{pmatrix}$.

If $\overrightarrow{OD}$ is perpendicular to the $y-$axis,

</td>
<td>This part is well done.</td>
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</table>

	$$\overrightarrow{OD}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}=0$$ $$\begin{pmatrix}3+\lambda\\-2+4\lambda\\5-2\lambda\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}=0 \quad\Rightarrow\quad \lambda=\frac{1}{2}$$	
(c) **[2]**	$$\overrightarrow{OD}=\overrightarrow{OA}+\frac{1}{2}\overrightarrow{OB}=\begin{pmatrix}3\\-2\\5\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1\\4\\-2\end{pmatrix}=\begin{pmatrix}3.5\\0\\4\end{pmatrix}$$ Area of ΔOAD $$=\frac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OD}\right\|$$ $$=\frac{1}{2}\left\|\begin{pmatrix}3\\-2\\5\end{pmatrix}\times\begin{pmatrix}3.5\\0\\4\end{pmatrix}\right\|$$ $$=\frac{1}{2}\left\|\begin{pmatrix}-8\\5.5\\7\end{pmatrix}\right\|$$ $$=\frac{1}{2}\sqrt{64+30.25+49}$$ $$=\frac{1}{4}\sqrt{573}$$	This part is well done.

3 **(a)** Write down constants A and B such that for all values of x, $x-1=A(2x-4)+B$. [1]

(b) By expressing x^2-4x+5 in completed square form, find

$$\int \frac{x-1}{x^2-4x+5}\,\mathrm{d}x.$$ [4]

(c) Hence, without using a calculator, show that

$$\int_{-2}^{2} \frac{|x-1|}{x^2-4x+5}\,\mathrm{d}x = \frac{\pi}{2}+\frac{1}{2}\ln\frac{p}{q}-\tan^{-1}q,$$

where p and q are exact real constants to be determined. [4]

(a) **[1]**	From $x-1=A(2x-4)+B$, Compare coefficients of x: $2A=1 \quad \Rightarrow A=\frac{1}{2}$ Compare constants: $-4A+B=-1 \quad \Rightarrow B=1$ Hence $A=\frac{1}{2}$, $B=1$.	This part is well done.
(b) **[4]**	$\int \frac{x-1}{x^2-4x+5}\,\mathrm{d}x = \int \frac{\frac{1}{2}(2x-4)+1}{x^2-4x+5}\mathrm{d}x$, from **(a)** $=\frac{1}{2}\int \frac{2x-4}{x^2-4x+5}\mathrm{d}x+\int \frac{1}{x^2-4x+5}\mathrm{d}x$ $=\frac{1}{2}\ln\left\|x^2-4x+5\right\|+\int \frac{1}{(x-2)^2+1}\mathrm{d}x$ $=\frac{1}{2}\ln\left(x^2-4x+5\right)+\tan^{-1}(x-2)+c$, since $x^2-4x+5>0$ for all real values of x.	
(c) **[4]**	Note that $\|x-1\|=\begin{cases}-(x-1), & \text{if } x\le 1,\\ x-1, & \text{if } x>1.\end{cases}$ $\int_{-2}^{2}\frac{\|x-1\|}{x^2-4x+5}\,\mathrm{d}x$ $=-\int_{-2}^{1}\frac{x-1}{x^2-4x+5}\,\mathrm{d}x+\int_{1}^{2}\frac{x-1}{x^2-4x+5}\,\mathrm{d}x$ $=-\frac{1}{2}\left[\ln(x^2-4x+5)\right]_{-2}^{1}-\left[\tan^{-1}(x-2)\right]_{-2}^{1}$ $+\frac{1}{2}\left[\ln(x^2-4x+5)\right]_{1}^{2}+\left[\tan^{-1}(x-2)\right]_{1}^{2}$ $=-\frac{1}{2}\ln 2+\frac{1}{2}\ln 17+\frac{1}{2}\ln 1-\frac{1}{2}\ln 2$ $-\tan^{-1}(-1)+\tan^{-1}(-4)+\tan^{-1}(0)-\tan^{-1}(-1)$ $=-\ln 2+\frac{1}{2}\ln 17-2\tan^{-1}(-1)-\tan^{-1}(4)$ $=\frac{1}{2}\ln\frac{17}{4}+2\tan^{-1}(1)-\tan^{-1}(4)$ $=\frac{\pi}{2}+\frac{1}{2}\ln\frac{17}{4}-\tan^{-1}(4)$, where $p=17$, $q=4$.	This part is not well-done. Not many are able to split the limits correctly. Similar question: Assignment 8B Qn 4

4 The functions f and g are defined by $\mathrm{f}: x \mapsto \dfrac{2x+14}{3x-2},\ x \in \mathbb{R},\ x > \dfrac{2}{3},$

$$\mathrm{g}: x \mapsto \frac{\sin x}{x},\ x \in \mathbb{R},\ x > 0.$$

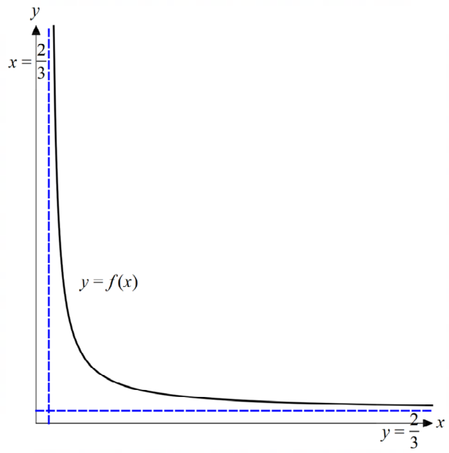
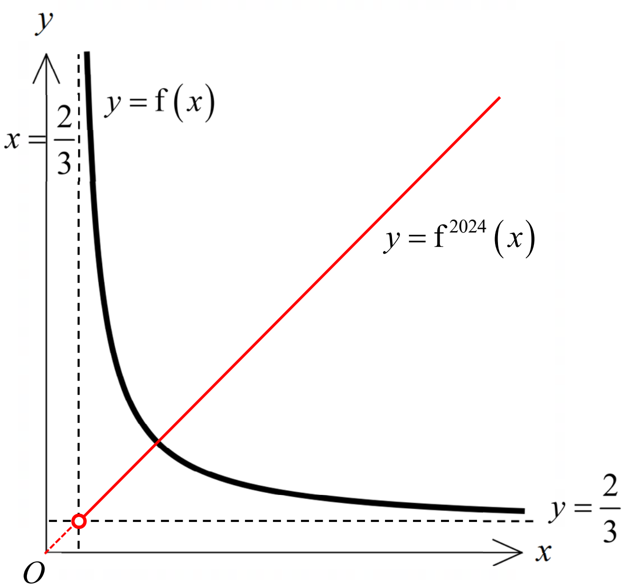
(a) Find $\mathrm{f}^{-1}(x)$ and state its domain. [3]

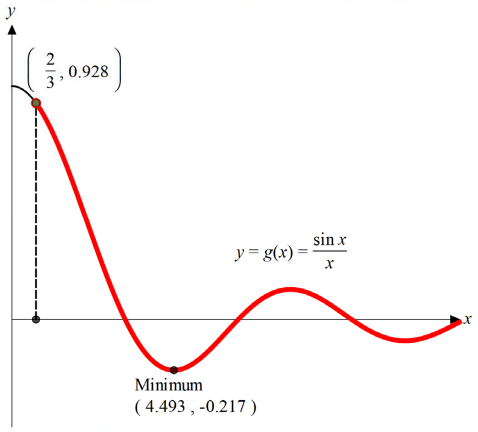
(b) Explain why $\mathrm{f}^{2024}(x) = x$ for $x > \dfrac{2}{3}$. [1]

(c) On the same diagram, sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{2024}(x)$, giving the equations of any asymptotes. [3]

(d) Explain why the composite function gf exists. [1]

(e) Find the range of gf . [2]

(a) **[3]**	Let $y = \dfrac{2x+14}{3x-2}$. $y(3x-2) = 2x+14$ $3xy - 2y = 2x+14$ $3xy - 2x = 2y+14$ $x = \dfrac{2y+14}{3y-2}$ $\mathrm{f}^{-1}(x) = \dfrac{2x+14}{3x-2}$ $D_{\mathrm{f}^{-1}} = R_{\mathrm{f}} = \left(\dfrac{2}{3}, \infty\right)$. 	Quite a number of students did not realise that the graph of $y = \dfrac{2x+14}{3x-2}$ has the horizontal asymptote $y = \frac{2}{3}$. Thus, not able to give the correct range of f (which is also the domain of the inverse function).
(b) **[1]**	Since $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ for $x > \dfrac{2}{3}$, then $\mathrm{f}^2(x) = x$. Hence, $\mathrm{f}^{2024}(x) = \mathrm{f}^{2022}\left(\mathrm{f}^2(x)\right)$ $= \mathrm{f}^{2022}(x)$, from above $= \mathrm{f}^{2020}\left(\mathrm{f}^2(x)\right)$ $= \mathrm{f}^{2020}(x)$ $\vdots$ $= \mathrm{f}^2(x) = x$	Similar question: Tut 3 Qn 9(iii), Chap 3 Example 11(b)
(c) **[3]**		Note that $D_{\mathrm{f}^{2024}} = D_{\mathrm{f}} = \left(\frac{2}{3}, \infty\right)$. So, we only drew the graph for $x > \frac{2}{3}$ Many also drew the graph of f in the 3rd quadrant. Note that domain of f is given to be $\left(\frac{2}{3}, \infty\right)$.

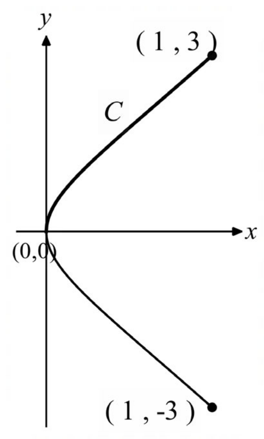
(d) **[1]**	Since $R_f = \left(\frac{2}{3}, \infty\right) \subseteq (0, \infty) = D_g$, so the composite function gf exists.	
(e) **[2]**	$D_f = \left(\dfrac{2}{3}, \infty\right) \xrightarrow{f} R_f = \left(\dfrac{2}{3}, \infty\right) \xrightarrow{g} R_{gf} = [-0.217, 0.928)$ (3 sf) 	Not many students are able to give the correct range.

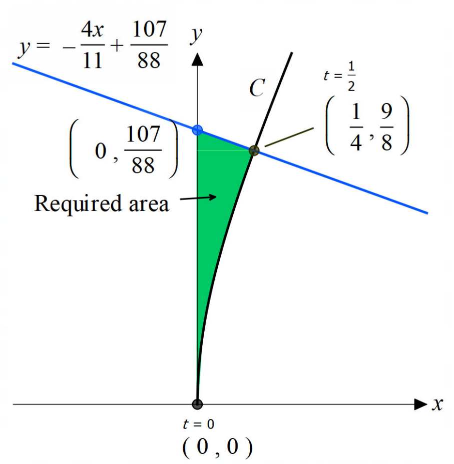
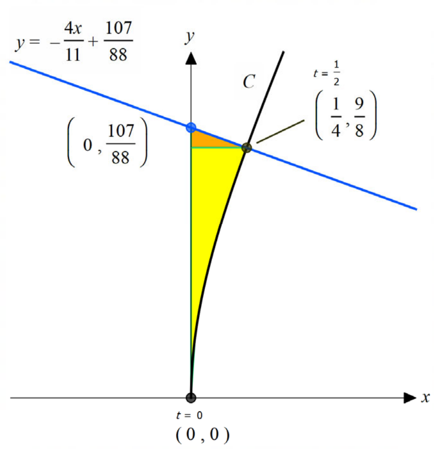
5 The curve C is defined by the parametric equations $x=t^2$ and $y=2t+t^3$, where $-1\le t\le 1$.

(a) Sketch the graph of C. [2]

(b) Without using a calculator, find the equation of the normal to the curve C at the point where $t=\dfrac{1}{2}$. [4]

(c) Find the exact area bounded by the curve C, the positive y-axis and the normal found in part **(b)**. [4]

(a) **[2]**		Some students used the default window setting and only obtain the graph in the first quadrant. There is a need to change $t_{\min}=-1$ and $t_{\max}=1$ as defined in the question. Coordinates of the end-points are to be stated on the graph.
(b) **[4]**	**<u>Method 1</u>** $x=t^2 \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}t}=2t, \qquad y=2t+t^3 \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}t}=2+3t^2.$ $\therefore \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2+3t^2}{2t}$ When $t=\dfrac{1}{2}, x=\dfrac{1}{4}, y=\dfrac{9}{8}, \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2+3\left(\frac{1}{4}\right)}{1}=\dfrac{11}{4}$ Gradient of normal = $-\dfrac{4}{11}$ Equation of normal is $y-\dfrac{9}{8}=-\dfrac{4}{11}\left(x-\dfrac{1}{4}\right)$ $y=-\dfrac{4}{11}x+\dfrac{1}{11}+\dfrac{9}{8}$ $\therefore y=-\dfrac{4}{11}x+\dfrac{107}{88}$ **<u>Method 2</u>** $y=2t+t^3 \Rightarrow y^2=4t^2+4t^4+t^6=4x+4x^2+x^3.$ Differentiate with respect x, $2y\dfrac{\mathrm{d}y}{\mathrm{d}x}=4+8x+3x^2 \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{4+8x+3x^2}{2y}.$ When $t=\dfrac{1}{2}, x=\dfrac{1}{4}, y=\dfrac{9}{8}, \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2+8\left(\frac{1}{4}\right)+3\left(\frac{1}{4}\right)^2}{2\left(\frac{9}{8}\right)}=\dfrac{11}{4}.$	This part is well done.

	Equation of normal is $y-\frac{9}{8}=-\frac{4}{11}\left(x-\frac{1}{4}\right)$ $y=-\frac{4}{11}x+\frac{1}{11}+\frac{9}{8}$ $\therefore y=-\frac{4}{11}x+\frac{107}{88}$.	
(c) **[4]**	 **Method 1** **(Area under graph to y-axis, parametric)** To find the point of intersection between normal and the y-axis, when $x=0$, $y=\frac{107}{88}$.  Required Area = yellow region + orange region $=\int_0^{\frac{9}{8}} x\ dy+\frac{1}{2}\times\left(\frac{107}{88}-\frac{9}{8}\right)\times\frac{1}{4}$ $=\int_0^{\frac{1}{2}} t^2\left(2+3t^2\right)dt+\frac{1}{88}$ $=\left[\frac{2t^3}{3}+\frac{3t^5}{5}\right]_0^{\frac{1}{2}}+\frac{1}{88}$ $=\frac{1}{12}+\frac{3}{160}+\frac{1}{88}$ $=\frac{599}{5280}$ units^2	This part is not well done.

Method 2
(Area under graph to *x*-axis, parametric)

Required Area

$= (\text{yellow} + \text{orange region}) - \text{orange region}$

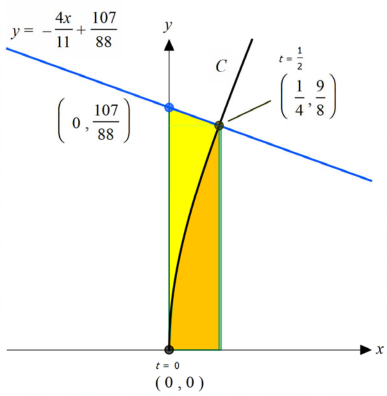
$$= \frac{1}{2}\left(\frac{107}{88}+\frac{9}{8}\right)\left(\frac{1}{4}\right)-\int_0^{\frac{1}{4}} y \, \mathrm{d}x$$

$$= \frac{103}{352}-\int_0^{\frac{1}{2}}\left(2t+t^3\right)\left(2t\right)\mathrm{d}t$$

$$= \frac{103}{352}-\left[\frac{4t^3}{3}+\frac{2t^5}{5}\right]_0^{\frac{1}{2}}$$

$$= \frac{103}{352}-\left(\frac{4}{24}+\frac{2}{160}\right)=\frac{103}{352}-\frac{43}{240}$$

$$= \frac{599}{5280} \text{ units}^2$$



Method 3
(Area under graph to *x*-axis, cartesian)

The cartesian equation of the curve in the first quadrant is $y = 2x^{\frac{1}{2}} + x^{\frac{3}{2}}$.

Required area

$= (\text{yellow} + \text{orange region}) - \text{orange region}$

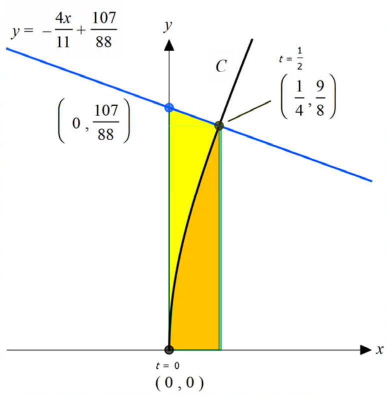
$$= \frac{1}{2}\left(\frac{107}{88}+\frac{9}{8}\right)\left(\frac{1}{4}\right)-\int_0^{\frac{1}{4}} y \, \mathrm{d}x$$

$$= \frac{103}{352}-\int_0^{\frac{1}{4}} 2x^{\frac{1}{2}}+x^{\frac{3}{2}} \, \mathrm{d}x$$

$$= \frac{103}{352}-\left[\frac{4x^{\frac{3}{2}}}{3}+\frac{2x^{\frac{5}{2}}}{5}\right]_0^{\frac{1}{4}}$$

$$= \frac{103}{352}-\left(\frac{1}{6}+\frac{1}{80}\right)=\frac{103}{352}-\frac{43}{240}$$

$$= \frac{599}{5280} \text{ units}^2$$



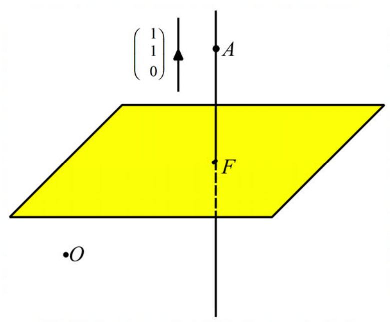
6 The point A has coordinates $(1.5,\ 0,\ 2)$, and the planes π_1 and π_2 have equations $x+y=4$ and $3x+2y-5z=7$ respectively.

(a) π_1 and π_2 intersect in a line l. Find a vector equation of l. [2]

(b) Find the coordinates of the foot of perpendicular from A to π_1. [4]

(c) The point B is the mirror image of A in π_1. Find the coordinates of B and determine if it lies on π_2. [3]

(d) Find the cartesian equation of the plane π_3, which is the reflection of π_2 in π_1. [3]

<table>
<tr>
<td>(a)
[2]</td>
<td>$x+y=4$ --- (1)
$3x+2y-5z=7$ --- (2)
Using GC, $l:\mathbf{r}=\begin{pmatrix}-1\\5\\0\end{pmatrix}+t\begin{pmatrix}5\\-5\\1\end{pmatrix},\ t\in\mathbb{R}$</td>
<td>Many are able to use the GC app (PlySmlt2) to obtain the correct answer.</td>
</tr>
<tr>
<td>(b)
[4]</td>
<td>$\overrightarrow{OA}=\begin{pmatrix}1.5\\0\\2\end{pmatrix},\ \pi_1:\mathbf{r}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}=4$
Let F be the foot of perpendicular from A to π_1.
$l_{AF}:\mathbf{r}=\begin{pmatrix}1.5\\0\\2\end{pmatrix}+s\begin{pmatrix}1\\1\\0\end{pmatrix},\ s\in\mathbb{R}$


l_{AF} intersects π_1 at F.
$\begin{pmatrix}1.5+s\\s\\2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}=4$
$1.5+s+s=4$
$2s=2.5$
$s=\frac{5}{4}$
$\therefore$ Coordinates of F is $(2.75,\ 1.25,\ 2)$ or $\left(\frac{11}{4},\ \frac{5}{4},\ 2\right)$.</td>
<td>A few students tried to imitate the method for finding the foot of perpendicular from a point to a line onto this question, but was unsuccessful.

Note that
$l_{AF}:\mathbf{r}=\begin{pmatrix}1.5\\0\\2\end{pmatrix}+s\begin{pmatrix}1\\1\\0\end{pmatrix}$
represents the equation of the line AF. So,
$\overrightarrow{AF}\neq\begin{pmatrix}1.5\\0\\2\end{pmatrix}+s\begin{pmatrix}1\\1\\0\end{pmatrix}$!!
The vector $\mathbf{r}$ is position vector of a point on the line AF. In other words, since F lies on l_{AF}, we can write
$\overrightarrow{OF}=\begin{pmatrix}1.5\\0\\2\end{pmatrix}+s\begin{pmatrix}1\\1\\0\end{pmatrix}$.

Similar question:
Tut 4D Qn 7(i),
Assignment 4D Qn 1(ii)</td>
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<tr><td>(c)
[3]</td><td>

$\overrightarrow{OF}=\dfrac{\overrightarrow{OA}+\overrightarrow{OB}}{2}$

$\begin{pmatrix}2.75\\1.25\\2\end{pmatrix}=\dfrac{\begin{pmatrix}1.5\\0\\2\end{pmatrix}+\overrightarrow{OB}}{2} \Rightarrow \overrightarrow{OB}=\begin{pmatrix}4\\2.5\\2\end{pmatrix}$

$\therefore$ Coordinates of B is $(4, 2.5, 2)$.

Sub $\begin{pmatrix}4\\2.5\\2\end{pmatrix}$ into π_2, $\begin{pmatrix}4\\2.5\\2\end{pmatrix}\cdot\begin{pmatrix}3\\2\\-5\end{pmatrix}=12+5-10=7$

$\therefore B$ lies on π_2.

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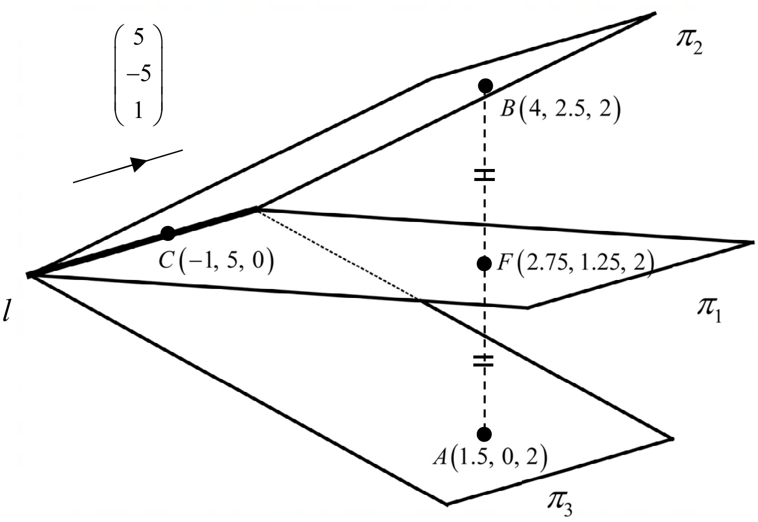
<tr><td>(d)
[3]</td><td>



Let $\overrightarrow{OC}=\begin{pmatrix}-1\\5\\0\end{pmatrix}$, where C lies on l.

Since A and l are on π_3, a vector parallel to π_3 is given by $\overrightarrow{CA}=\begin{pmatrix}1.5\\0\\2\end{pmatrix}-\begin{pmatrix}-1\\5\\0\end{pmatrix}=\begin{pmatrix}2.5\\-5\\2\end{pmatrix}$.

A normal to $\pi_3=\begin{pmatrix}2.5\\-5\\2\end{pmatrix}\times\begin{pmatrix}5\\-5\\1\end{pmatrix}=\begin{pmatrix}5\\7.5\\12.5\end{pmatrix}=2.5\begin{pmatrix}2\\3\\5\end{pmatrix}$.

Now, $\begin{pmatrix}1.5\\0\\2\end{pmatrix}\cdot\begin{pmatrix}2\\3\\5\end{pmatrix}=3+10=13$, so $\pi_3:\mathbf{r}\cdot\begin{pmatrix}2\\3\\5\end{pmatrix}=13$

and hence the cartesian equation is $2x+3y+5z=13$.

</td><td>This part is not well done.

It is always good to draw a diagram to help in visualizing the question.</td></tr>
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7 **(a)** Nathan is saving for a car that he intends to buy in the future, and he would like to save a minimum of \$130 000. He saves regularly in an account which offers no interest. He makes an initial deposit of \$$A$ on 31 January 2022. Each subsequent month, he deposits \$150 more than he deposited in the previous month. His final deposit is made on 30 June 2024. Find, to the nearest dollar, the smallest value of A so that he can save at least \$130 000. [2]

Nathan comes across a savings plan offered by a bank. Interest is added to the account at the end of each month at a fixed rate of 1.25% of the amount in the account at the beginning of that month under this savings plan.

(b) Nathan opened an account under this savings plan. He decides to deposit \$$X$ at the beginning of the first month and then a further \$$X$ at the beginning of the second and each subsequent month. He also decides that he will not draw any money out of the account, but just leave the money in the account for the interest to build up.

(i) Write down the amount in the bank account, including the interest, at the end of 1 month. [1]

(ii) Show that, at the end of n months, when the interest for the last month has been added, he will have a total of \$ $mX\left(1.0125^n-1\right)$ in his bank account, where m is an integer to be determined. [3]

(iii) After how many complete months will he have, for the first time, at least \$$13X$ in his bank account? [2]

(c) Nathan decides that, to assist him in his everyday expenses, he will withdraw the interest as soon as it has been added. With this decision, he deposits \$$Y$ at the beginning of each month. Show that, at the end of N months, he will have received a total of \$ $kYN\left(N+1\right)$ in interest, where k is an exact constant to be determined. [3]

(a) **[2]**	We have an AP with the following: First term = A Common difference = 150 Number of terms (number of months) = 30 So we have $\frac{30}{2}\left[2A+(30-1)150\right]\geq 130000$ $A\geq\frac{1}{2}\left[\frac{130000}{15}-(29)(150)\right]\approx 2158.33$ Hence smallest A = 2159 (nearest dollar).	
(b)(i) **[1]**	$1.0125X$.	
(ii) **[3]**	see table below	

Month	Amt at Beginning	Amt at the End
1	X	$1.0125X$
2	$1.0125X+X$	$(1.0125X+X)(1.0125)$ $=1.0125X+1.0125^2X$
3	$X+1.0125X+1.0125^2X$	$(X+1.0125X+1.0125^2X)(1.0125)$ $=1.0125X+1.0125^2X+1.0125^3X$
…	…	…
n	$X+1.0125X+1.0125^2X$ $+\ldots+1.0125^{n-1}X$	$(X+1.0125X+1.0125^2X$ $+\ldots+1.0125^{n-1}X)(1.0125)$ $=1.0125X+1.0125^2X$ $+1.0125^3X+\ldots+1.0125^nX$

	Hence, total amount in the account $=1.0125X+1.0125^2X+1.0125^3X+\ldots+1.0125^nX$ $=1.0125X\left(1+1.0125+1.0125^2+\ldots+1.0125^{n-1}\right)$ $=\dfrac{1.0125X\left(1.0125^n-1\right)}{1.0125-1}$ $=\dfrac{1.0125}{0.0125}X\left(1.0125^n-1\right)$ $=81X\left(1.0125^n-1\right)$ $=mX\left(1.0125^n-1\right)$, where $m=81$ (shown).	
(iii) **[2]**	To have at least $\$13X$ in the account, we need $81X\left(1.0125^n-1\right)\geq 13X$ $\Rightarrow 1.0125^n\geq\dfrac{13}{81}+1$ $\Rightarrow n\geq\dfrac{\ln\left(\frac{94}{81}\right)}{\ln\left(1.0125\right)}\approx 11.9819$ Alternatively, $81X\left(1.0125^n-1\right)\geq 13X\Rightarrow 81\left(1.0125^n-1\right)-13\geq 0$ Let $\mathrm{f}\left(n\right)=81\left(1.0125^n-1\right)-13$. Using GC, $\mathrm{f}\left(11\right)\approx -1.1396<0$ $\mathrm{f}\left(12\right)\approx 0.021116>0$ Hence after 12 complete months, he will have at least $\$13X$ in his account for the first time.	

(c)
[3]

Month	Amt (Beginning)	Amt (End)	Interest
1	Y	$Y+0.0125Y$	$0.0125Y$
2	$2Y$	$2Y+0.0125(2Y)$	$0.0125Y+0.0125(2Y)$
3	$3Y$	$3Y+0.0125(3Y)$	$0.0125Y+0.0125(2Y)$ $+0.0125(3Y)$
…	…	…	…
N	NY	$NY+0.0125(NY)$	$0.0125Y+0.0125(2Y)$ $+0.0125(3Y)+\ldots$ $+0.0125(NY)$

Hence total interest received

$$=0.0125Y+0.0125(2Y)+0.0125(3Y)+\ldots+0.0125(NY)$$
$$=0.0125Y[1+2+3+\ldots+N]$$
$$=0.0125Y\left[\frac{N(N+1)}{2}\right]$$
$$=0.00625YN(N+1) \quad \text{or} \quad \frac{1}{160}YN(N+1)$$
$$=kYN(N+1), \text{ where } k=\frac{1}{160} \text{ or } 0.00625 \text{ (Shown).}$$

8 From past data, it is known that the mean time taken by students at a school to complete a H2 Mathematics A-level examination paper was 170 minutes. In June 2024, Ms Tan believes that her students are not ready for the examination and will take more than 170 minutes to complete the paper. She gives the same examination paper to 50 students, assuming that these students have not seen the paper before. She notes the time taken by each student and carries out a test.

(a) State null and alternative hypotheses for the teacher's belief, defining any parameters you use. [2]

(b) Given that Ms Tan's students took an average of 172.1 minutes and the sample variance is 53.8 minutes^2, test, at the 5% significance level, whether the teacher's belief is supported. [4]

(a) **[2]**	Let μ (min) denote the **population** mean time taken by the students to complete the H2 Mathematics examination paper. $\text{H}_0 : \mu = 170, \quad \text{H}_1 : \mu > 170$	Most students used a sensible symbol (μ) for the **population** mean time taken. Almost all are able to state the hypotheses correctly, but not all defined the parameter μ. It is necessary to include the word population when defining μ. Similar question: Tut S5 Qn 5(i) & 6(ii)
(b) **[4]**	Let X min be the time taken by a randomly selected student. Perform a 1-tail test at 5% significance level. Unbiased estimate of population variance $= \frac{50}{49}(53.8) \approx 54.898$ Under H_0, since n = 50 is large, $\bar{X} \sim \text{N}\left(170, \frac{54.898}{50}\right)$ or $\bar{X} \sim \text{N}\left(170, \frac{53.8}{49}\right)$ approximately by Central Limit Theorem. Using a z-test, p-value $= \text{P}\left(\bar{X} \geq 172.1\right) = 0.0225$ (to 3s.f.) Since p-value = 0.0225 < 0.05, we reject H_0 and conclude there is sufficient evidence, at the 5% significance level, that the teacher's students take more than 170 minutes to complete the paper.	The variance 53.8 given in the question is the sample variance. Very few students used this sample variance to find the unbiased estimate of the population variance. Similar question: Tut S5 Qn 8(iii) & (iv)

9 The letters from the word PERIMETER are arranged in a row.

(a) Find the number of different arrangements of the nine letters. [1]

(b) Find the number of different arrangements if there are at least 6 letters between the two Rs. [3]

One of the Es is removed and the remaining letters are arranged randomly in a row.

(c) Find the probability that no adjacent letters are the same. [4]

(a) **[1]**	Number of arrangements $=\dfrac{9!}{3!2!}=30240$	This part is well done.
(b) **[3]**	Case 1: R_ _ _ _ _ _ R_ Case 2: R_ _ _ _ _ _ _R Case 3: _R_ _ _ _ _ _ R In each of the case, there are $\dfrac{7!}{3!}$ ways to arrange. So total number of ways $=\dfrac{7!}{3!}\times 3=2520$.	
(c) **[4]**	Total number of arrangements $=\dfrac{8!}{2!2!}=10080$ **Method 1** <u>Case 1</u>: Es together, Rs not together Number of ways to slot the 2 Rs into 6 spaces $= {}^6C_2$ Number of ways $={}^6C_2\times 5!=1800$ <u>Case 2</u>: Rs together, Es not together Number of ways $={}^6C_2\times 5!=1800$ <u>Case 3</u>: Rs together, Es together $=6!=720$ Required probability $=\dfrac{10080-1800-1800-720}{10080}=\dfrac{5760}{10080}=\dfrac{4}{7}$ **Method 2** <u>Case 1</u>: Es together Number of ways $=\dfrac{7!}{2!}=2520$ <u>Case 2</u>: Rs together Number of ways $=\dfrac{7!}{2!}=2520$ <u>Case 3</u>: Es together and Rs together Number of ways $=6!=720$ Required probability $=\dfrac{10080-\left(2520+2520-720\right)}{10080}=\dfrac{5760}{10080}=\dfrac{4}{7}$	This part is not well done.

10 A biased die in the form of a tetrahedron has its four faces labelled 1 to 4, with one number printed on each face. The die is tossed and X is the random variable representing the number on the face on which the die lands. The probability distribution of X is shown in the table below.

x	1	2	3	4
$\mathrm{P}(X=x)$	p	q	q	p

(a) State the numerical value of $\mathrm{E}(X)$. [1]

(b) Given that $\mathrm{Var}(X)=1.65$, find the values of p and q. [4]

A game is played by a player tossing this die once. If the die lands on the face with the number 4 printed on it, the player wins, otherwise the player loses. The random variable Y denotes the number of wins out of 50 games the player plays.

(c) State two assumptions needed for Y to be well modelled by a binomial distribution. [2]

Assume that Y follows a binomial distribution.

(d) Find the probability the player wins more than the expected number of games won. [3]

<table>
<tr><td>(a)
[1]</td><td>$\mathrm{E}(X)=\frac{2+3}{2}=2.5$, by symmetry.

Alternatively,
$\sum \mathrm{P}(X=x)=1 \Rightarrow 2p+2q=1 \Rightarrow p+q=\frac{1}{2}$.
$\mathrm{E}(X)=\sum x\mathrm{P}(X=x)=p+2q+3q+4p=5(p+q)=2.5$.</td><td>This part is well done</td></tr>
<tr><td>(b)
[4]</td><td>$\mathrm{E}(X)=p+2q+3q+4p=2.5$
$\Rightarrow p+q=\frac{2.5}{5}=0.5$ ---- (1)

Method 1
$\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$
$=(p+4q+9q+16p)-(2.5)^2=1.65$
$\Rightarrow 17p+13q=7.9$ ---- (2)

Solving (1) and (2), we have $p=0.35,\ q=0.15$.

Method 2
$\mathrm{Var}(X)=\mathrm{E}\left((X-\mathrm{E}(X))^2\right)$
$=\mathrm{E}\left((X-2.5)^2\right)$
$=(1-2.5)^2 p+(2-2.5)^2 q+(3-2.5)^2 q+(4-2.5)^2 p$
$=4.5p+0.5q=1.65-----(2)$
With $p+q=0.5$, we get $p=0.35,\ q=0.15$</td><td>This part is well done</td></tr>
</table>

(c) **[2]**	The event that the player wins in a game is independent of the event the player wins in another game. The probability of a player winning in a game remains as a constant of $p = 0.35$.	Common mistakes seen: • The **probability** that player wins in a game is **independent** of the **probability** that player wins in another game. *We do not say probability being independent of each other.* • The game has 2 outcomes – either win or lose *This is not an assumption as this is obviously implied in the question.* • There is a fixed number of games *This is not an assumption as this is obviously implied in the question.* Similar question: Chap S2B Example 5(i) & Example 6
(d) **[3]**	We have $Y \sim \mathrm{B}(50,\ 0.35)$ and $\mathrm{E}(Y) = 50(0.35) = 17.5$. $\mathrm{P}(Y > 17.5) = \mathrm{P}(Y \geq 18)$ $= 1 - \mathrm{P}(Y \leq 17)$ $= 0.49402$ (5 sf) $= 0.494$ (3 sf).	This part is well done

11 **In this question you should state clearly the values of the parameters of any normal distributions you use.**

A fruit seller sells two kinds of fruits, namely durians and soursops.

The masses, in kg, of a durian and a soursop are modelled as having normal distributions with means and standard deviations as shown in the following table.

	Mean mass (kg)	Standard deviation (kg)
Durian	2.2	0.4
Soursop	1.3	0.2

(a) Find the mass that is exceeded by 95% of the soursops. [1]

(b) Three durians are randomly chosen. Find the probability that two of these durians each has mass less than 2.3kg and one of the durians has mass more than 2.5kg. [3]

(c) Find the probability that the mass of a randomly chosen durian differs from twice the mass of a randomly chosen soursop by more than 300 grams. [3]

(d) Write down an assumption that you have used in part **(c)**. [1]

(e) Durians are sold at \$20 per kg. Find the probability that 5 randomly chosen durians cost at least \$225. [3]

(a) **[1]**	Let S be the mass of a soursop in kg. Then $S \sim \mathrm{N}(1.3, 0.2^2)$. Want to find *m* such that $\mathrm{P}\left(S > m\right) = 0.95$. From GC, $\mathrm{P}\left(S > 0.97103\right) = 0.95$. Therefore the required mass is 0.971 kg (3 s.f.).	This part is well done. Similar question: Tut S3 Qn 7(i) & 10 (i)
(b) **[3]**	Let D be the mass of a durian in kg. Then $D \sim \mathrm{N}(2.2, 0.4^2)$. Required probability = $3\left[\mathrm{P}(D < 2.3)\right]^2 \mathrm{P}(D > 2.5) = 0.244$ (3 s.f.)	Many students did not "multiply by 3". Leaving the answer as $\left[\mathrm{P}(D < 2.3)\right]^2 \mathrm{P}(D > 2.5)$ means the first 2 durians has mass less than 2.3kg and the 3rd durian has mass more than 2.5kg. As the durian that has mass more than 2.5kg can be any of the 3 durians, so there is a need to multiply by 3 (or $\frac{3!}{2!}$) Similar question: Tut S3 Qn 7(iv)
(c) **[3]**	$D - 2S \sim \mathrm{N}(-0.4, 0.4^2 + 4\left(0.2\right)^2)$ $D - 2S \sim \mathrm{N}(-0.4, 0.32)$ $\mathrm{P}\left(\lvert D - 2S\rvert > 0.3\right)$ $= \mathrm{P}\left(D - 2S > 0.3\right) + \mathrm{P}\left(D - 2S < -0.3\right)$ $= 0.678$(3 s.f.)	Most are able to get the distribution of $D - 2S$ correctly, but not many are able to deal with $\lvert D - 2S\rvert > 0.3$ Similar question: Tut S3 Qn 5 & 9, Chap S3 Example 7(iii), Example 10(ii)
(d) **[1]**	The mass of a durian is independent of the mass of a soursop.	In order to use the result $\mathrm{Var}(X \pm Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$, there is a need for the random variables *X* and *Y* to be

		independent of each other. See Chap S3 Sect 4.2. Similar question: Tut S3 Qn 7(iii) & Assignment S3 Qn 1(iii)
(e) **[3]**	Let $T = 20\left(D_1 + D_2 + D_3 + D_4 + D_5\right)$. Then, $\mathrm{E}(T) = 20(5)(2.2) = 220$ $\mathrm{Var}(T) = 20^2(5)(0.4^2) = 320$ Then, $T \sim \mathrm{N}\left(220,\ 320\right)$. $\mathrm{P}\left(T > 225\right) = 0.390$ (3 s.f.) **Alternatively,** If 5 randomly chosen durians cost at least \$225, they will be at least 11.25kg. Let $Y = D_1 + D_2 + D_3 + D_4 + D_5$. Then $Y \sim \mathrm{N}\left(5\times 2.2,\ 5\times 0.4^2\right)$, i.e. $Y \sim \mathrm{N}\left(11,\ 0.8\right)$. $\mathrm{P}\left(Y > 11.25\right) = 0.390$ (3 s.f.)	It is important to write down the distribution of the random variable, not just merely stating the mean and variance of the random variable (i.e. stating the values of E(T) and Var(T) without writing $T \sim \mathrm{N}\left(220,\ 320\right)$) Similar question: Tut S3 Qn 9, Chap S3 Example 8(iii), 10(iii)