RAFFLES INSTITUTION

2022 Year 6

H3 9820 Test 2

Time allocated: 1 hour 50 minutes **Total Marks: 60**

Instructions: Write your name and CT group on all the work you hand in.
Answer **all** questions.

1 **(a)** Expand and simplify $(a-b)\left(a^n + a^{n-1}b + a^{n-2}b^2 + ... + ab^{n-1} + b^n\right)$. [2]

(b) The prime number 3 has the property that it is one less than a perfect square. Determine all prime numbers with this property, justifying your answer. [2]

(c) Find all prime numbers that are one more than a perfect cube, justifying your answer. [3]

(d) Is $3^{2021} - 2^{2021}$ a prime number? Explain your reasoning carefully. [2]

(e) Is there a positive integer k for which $k^3 + 2k^2 + 2k + 1$ is a perfect cube? Explain your reasoning carefully. [3]

2 **(a)** Suppose that a, b and c are positive real numbers such that the polynomial

$$\mathrm{f}(x) = x^3 - 3ax^2 + 3bx - c$$

has three positive real roots α, β and γ.

(i) Express a, b and c in terms of α, β and γ. [3]

(ii) Show that $\sqrt{b} \geq \sqrt[3]{c}$. [2]

(iii) Use the graph of $y = \mathrm{f}(x)$ to explain why the polynomial $\mathrm{f}'(x)$ has 2 positive roots. [2]

(iv) Hence by considering $\mathrm{f}'(x)$, show that $a \geq \sqrt{b}$. [2]

(b) Let A, B and C be the angles of a triangle.

(i) Show that $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$. [3]

(ii) Hence, using the results established in **(a)** and **(b)(i)**, show that

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \geq \sqrt{3} \quad \text{and} \quad \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \leq \frac{\sqrt{3}}{9}.$$ [3]

3 A sequence $x_1, x_2, ...$ of real numbers is defined by $x_{n+1} = x_n^{\ 2} - 2$ for $n \geq 1$ and $x_1 = a$.

(a) Show that if $a > 2$ then $x_n \geq 2 + 4^{n-1}(a-2)$. [5]

(b) Show also that $x_n \to \infty$ as $n \to \infty$ if and only if $|a| > 2$. [5]

4 **Throughout this question, no marks will be awarded for any use of the exponential or the logarithmic function.**

For positive real numbers *x*, define $\mathrm{F}(x) = \int_1^x \frac{1}{t}\,\mathrm{d}t$.

(a) Show that F is a strictly increasing function. [1]

(b) Show in any order, that for all positive real numbers a, b,

(i) $\mathrm{F}(ab) = \mathrm{F}(a) + \mathrm{F}(b)$,

(ii) $\mathrm{F}\left(\frac{a}{b}\right) = \mathrm{F}(a) - \mathrm{F}(b)$.

[4]

(c) If there exists a real number L such that $\lim_{x\to\infty} \mathrm{F}(x) = L$, state $\lim_{x\to\infty} \mathrm{F}(2x)$. Hence deduce that $\lim_{x\to\infty} \mathrm{F}(x) = +\infty$. [3]

(d) Show that $\mathrm{F}\left(\frac{1}{x}\right) = -\mathrm{F}(x)$ for all positive real numbers *x* and hence find $\lim_{x\to 0^+} \mathrm{F}(x)$, explaining your reasoning clearly. [2]

(e) Show also that $\mathrm{F}(2) < 1 < \mathrm{F}(3)$. [3]

5 Let $S = \{1, 2, 3, ..., 2n-1\}$. Remove at least $n-1$ numbers from S using the following rules:

- If the number $s \in S$ is removed and $2s \in S$, then $2s$ must be removed,
- If the numbers $s, t \in S$ are removed and $s + t \in S$, then $s + t$ must be removed.

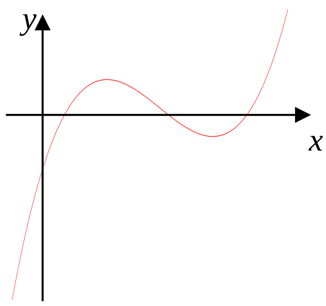
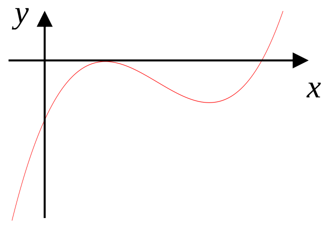
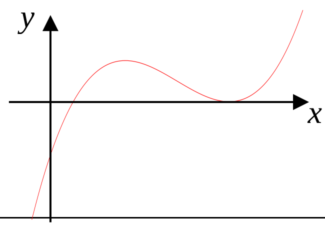
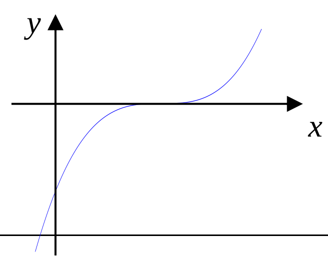
After all the possible numbers are removed from *S*, let *T* denote the sum of the remaining numbers.

(a) Find the smallest possible value of *T*. [3]

(b) Find the largest possible value of *T*. [7]

[END OF TEST]

1 (a) [2]	$\left(a-b\right)\left(a^n+a^{n-1}b+a^{n-2}b^2+\ldots+ab^{n-1}+b^n\right)$ $=a^{n+1}-a^nb+a^nb-a^{n-1}b^2+a^{n-1}b^2-\ldots-ab^n+ab^n-b^{n+1}$ $=a^{n+1}-b^{n+1}$
(b) [2]	Since $(n+1)(n-1)=n^2-1=p$, we must have $n-1=1$ since $n-1<n+1$. This means that n = 2 and the only prime with this property is $2^2-1=3$.
(c) [3]	We want $n^3+1=(n+1)(n^2-n+1)=p$ and therefore $n+1=1$ or $n^2-n+1=1$. In the first case we must have n = 0 which doesn't lead to a prime, and in the second we have either n = 0 or n = 1. For n = 1 we get $1^3+1=2$, a prime as desired.
(d)	We have $3^{2021}-2^{2021}=\left(3^{43}\right)^{47}-\left(2^{43}\right)^{47}$ $=\left(3^{43}-2^{43}\right)\left(\left(3^{43}\right)^{46}+\left(3^{43}\right)^{45}\left(2^{43}\right)+\ldots+\left(3^{43}\right)\left(2^{43}\right)^{45}+\left(2^{43}\right)^{46}\right)$ and since each term is greater than 1, $3^{2021}-2^{2021}$ is not a prime number.
(e)	Note that for positive k, $k^3<k^3+2k^2+2k+1<k^3+3k^2+3k+1$ and since the expression lies between 2 consecutive cubes, it cannot be a perfect cube.

2 (a) (i) [3]	$\mathrm{f}(x)=(x-\alpha)(x-\beta)(x-\gamma)=x^3-3ax^2+3bx-c$ Compare the coefficients of $x^2:\quad \alpha+\beta+\gamma=3a\Rightarrow a=\dfrac{\alpha+\beta+\gamma}{3}$ $x:\quad \alpha\beta+\beta\gamma+\gamma\alpha=3b\Rightarrow b=\dfrac{\alpha\beta+\beta\gamma+\gamma\alpha}{3}$ constant term: $c=\alpha\beta\gamma$
(a) (ii) [2]	Since α, β and γ are positive, $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$ are positive. By AM-GM inequality, $b=\dfrac{\alpha\beta+\beta\gamma+\gamma\alpha}{3}\geq\sqrt[3]{(\alpha\beta\gamma)^2}=\sqrt[3]{c^2}$ Since b and c are positive real numbers, $\sqrt{b}\geq\sqrt[3]{c}$.
(a) (iii) [2]	The possible cases for the graph of polynomial $\mathrm{f}(x)$ are (i)  (ii)  (iii)  (iv) 

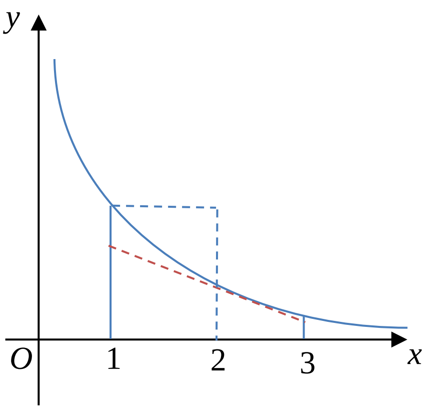
	Since the polynomial $\mathrm{f}(x)$ has 3 roots, the polynomial $\mathrm{f}'(x)$ which is quadratic, has 2 roots. In addition, the roots of the polynomial $\mathrm{f}'(x)$ are the *x*-coordinates of the stationary points of the polynomial $\mathrm{f}(x)$. As the stationary points occur between the roots (α, β and γ) of the polynomial $\mathrm{f}(x)$ (as seen in (i)) or are one of the roots themselves (as seen in (ii), (iii) and (iv)) and since these roots α, β and γ are positive, the roots of the polynomial $\mathrm{f}'(x)$ which are the *x*-coordinates of the stationary points of the polynomial $\mathrm{f}(x)$ are positive. Hence the polynomial $\mathrm{f}'(x)$ has 2 positive roots.
(a) (iv) [2]	$\mathrm{f}'(x)=3x^2-6ax+3b$ Let the roots of polynomial $\mathrm{f}'(x)$ be p and q. Then $p+q=2a$ and $pq=b$. Since p and q are positive, by AM-GM inequality, $a=\dfrac{p+q}{2}\geq\sqrt{pq}=\sqrt{b}$. *Alternative* $\mathrm{f}'(x)=3x^2-6ax+3b=3(x^2-2ax+b)$ Since the roots of polynomial $\mathrm{f}'(x)$ are real, Discriminant $=4a^2-4b\geq 0\Rightarrow a\geq\sqrt{b}$.
(b) (i) **(b) (ii)**	Since *A*, *B* and *C* be the angles of a triangle, $A+B+C=\pi$ $\dfrac{A+B+C}{2}=\dfrac{\pi}{2}$ $\tan\left(\dfrac{A}{2}+\dfrac{B}{2}\right)=\tan\left(\dfrac{\pi}{2}-\dfrac{C}{2}\right)$ $\dfrac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\cot\dfrac{C}{2}$ $\tan\dfrac{A}{2}\tan\dfrac{C}{2}+\tan\dfrac{B}{2}\tan\dfrac{C}{2}=1-\tan\dfrac{A}{2}\tan\dfrac{B}{2}$ $\tan\dfrac{A}{2}\tan\dfrac{B}{2}+\tan\dfrac{B}{2}\tan\dfrac{C}{2}+\tan\dfrac{C}{2}\tan\dfrac{A}{2}=1$ Let $\alpha=\tan\dfrac{A}{2},\beta=\tan\dfrac{B}{2}$ and $\gamma=\tan\dfrac{C}{2}$. As *A*, *B* and *C* be the angles of a triangle, $\dfrac{A}{2},\dfrac{B}{2}$ and $\dfrac{C}{2}$ are acute angles and hence $\alpha=\tan\dfrac{A}{2},\beta=\tan\dfrac{B}{2}$ and $\gamma=\tan\dfrac{C}{2}$ are positive.

	$a \geq \sqrt{b}$ $\frac{\alpha+\beta+\gamma}{3} \geq \sqrt{\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{3}}$ From **(a)(iv)**, $\frac{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}{3} \geq \sqrt{\frac{\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}}{3}} = \frac{1}{\sqrt{3}}$ $\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2} \geq \sqrt{3}$ From **(a)(ii)**, $\sqrt{b} \geq \sqrt[3]{c}$ $\sqrt{\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{3}} \geq \sqrt[3]{\alpha\beta\gamma}$ $\sqrt{\frac{\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}}{3}} \geq \sqrt[3]{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}$ $\frac{1}{\sqrt{3}} \geq \sqrt[3]{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}$ $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \leq \left(\frac{1}{\sqrt{3}}\right)^3 = \frac{\sqrt{3}}{9}$

3 (a)	Let P_n be the statement $x_n \geq 2+4^{n-1}(a-2)$ for $a>2$. Base case P_1 : When $n=1$, $2+4^{n-1}(a-2)=2+a-2=a=x_1$. So P_1 is true. WTS P_k is true $\Rightarrow$ P_{k+1} is true. Suppose $x_k \geq 2+4^{k-1}(a-2)$ for some $k\in\mathbb{Z}^+$ (and $x_k>0$ as $a>2$) Then $x_{k+1}=x_k^{\ 2}-2 \geq \left[2+4^{k-1}(a-2)\right]^2-2$ $=4+2\times2\times4^{k-1}(a-2)+4^{2k-2}(a-2)^2-2$ $=2+4^k(a-2)+4^{2k-2}(a-2)^2$ $>2+4^k(a-2)$ Hence P_{k+1} is true. Thus by induction $x_n \geq 2+4^{n-1}(a-2)$ for positive integer *n*.

(b)	If $\lvert x_k \rvert \le 2$, then $0 \le \lvert x_k \rvert^2 \le 4$, so $-2 \le \lvert x_k \rvert^2 - 2 \le 2$, that is $-2 \le x_{k+1} \le 2$. If $\lvert a \rvert \le 2$, $\lvert x_1 \rvert \le 2$ and thus by induction $-2 \le x_n \le 2$, that is $x_n \not\to \infty$ For the case where $\lvert a \rvert > 2$: Regardless of whether a is positive or negative, $a^2 - 2 = \lvert a \rvert^2 - 2$, hence it suffices to consider $a > 2$ for the behavior of all terms after x_1. Therefore, from part (i), we know $x_n \ge 2 + 4^{n-1}\left(\lvert a \rvert - 2\right)$ for $n \ge 2$, and thus $x_n \to \infty$ as $n \to \infty$; Hence we have shown $x_n \to \infty$ as $n \to \infty$ if and only if $\lvert a \rvert > 2$.

4 (a) [1]	By the Fundamental Theorem of Calculus, $\mathrm{F}'(x) = \frac{1}{x} > 0$ Alternatively, if $a < b$, since $\frac{1}{t} > 0$ for positive t, then $\mathrm{F}(b) - \mathrm{F}(a) = \int_a^b \frac{1}{t}\,\mathrm{d}t > 0 \Rightarrow \mathrm{F}(b) > \mathrm{F}(a)$ and thus F is strictly increasing.
(b) [4]	We want to show $\int_1^{ab} \frac{1}{t}\,\mathrm{d}t = \int_1^{a} \frac{1}{t}\,\mathrm{d}t + \int_1^{b} \frac{1}{t}\,\mathrm{d}t.$ Note that $\int_1^{ab} \frac{1}{t}\,\mathrm{d}t = \int_1^{a} \frac{1}{t}\,\mathrm{d}t + \int_a^{ab} \frac{1}{t}\,\mathrm{d}t$ Using the substitution $t = au$ $\int_a^{ab} \frac{1}{t}\,\mathrm{d}t = \int_1^{b} \frac{1}{au}\,a\,\mathrm{d}u = \int_1^{b} \frac{1}{u}\,\mathrm{d}u$ and hence $\int_1^{ab} \frac{1}{t}\,\mathrm{d}t = \int_1^{a} \frac{1}{t}\,\mathrm{d}t + \int_1^{b} \frac{1}{t}\,\mathrm{d}t$. Likewise, $\int_1^{\frac{a}{b}} \frac{1}{t}\,\mathrm{d}t = \int_1^{a} \frac{1}{t}\,\mathrm{d}t + \int_a^{\frac{a}{b}} \frac{1}{t}\,\mathrm{d}t$ Using the substitution $t = \frac{a}{w}$ $\int_a^{\frac{a}{b}} \frac{1}{t}\,\mathrm{d}t = \int_1^{b} \frac{w}{a}\left(-\frac{a}{w^2}\right)\mathrm{d}w = -\int_1^{b} \frac{1}{w}\,\mathrm{d}w$ And hence $\int_1^{\frac{a}{b}} \frac{1}{t}\,\mathrm{d}t = \int_1^{a} \frac{1}{t}\,\mathrm{d}t - \int_1^{b} \frac{1}{t}\,\mathrm{d}t$

	Alternatively, to show $\mathrm{F}\left(\frac{a}{b}\right)=\mathrm{F}(a)-\mathrm{F}(b)$, it suffices to show $\mathrm{F}\left(\frac{1}{x}\right)=-\mathrm{F}(x)$. This is equivalent to $\int_1^{\frac{1}{x}}\frac{1}{t}\,\mathrm{d}t=-\int_1^{x}\frac{1}{t}\,\mathrm{d}t$. Here we use the substitution $t=\frac{1}{v}$. Then $\int_1^{\frac{1}{x}}\frac{1}{t}\,\mathrm{d}t=\int_1^{v}v\left(-\frac{1}{v^2}\right)\mathrm{d}v=-\int_1^{v}\frac{1}{v}\,\mathrm{d}v$ and we are done. Therefore $\mathrm{F}\left(\frac{a}{b}\right)=\mathrm{F}(a)+\mathrm{F}\left(\frac{1}{b}\right)=\mathrm{F}(a)-\mathrm{F}(b)$.
(c)	$\lim_{x\to\infty}\mathrm{F}(2x)=L$ Then this means $\lim_{x\to\infty}(\mathrm{F}(2)+\mathrm{F}(x))=L\Rightarrow F(2)+L=L\Rightarrow F(2)=0$ which is a contradiction since $\int_1^2\frac{1}{t}\,\mathrm{d}t>0$. Hence $\lim_{x\to\infty}\mathrm{F}(x)=+\infty$ as F is strictly increasing.
(d)	Since $\mathrm{F}(1)=\int_1^1\frac{1}{t}\,\mathrm{d}t=0$, $\mathrm{F}\left(\frac{1}{x}\right)=\mathrm{F}(1)-\mathrm{F}(x)=-\mathrm{F}(x)$. Therefore we have $\lim_{x\to0^+}\mathrm{F}(x)=-\lim_{x\to0^+}\mathrm{F}\left(\frac{1}{x}\right)=-\lim_{x\to\infty}\mathrm{F}(x)=-\infty$.
(e)	Consider the area under the graph of $y=\frac{1}{x}$ between x = 1 and x = 2. Since the curve is convex, this area F(2), is less than the area of the rectangle shown, which has area 1. F(3) would then be less than the area of the trapezium shown, which is 1, since the base is 2 and the midline has height $\frac{1}{2}$. 

5 (a) [3]	Remove 1 from S. Then 2 is removed. (by rule (I)) $1+2=3,\ 1+3=4,\ ...,\ 1+(2n-2)=2n-1$ are removed. (by rule (II)) Since no remaining number in S, the smallest sum $=0$.
(b) [7]	Let R be the sum of the removed numbers. The problem is equivalent to finding the minimum value of R. Let $r_1<r_2<r_3<...<r_p \in [1,2n-1]$ be the removed numbers. By assumption, $p \geq n-1$ To avoid situation in part (a), $r_1>1$, then $r_1+r_p \geq 2n$, otherwise $r_1+r_p \leq 2n-1$ should be removed by rule (II), then $r_p<r_1+r_p$. This contradicts the assumption that r_p is the greatest removed number as above. Next $r_1+r_{p-1} \geq r_p$, otherwise r_1+r_{p-1} should be removed by rule (II) which implies that $r_{p-1}<r_1+r_{p-1}<r_p$ and it is impossible. Thus $r_1+r_{p-1} \geq r_p$. Since $r_2+r_{p-1}>r_1+r_{p-1} \geq r_p$, this implies that $r_2+r_{p-1} \geq 2n$, otherwise r_2+r_{p-1} should be removed, but it is greater than r_p which contradicts the assumption that r_p is the greatest removed number as above. Using same argument as above, we deduce that $r_i+r_{p+1-i} \geq 2n$, for all $1 \leq i \leq \frac{p+1}{2}$. It follows that $2R=(r_1+r_p)+(r_2+r_{p-1})+...+(r_p+r_1) \geq p.2n \geq (n-1).2n$ or $R=\sum_{i=1}^{p} r_i \geq n(n-1)$. The equality occurs if and only $r_i+r_{p+1-i}=2n$ for all $1 \leq i \leq \frac{p+1}{2}$. Using the above condition of $r_2+r_{p-1}=2n$, $2n=r_2+r_{p-1}>r_1+r_{p-1} \geq r_p$. This implies that $r_1+r_{p-1}=r_p$ since r_p is the greatest removed number. $r_p=2r_1+r_{p-2}=...=pr_1$ which implies $2n=r_1+r_p=(1+p)r_1 \geq nr_1$ since $p \geq n-1$ (assumption as above). Hence $r_1 \leq 2$. Combining $r_1>1$ and $r_1 \leq 2$ obtains $r_1=2$ and $p=n-1$. By rule (II), $r_i=2i$ for $1 \leq i \leq n-1$ In conclusion, the removed numbers must be 2, 4, …, $2n-2$ and the maximal sum of the remaining numbers is $1+3+...+2n-1=n^2$.