XINMIN SECONDARY SCHOOL
新民中学
SEKOLAH MENENGAH XINMIN
Preliminary Examination 2024

CANDIDATE NAME: Mark Scheme

CLASS: P 2

INDEX NUMBER:

ADDITIONAL MATHEMATICS 4049/02

Secondary 4 Express

26 August 2024

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Vetter: Ms Low Yan Jin
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2 hour 15 minutes

Candidates answer on the Question Paper

READ THESE INSTRUCTIONS FIRST

Write your name, index number and class in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
The number of marks is given in brackets [ ] at the end of each question or part question.

If working is needed for any question it must be shown in the space below the question.
Omission of essential working will result in loss of marks.
The total of the marks for this paper is 90.

The use of an approved scientific calculator is expected, where appropriate.
If the degree of accuracy is not specified in the question and if the answer is not exact, give the answer to three significant figures. Give answers in degrees to one decimal place.
For π, use either your calculator value or 3.142.

Errors	Qn No.	Errors	Qn No.
Accuracy		Simplification	
Brackets		Units	
Geometry		Marks Awarded	
Presentation		Marks Penalised	

For Examiner's Use
90

Parent's/Guardian's Signature:

This document consists of **19** printed pages and **1** blank page.

[Turn over

1. ALGEBRA

Quadratic Equation

For the quadratic equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \text{ ,}$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for $\Delta\, ABC$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \tfrac{1}{2} bc \sin A$$

Answer **all** the questions.

1 A company purchased a colour copier machine at a cost of \$8500. The value of this machine decreases with time such that its value, \$$V$, after t months of usage is given by $V = 8500e^{-kt}$, where k is a constant.

(a) The value of the copier machine is expected to fall to \$6400 after 8 months of usage. Estimate the value, to the nearest dollar, of the machine after 2 years of usage. [4]

At $t=8$,

$6400 = 8500e^{-8k}$ [M1] sub in values

$e^{-8k} = \frac{64}{85}$

$-8k = \ln\frac{64}{85}$

$k = -\frac{1}{8}\ln\frac{64}{85}$

$= 0.035471$

either [M1] value of k

At $t=24$,

$V = 8500e^{-0.035471 \times 24}$ [M1] sub in values, allow ecf

$= \$3628.29$

$= \$3628$ [A1] nearest dollar

(b) Copier machines are to be replaced when its value reaches $\frac{1}{7}$ of its initial value.

The company's manager, Mrs Lee, claims that the machine will last for at least 5 years before a replacement is due. Showing all necessary working, explain whether you agree with Mrs Lee. [2]

$\frac{1}{7}(\cancel{8500}) = \cancel{8500}e^{-0.035471t}$

$\ln\frac{1}{7} = -0.035471t$

$t = \frac{\ln\frac{1}{7}}{-0.035471}$

$= 54.859$ months

either (allow ecf) [M1] value of t in months

≈ 4.57 years < 5 years

Disagree, it will require a replacement in less than 5 years.

*must show comparison w respect to context

[A1] elaboration/reference to 5years explained.

[Alternative]

$\text{Value}_0 = \frac{1}{7} \times 8500$

$= \$1214.29$

$\text{Value}_n = 8500e^{(-0.035471)(60)}$

$= \$1011.88$

[M1] allow ecf

Disagree, since \$1011.88 < \$1214.29, it will require replacement in less than 5 years.

2 **(a)** Differentiate $2x\sin\frac{x}{2}$. [2]

Penalise under [presentation] if '$\frac{dy}{dx}$' is written without defining Let $y = 2x\sin\frac{x}{2}$.

$$\frac{d}{dx}\left(2x\sin\frac{x}{2}\right) = 2x\cdot\frac{1}{2}\cos\frac{x}{2} + 2\sin\frac{x}{2} \quad \text{[M1] product rule}$$

$$= x\cos\frac{x}{2} + 2\sin\frac{x}{2} \quad \text{[A1]}$$

(b) Use the result in part **(a)** to evaluate $\int_0^{\pi} 3x\cos\frac{x}{2}\,dx$, leaving your answer as an exact value in the form $a\pi - b$, where a and b are constants. [4]

From (a),

$$\int_0^{\pi} x\cos\frac{x}{2} + 2\sin\frac{x}{2}\,dx = \left[2x\sin\frac{x}{2}\right]_0^{\pi}$$

$$\int_0^{\pi} x\cos\frac{x}{2}\,dx = \left[2x\sin\frac{x}{2}\right]_0^{\pi} - \int_0^{\pi} 2\sin\frac{x}{2}\,dx \quad \text{— [M1] —}$$

$$\therefore \int_0^{\pi} 3x\cos\frac{x}{2}\,dx = 6\left[x\sin\frac{x}{2}\right]_0^{\pi} - 6\left[-2\cos\frac{x}{2}\right]_0^{\pi} \quad \text{— [M1] —}$$

for $\int_0^{\pi}\sin\frac{x}{2}\,dx$ or $\int_0^{\pi} 2\sin\frac{x}{2}\,dx$

$$= 6\left[x\sin\frac{x}{2} + 2\cos\frac{x}{2}\right]_0^{\pi}$$

$$= 6\left(\pi\sin\frac{\pi}{2} + 2\cos\frac{\pi}{2} - 2\cos 0\right) \quad \text{— [M1]}$$

definite integral value

$$= 6(\pi - 2)$$

$$= 6\pi - 12 \quad \text{[A1]}$$

[Alternative]

$$\int_0^{\pi} x\cos\frac{x}{2}\,dx = \left[2x\sin\frac{x}{2}\right]_0^{\pi} - \int_0^{\pi} 2\sin\frac{x}{2}\,dx$$

$$= \left(2\pi\sin\frac{\pi}{2} - 0\right) - \left[-4\cos\frac{x}{2}\right]_0^{\pi}$$

$$= 2\pi\sin\frac{\pi}{2} + \left(4\cos\frac{\pi}{2} - 4\cos 0\right)$$

$$= 2\pi - 4$$

$$\therefore \int_0^{\pi} 3x\cos\frac{x}{2}\,dx = 3(2\pi - 4)$$

$$= 6\pi - 12$$

3 It is given that $f(x) = 2x^3 + px^2 + qx + 3$, where p and q are constants, has a factor of $2x-1$ and leaves a remainder of -75 when divided by $x+2$.

(a) Show that $p = -15$ and $q = 1$. [4]

-1m from qn if students do not show "= 0" at all.

$f(\tfrac{1}{2}) = 0$

$2(\tfrac{1}{2})^3 + p(\tfrac{1}{2})^2 + q(\tfrac{1}{2}) + 3 = 0$ [M1] factor theorem

$\tfrac{1}{4}p + \tfrac{1}{2}q + \tfrac{13}{4} = 0$

$p + 2q + 13 = 0$

$p = -2q - 13$ —— ①

$f(-2) = -75$

$2(-2)^3 + p(-2)^2 + q(-2) + 3 = -75$ [M1] Remainder theorem

$4p - 2q - 13 = -75$

$2p = q - 31$ —— ②

Sub ① into ②:

$2(-2q-13) = q - 31$

$-4q - 26 = q - 31$

$5q = 5$

$q = 1$

[A1] for 1st unknown shown through substitution or elimination method.

$\therefore p = -2(1) - 13$

$= -15$ [A1]

(b) Solve the equation $f(x) = 0$. [4]

-1m from qn if no "= 0" at all.

Note: If both [M1] not awarded, no [A2]

$f(x) = 2x^3 - 15x^2 + x + 3$

Method #1: ⟷ either [M1] ⟷ Method #2

Method #1:

$$\begin{array}{r} x^2 - 7x - 3 \\ 2x-1 \enclose{longdiv}{2x^3 - 15x^2 + x + 3} \\ -(2x^3 - x^2) \\ \hline -14x^2 + x \\ -(-14x^2 + 7x) \\ \hline -6x + 3 \\ -(-6x+3) \\ \hline 0 \end{array}$$

Method #2:

Let $f(x) = (2x-1)(Ax^2 + Bx + C)$

comparing coeff. of x^3 and constant:

$\Rightarrow A = 1,\ C = -3$

comparing coeff. of x:

$1 = 2(-3) - B$

$B = -6 - 1$

$= -7$

$\Rightarrow f(x) = (2x-1)(x^2 - 7x - 3)$

$\Rightarrow (2x-1)(x^2-7x-3) = 0$ [M1]

$2x - 1 = 0$ or $x = \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(-3)}}{2(1)}$

$x = \tfrac{1}{2}$ [A1]

$= 7.4051$ or -0.40512

$= 7.41$ or -0.405

OR $x = \dfrac{7 \pm \sqrt{61}}{2}$

[A1] only awarded when quadratic formula shown.

(c) Hence, solve the equation $2k\sqrt{k} + pk + q\sqrt{k} + 3 = 0$. [2]

[Alternative]

Let $k = x^2$

⋮

$k = \tfrac{1}{4},\ 54.8,\ 0.164$

Let $x = \sqrt{k}$,

$\sqrt{k} = \tfrac{1}{2}$ OR $\sqrt{k} = 7.4051$ OR $\sqrt{k} = -0.40512$ (rej.)

$k = \tfrac{1}{4}$ [B1]

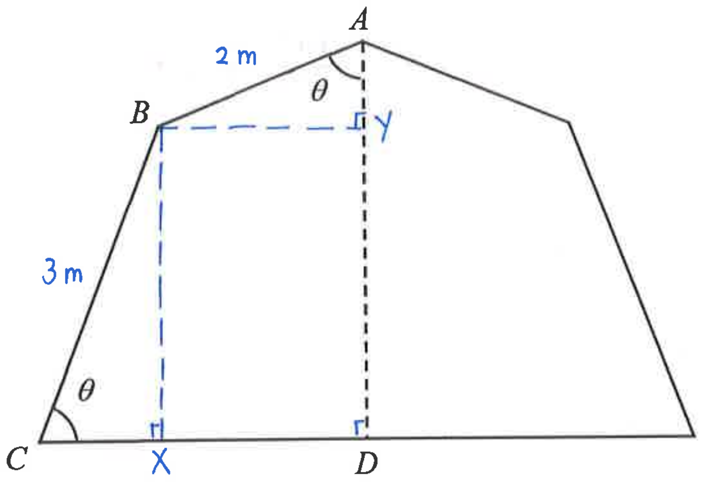
$k = 7.4051^2$

$= 54.8$

[A1] both must show rejected value. (since $\sqrt{k} > 0$)

angle BAD = angle BCD = θ.

4 The diagram shows a vertical cross section of a tent in which $AB = 2$ m, $BC = 3$ m and ∧ It is symmetrical about its vertical height AD and it is set up on horizontal ground.



(a) Show that $AD = 3\sin\theta + 2\cos\theta$. [2]

must be shown.

$*\cos\theta = \dfrac{AY}{2}$

$AY = 2\cos\theta$

either [M1]

$*\sin\theta = \dfrac{BX}{3}$

$BX = 3\sin\theta$

$AD = AY + YD$

$= AY + BX$

$= 2\cos\theta + 3\sin\theta$ [A1] shown

(b) Express AD in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. [3]

Let $R\cos(\theta - \alpha) = 2\cos\theta + 3\sin\theta$

$R = \sqrt{2^2 + 3^2}$ [M1]

$= \sqrt{13}$

$\tan\alpha = \dfrac{3}{2}$

$\alpha = \tan^{-1}\left(\dfrac{3}{2}\right)$ [M1]

$= 56.309°$

$= 56.3°$

$\Rightarrow AD = \sqrt{13}\cos(\theta - 56.3°)$ [A1]

OR $3.61\cos(\theta - 56.3°)$

(c) Given that the vertical height of the tent is 3.45 m, calculate the value of θ. [2]

$$\sqrt{13}\cos(\theta - 56.309°) = 3.45$$

$$\cos(\theta - 56.309°) = \frac{3.45}{\sqrt{13}} \quad \text{or } 0.95685$$

[M1] allow ecf (max 1m) for error from (b)

$$\text{ref.} \measuredangle = \cos^{-1}\left(\frac{3.45}{\sqrt{13}}\right)$$

$$= 16.891°$$

since θ is acute,

$$\theta - 56.309° = 16.891°$$

$$\theta = 16.891° + 56.309°$$

$$= 73.2°$$ [A1]

(d) Find the value of θ for which AD is a maximum. [2]

Maximum value of AD occurs at:

$$\cos(\theta - 56.309°) = 1$$ [M1] allow ecf from (b)

$$\theta - 56.309° = 0°$$

$$\therefore \theta = 56.3°$$ [A1]

[Alternative]

$$\frac{d}{dx}(3\sin\theta + 2\cos\theta)$$

$$= 3\cos\theta - 2\sin\theta$$

$$3\cos\theta - 2\sin\theta = 0$$

$$2\tan\theta = 3$$

[M1] allow ecf
need to show $\frac{d}{dx}\ldots = 0$ for maximum.

⋮

$\theta = 56.3°$ (acute) [A1]

5 **(a)** Given that $\log_p A^2 = 10$ and $\log_p B = 2$, find the value of $\log_A pB$. [3]

$2\log_p A = 10$

$\log_p A = 5$

$\log_p B = 2$

$p^2 = B$

either [M1]

$\log_A pB = \dfrac{\log_p pB}{\log_p A}$ [M1] change base

$= \dfrac{\log_p (p \times p^2)}{5}$

$= \dfrac{\log_p (p^3)}{5}$

$= \dfrac{3}{5}$ [A1]

[Alternative ①]

$2\log_p A = 10$

$\log_p A = 5$ [M1]

$\log_A pB = \dfrac{\log_p pB}{\log_p A}$ [M1]

$= \dfrac{\log_p p + \log_p B}{5}$

$= \dfrac{1+2}{5}$

$= \dfrac{3}{5}$ [A1]

[Alternative ②]

$2\log_p A = 10$

$\log_p A = 5$

$\log_A p = \dfrac{\log_p p}{\log_p A} = \dfrac{1}{5}$

either [M1]

$\log_A B = \dfrac{\log_p B}{\log_p A} = \dfrac{2}{5}$ [M1]

$\therefore \log_A pB = \log_A p + \log_A B$

$= \dfrac{1}{5} + \dfrac{2}{5}$

$= \dfrac{3}{5}$ [A1]

[Alternative ③]
change to base A

$\log_p A = 5$

$\log_A p = \dfrac{1}{5}$

either [M1]

$\dfrac{\log_A B}{\log_A p} = 2$ [M1]

$\log_A B = 2 \times \dfrac{1}{5} = \dfrac{2}{5}$

$\log_A p + \log_A B = \dfrac{1}{5} + \dfrac{2}{5} = \dfrac{3}{5}$ [A1]

(b) Solve $3^x = 6 - 5\left(3^{-x}\right)$. [4]

$3^x = 6 - \dfrac{5}{3^x}$

$3^{2x} = 6(3^x) - 5$ [M1] multiply by 3^x throughout eqn

$3^{2x} - 6(3^x) + 5 = 0$

Let $u = 3^x$,

$u^2 - 6u + 5 = 0$

$(u-1)(u-5) = 0$

[M1] both (substitution & factorise)

$u = 1$ or $u = 5$

$\Rightarrow 3^x = 3^0$ $\qquad \Rightarrow 3^x = 5$

$\therefore x = 0$ [A1]

$\lg 3^x = \lg 5$

$x = \dfrac{\lg 5}{\lg 3}$

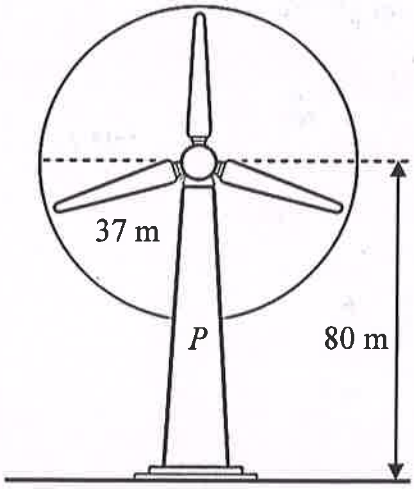
$= 1.46$ [A1]

(accept $x = \log_3 5$)

(-1) deduct [A1] if any x values rejected.

(P)[1] lg written as log / $\log_{10}$

6 The diagram shows a wind turbine with propeller-like blades that have a length of 37 m each. Wind turns the blades that spin around a rotor in the centre to generate electricity. The height, h m, of the tip of each blade above the ground, t seconds after leaving a particular point P, can be modelled by $h = a - 37\cos bt$, where a and b are constants.



The centre of the wind turbine's rotor is 80 m from the ground and, on average, the blades rotate in an anti-clockwise direction at a rate of 1 revolution every 8π seconds.

(a) Explain why $a = 80$ and show that $b = \frac{1}{4}$. [2]

$\boxed{\text{For } a = 80}$

$a \neq$ amplitude

at point P, $t = 0$

$h = 80 - 37$

$= 43$ m

$\Rightarrow 43 = a - 37\cos(0)$

$\therefore a = 80$ [A1]

[Alternatives]

Accept all other equivalent answers: if $b = \frac{1}{4}$ found first.

- when $t = 2\pi$, $h = 80$
- when $t = 4\pi$, $h = 117$
- when $t = 6\pi$, $h = 80$

$\boxed{\text{OR}}$ max. $h = 117$
min. $h = 43$
$a = \frac{117 + 43}{2} = 80$ [A1]

$\boxed{\text{For } b = \frac{1}{4}}$

Period $= 8\pi$

$\frac{2\pi}{b} = 8\pi$

$b = \frac{2\pi}{8\pi}$

$= \frac{1}{4}$ [A1]

$\boxed{\text{OR}}$ $\cos(2\pi b) = 0$

$b = \frac{\frac{\pi}{2}}{2\pi} = \frac{1}{4}$

(b) Find the time taken, in seconds, for the blade to first reach a height of 89 m above ground after leaving P. [3]

$h = 80 - 37\cos\frac{t}{4}$

$80 - 37\cos\frac{t}{4} = 89$

$\cos\frac{t}{4} = \frac{89 - 80}{-37}$

$\cos\frac{t}{4} = -\frac{9}{37}$ [M1]

$\underset{(\text{ref.}\angle)}{\alpha} = \cos^{-1}\left(\frac{9}{37}\right)$

$= 1.3250$ rad.

$\therefore \frac{t}{4} = \pi - 1.3250,\ \pi + 1.3250$

$= 1.8165,\ 4.4665$ [M1]

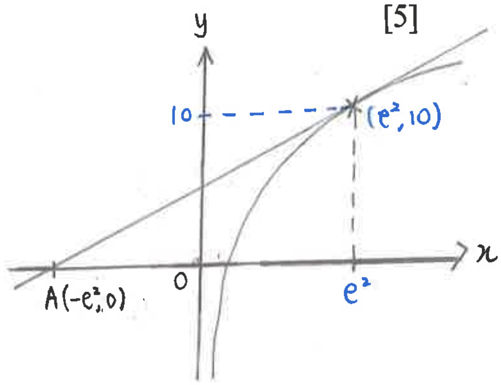
$t = 7.27,\ 17.9$ (rej.)

[A1]

* since first t value,
benefit of doubt:
accept $\frac{t}{4} = \cos^{-1}\left(-\frac{9}{37}\right)$
To take note of α (ref. $\angle$) misconception.

7 The equation of a curve is $y = 5\ln x$. The tangent to the curve at $x = e^2$ intersects the x-axis at A.

(a) Show that the coordinates of A are $(-e^2, 0)$. [5]



at $x = e^2$, $y = 5\ln e^2$

$= 5(2\ln e)$

$= 10$ [M1]

$\frac{dy}{dx} = 5\left(\frac{1}{x}\right)$

$= \frac{5}{x}$ [M1]

$m_{(\text{at } x = e^2)} = \frac{5}{e^2}$

$\therefore$ Eq$^{\underline{n}}$ of tangent : $y - 10 = \frac{5}{e^2}(x - e^2)$

$y = \frac{5}{e^2}x + 5$ [M1] allow ecf

$\Rightarrow$ Intersects x-axis, $y = 0$.

$\therefore \frac{5}{e^2}x + 5 = 0$ [M1] allow ecf.

$\frac{5}{e^2}x = -5$

$\frac{x}{e^2} = -1$

$\therefore x = -e^2$

Solve for x [A1]

$\Rightarrow$ coordinates of $A(-e^2, 0)$. (shown)

$\boxed{\text{OR}}$ Let $A(x, 0)$,

$\frac{10 - 0}{e^2 - x} = \frac{5}{e^2}$ [M1]

$10e^2 = 5(e^2 - x)$ [M1]

$-5x = 5e^2$

$x = \frac{5e^2}{-5}$

$= -e^2$

$\therefore A(-e^2, 0)$

[A1]

(b) Find the area bounded by the tangent, the line $x = e^2$ and the x-axis. [2]

Area of bounded region

$= \frac{1}{2} \times 2e^2 \times 10$ [M1]

$= 10e^2$ units^2 [A1]

OR 73.9 units^2

[Alternative]

Area $= \int_{-e^2}^{e^2} \frac{5}{e^2}x + 5 \, dx$

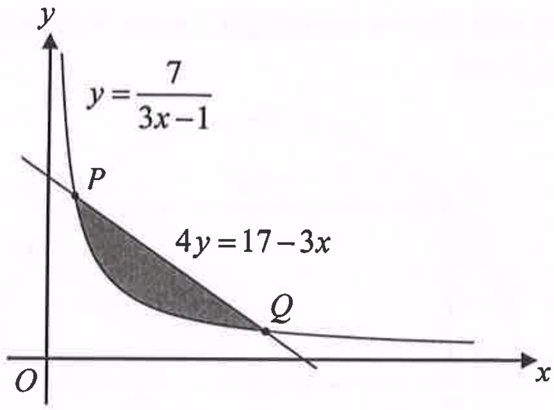
$= \left[\frac{5}{2e^2}x^2 + 5x\right]_{-e^2}^{e^2}$ [M1]

$= \left(\frac{5}{2}e^2 + 5e^2\right) - \left(\frac{5}{2}e^2 - 5e^2\right)$

$= 5e^2 + 5e^2$

$= 10e^2$ units^2 [A1]

8



The diagram shows part of the curve $y=\dfrac{7}{3x-1}$ and the line $4y=17-3x$, where the curve intersects the line at points P and Q.

Find, showing all necessary working, the area of the shaded region that can be expressed in the form $a-b\ln 7$, where a and b are constants. [6]

$y=\dfrac{7}{3x-1}$ —①

$4y=17-3x$

$y=\dfrac{17-3x}{4}$ —②

sub ② into ①:

$\dfrac{17-3x}{4}=\dfrac{7}{3x-1}$ [M1]

$(17-3x)(3x-1)=28$

$51x-17-9x^2+3x=28$

$9x^2-54x+45=0$

$x^2-6x+5=0$

$(x-1)(x-5)=0$

overall −1m for Q8 for lack of equivalence or not factorised (eg. skip steps for ÷9)

$x=1$ or 5 [M1]

Area of shaded region

$=\displaystyle\int_1^5 \frac{17-3x}{4}-\frac{7}{3x-1}\,dx$

$=\underbrace{\left[\frac{17}{4}x-\frac{3}{8}x^2\right]_1^5}_{\text{[M1] area under line}}-\underbrace{\frac{7}{3}\left[\ln(3x-1)\right]_1^5}_{\text{[M1] area under curve}}$

$=\left[\frac{17}{4}(5)-\frac{3}{8}(5)^2\right]-\left(\frac{17}{4}-\frac{3}{8}\right)$ OR $-\frac{7}{3}\left[\ln 14-\ln 2\right]$

[M1] either definite integral value

$=\dfrac{95}{8}-\dfrac{31}{8}-\dfrac{7}{3}\ln\left(\dfrac{14}{2}\right)$

$=8-\dfrac{7}{3}\ln 7$ [A1]

[Alternative]

Area $=\underbrace{\frac{1}{2}(3.5+0.5)(5-1)}_{\text{area of trapezium [M1]}}-\displaystyle\int_1^5\frac{7}{3x-1}\,dx$

$=\frac{1}{2}(4)(4)-\underbrace{\frac{7}{3}\left[\ln(3x-1)\right]_1^5}_{\text{[M1] area under curve}}$

$=8-\underline{\frac{7}{3}(\ln 14-\ln 2)}$ [M1]

$=8-\frac{7}{3}\ln\left(\frac{14}{2}\right)$

$=8-\frac{7}{3}\ln 7$

[Alternative] against the y-axis

Area $=\displaystyle\int_{0.5}^{3.5}\frac{17-4y}{3}-\frac{7+y}{3y}\,dy$

$=\underbrace{\left[\frac{17}{3}-\frac{4}{3}y\right]_{0.5}^{3.5}}_{\text{[M1]}}-\underbrace{\left[\frac{7}{3}\ln y+\frac{1}{3}y\right]_{0.5}^{3.5}}_{\text{[M1]}}$

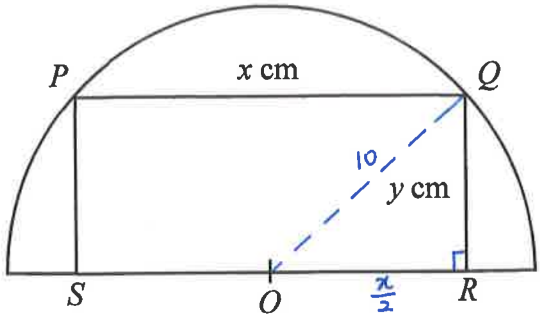
$=\left(\frac{119}{6}-\frac{49}{6}\right)-\left(\frac{17}{6}-\frac{1}{6}\right)-\left[\frac{7}{3}\ln 3.5+\frac{7}{6}-\frac{7}{3}\ln 0.5-\frac{1}{6}\right]$

[M1] either

$=9-\left(1+\frac{7}{3}\ln 7\right)$

$=8-\frac{7}{3}\ln 7$ [A1]

9 $PQRS$ is a rectangle with $PQ = x$ cm and $QR = y$ cm. It is inscribed in a semicircle with centre O and radius 10 cm.



(a) Show that the area of the rectangle, A cm^2, is given by $A = \frac{x}{2}\sqrt{400 - x^2}$. [2]

$y^2 + \left(\frac{x}{2}\right)^2 = 10^2$

$y = \sqrt{100 - \frac{x^2}{4}}$ [M1]

(since $y > 0$)

$\therefore$ Area $= xy$

$= x\sqrt{100 - \frac{x^2}{4}}$

$= x\sqrt{\frac{400 - x^2}{4}}$ manipulation must be shown

$= \frac{x}{2}\sqrt{400 - x^2}$ [A1] shown

OR

Area $= xy$

$= x\sqrt{100 - \frac{x^2}{4}}$

Must be shown. $= \frac{x}{2}(2)\sqrt{100 - \frac{x^2}{4}}$

$= \frac{x}{2}\sqrt{4}\sqrt{100 - \frac{x^2}{4}}$ [A1]

$= \frac{x}{2}\sqrt{400 - x^2}$

OR

$y^2 = 100 - \frac{x^2}{4}$

$4y^2 = 400 - x^2$

$2y = \sqrt{400 - x^2}$ [M1]

$y = \frac{1}{2}\sqrt{400 - x^2}$ sub

Area $= xy$

$= x\left(\frac{1}{2}\sqrt{400 - x^2}\right)$ [A1]

$= \frac{x}{2}\sqrt{400 - x^2}$

(b) Given that x can vary, find the value of x for which the area of the rectangle is stationary. Leave your answer in the form $a\sqrt{b}$, where a and b are constants. [4]

$\frac{dA}{dx} = \frac{1}{2}x \cdot \frac{1}{2}(400 - x^2)^{-\frac{1}{2}}(-2x) + \frac{1}{2}(400 - x^2)^{\frac{1}{2}}$ [M1] OR Quotient rule

$= -\frac{x^2}{2\sqrt{400 - x^2}} + \frac{\sqrt{400 - x^2}}{2}$

$= \frac{-x^2 + 400 - x^2}{2\sqrt{400 - x^2}}$ any acceptable [M1]

$= \frac{200 - x^2}{\sqrt{400 - x^2}}$

Ⓟ$^{-1}$ for any errors in expressions for $\frac{dA}{dx}$

Not accepted: $\frac{dA}{dx}\left(\frac{x}{2}\sqrt{400 - x^2}\right)$ or $\frac{dy}{dx}$ etc ...

When $\frac{dA}{dx} = 0$,

$\frac{200 - x^2}{\sqrt{400 - x^2}} = 0$ [M1] For equating $\frac{dA}{dx} = 0$, using their $\frac{dA}{dx}$ above. ecf allowed.

$200 = x^2$

$x = \sqrt{200}$ (since $x > 0$)

$= 10\sqrt{2}$ [A1]

(accept $5\sqrt{8}$ as per $a\sqrt{b}$ form)

(c) Explain why the value of x in (b) gives the largest possible value of A and hence, find the maximum area of the rectangle. [3]

⇒ Proving/finding maximum area

(P)$^{-1}$ $\frac{dy}{dx}$ stated instead of $\frac{dA}{dx}$

Method #1 (2nd derivative test)

$$\frac{d^2A}{dx^2} = (200-x^2)\cdot -\frac{1}{2}(400-x^2)^{-\frac{3}{2}}(-2x) + (-2x)(400-x^2)^{-\frac{1}{2}}$$

$$= \frac{x(200-x^2)}{(400-x^2)^{\frac{3}{2}}} - \frac{2x}{(400-x^2)^{\frac{1}{2}}}$$

[M1] any equivalent $\frac{d^2A}{dx^2}$ expression

ecf allowed from (b)

$$= \frac{200x - x^3 - 800x + 2x^3}{(400-x^2)^{\frac{3}{2}}}$$

$$= \frac{x^3 - 600x}{(400-x^2)^{\frac{3}{2}}}$$

at $x = 10\sqrt{2}$,

$$\frac{d^2A}{dx^2} = \frac{(10\sqrt{2})^3 - 600(10\sqrt{2})}{(400-200)^{\frac{3}{2}}}$$

$$= -2 < 0$$

Since $\frac{d^2A}{dx^2} < 0$, maximum area occurs at $x = 10\sqrt{2}$.

[A1] $\frac{d^2A}{dx^2}$ value must be shown

$$\therefore \text{Max. area} = \frac{10\sqrt{2}}{2}\sqrt{400-(10\sqrt{2})^2}$$

$$= 5\sqrt{2}\cdot\sqrt{200}$$

$$= 5\sqrt{400}$$

$$= 100 \text{ cm}^2$$ [A1]

Method #2 (1st derivative test)

x	$(10\sqrt{2})^-$	$10\sqrt{2}$	$(10\sqrt{2})^+$
$\frac{dA}{dx}$	>0	$=0$	<0
sketch of tangent	/	—	\

[M1]

By the 1st derivative test,
⇒ maximum area occurs at $x = 10\sqrt{2}$ [A1]

$$\therefore \text{Max. area} = \frac{10\sqrt{2}}{2}\sqrt{400-(10\sqrt{2})^2}$$

$$= 100 \text{ cm}^2$$ [A1]

Explanations NOT accepted:

'$x = 10\sqrt{2}$ is a maximum point'.

'$x = 10\sqrt{2}$ is maximum' ← any unclear/inaccurate indication of the meaning of x.

10 AB is a chord of the circle C_1, where the coordinates of A and B are $(2, 5)$ and $(6, 3)$ respectively. The line $y = 5 - x$ passes through the centre of the circle.

(a) Find the coordinates of the centre of C_1. [4]

Method#1

Mid-point of AB $= \left(\frac{2+6}{2}, \frac{5+3}{2}\right)$

$= (4, 4)$ [M1]

$m_{AB} = \frac{5-3}{2-6}$

$= -\frac{1}{2}$

$\therefore m_{\perp} = 2$ [M1]

Eqn of ⊥ bisector:

$y - 4 = 2(x-4)$

$y = 2x - 4$ —①

$y = 5 - x$ —②

Sub ① into ②:

$2x - 4 = 5 - x$ [M1] intersection of ⊥ bisector and line

allow ecf

$3x = 9$

$x = 3$

$\therefore y = 5 - 3$

$= 2$

⇒ centre: $(3, 2)$ [A1]

Method #2

Let centre of C_1 be (a, b)

Length of AC_1 = Length of BC_1

$\sqrt{(5-b)^2 + (2-a)^2} = \sqrt{(3-b)^2 + (6-a)^2}$ [M1]

$25 - 10b + \cancel{b^2} + 4 - 4a + \cancel{a^2} = 9 - 6b + \cancel{b^2} + 36 - 12a + \cancel{a^2}$

$29 - 10b - 4a = 45 - 6b - 12a$

$0 = 16 + 4b - 8a$

$b - 2a + 4 = 0$

$b = 2a - 4$ —①

For line $y = 5 - x$,

at (a, b),

$b = 5 - a$ —② [M1]

Sub ① into ②:

$2a - 4 = 5 - a$ [M1]

$3a = 9$

$a = 3$

$\therefore b = 5 - 3$

$= 2$

⇒ centre $(3, 2)$ [A1]

(b) Find the equation of the circle in the form $x^2 + y^2 + px + qy + r = 0$, where p, q and r are integers. [3]

Method #1

radius $= \sqrt{3^2+1^2}$

$= \sqrt{10}$ [M1]

$(x-3)^2 + (y-2)^2 = (\sqrt{10})^2$ allow ecf [M1]

$x^2 - 6x + 9 + y^2 - 4y + 4 = 10$

$x^2 + y^2 - 6x - 4y + 3 = 0$ [A1]

Method #2

radius $= \sqrt{3^2+1^2}$

$= \sqrt{10}$ [M1]

using $x^2+y^2+2gx+2fy+c=0$,

centre of circle : $C(-g,-f)$

$\Rightarrow -g = 3$, $g = -3$, $2g = -6$

$\Rightarrow -f = 2$, $f = -2$, $2f = -4$

radius $= \sqrt{f^2+g^2-c}$

$\sqrt{10} = \sqrt{(-2)^2+(-3)^2-c}$

$10 = 13 - c$

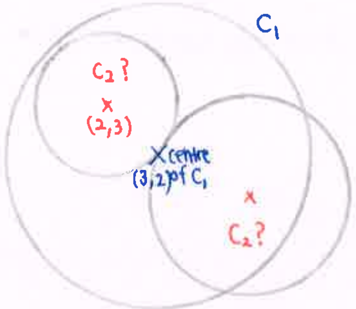
$c = 3$

either values of both p and q [M1] or value of r. allow ecf

$\therefore x^2+y^2-6x-4y+3=0$ [A1]

(c) Another circle C_2 with centre $(2, 3)$ passes through the centre of C_1.

Explain if the C_2 lies entirely within C_1. [2]



For C_2 to lie entirely within C_1,

diameter of C_2 < radius of C_1

radius of $C_2 = \sqrt{1^2+1^2}$

$= \sqrt{2}$ [M1] radius of C_2

$\therefore$ diameter of $C_2 = 2\sqrt{2} < \sqrt{10}$

Since the diameter of circle C_2 is shorter than the radius of circle C_1, it lies entirely within C_1. [A1] explained with comparison.

11 **(a)** Prove that $\dfrac{\sin 2x - \cos 2x + 1}{\sin 2x + \cos 2x + 1} = \tan x$ [4]

$$LHS = \frac{2\sin x\cos x - (1 - 2\sin^2 x) + 1}{2\sin x\cos x + (2\cos^2 x - 1) + 1}$$

[M1] double angle formula for $\sin 2x$

[M1] either $(2\cos^2 x - 1)$ selected in numerator or $(1 - 2\sin^2 x)$ selected in denominator

$$= \frac{2\sin x\cos x + 2\sin^2 x}{2\sin x\cos x + 2\cos^2 x}$$

$$= \frac{2\sin x(\cancel{\cos x + \sin x})}{2\cos x(\cancel{\sin x + \cos x})}$$

[M1] factorisation

$$= \frac{\sin x}{\cos x} = \tan x$$

[A1] shown.

Alternative

$$\frac{\underbrace{2\sin x\cos x}_{[M1]} - (\cos^2 x - \sin^2 x) + 1}{2\sin x\cos x + (\cos^2 x - \sin^2 x) + 1}$$

$$= \frac{2\sin x\cos x - \cos^2 x + \sin^2 x + (\sin^2 x + \cos^2 x)}{2\sin x\cos x + \cos^2 x - \sin^2 x + (\sin^2 x + \cos^2 x)}$$

either [M1]

$$= \frac{2\sin x\cos x + 2\sin^2 x}{2\sin x\cos x + 2\cos^2 x} = \frac{2\sin x(\cos x + \sin x)}{2\cos x(\sin x + \cos x)}$$

[M1]

$$= \frac{\sin x}{\cos x} = \tan x$$

[A1]

Alternative:

$$LHS = \frac{2\sin x\cos x - (2\cos^2 x - 1) + 1}{2\sin x\cos x + (1 - 2\sin^2 x) + 1}$$

[M1] double angle formula for $\sin 2x$

$$= \frac{2\sin x\cos x - 2\cos^2 x + 2}{2\sin x\cos x - 2\sin^2 x + 2}$$

$$= \frac{\cancel{2}(\sin x\cos x - \cos^2 x + 1)}{\cancel{2}(\sin x\cos x - \sin^2 x + 1)}$$

$$= \frac{\sin x\cos x - (1 - \sin^2 x) + 1}{\sin x\cos x - (1 - \cos^2 x) + 1}$$

[M1] either numerator: $\cos 2x = 2\cos^2 x - 1$ selected first, then $\cos^2 x = 1 - \sin^2 x$

or denominator: $\cos 2x = 1 - 2\sin^2 x$ selected first, then $\sin^2 x = 1 - \cos^2 x$.

$$= \frac{\sin x\cos x + \sin^2 x}{\sin x\cos x + \cos^2 x}$$

$$= \frac{\sin x(\cancel{\cos x + \sin x})}{\cos x(\cancel{\sin x + \cos x})}$$

[M1] factorisation

$$= \frac{\sin x}{\cos x} = \tan x$$

[A1] shown

(b) Hence, solve the equation $\frac{\sin 2x - \cos 2x + 1}{\sin 2x + \cos 2x + 1} = 5 - 2\sec^2 x$ for $0° < x < 360°$. [4]

Hence, $\tan x = 5 - 2\sec^2 x$

$\tan x = 5 - 2(1 + \tan^2 x)$ [M1] $\sec^2 x = 1 + \tan^2 x$

award M1 even if solution from (a) is applied incorrectly

$2\tan^2 x + \tan x - 3 = 0$

$(2\tan x + 3)(\tan x - 1) = 0$ [M1]

$\tan x = -\frac{3}{2}$ or $\tan x = 1$

$\alpha = \tan^{-1}\left(\frac{3}{2}\right)$ $\alpha = 45°$

$= 56.309°$ $\therefore x = 45°, 225°$ [A1]

$\therefore x = 180° - \alpha, 360° - \alpha$

$= 180° - 56.309°, 360° - 56.309°$

$= 123.7°, 303.7°$ [A1]

$\Rightarrow x = 45°, 123.7°, 225°, 303.7°$. (u)$^{-1}$ degrees must be stated.

[A1] deducted for <1dp answers. (eg. 123°/124° or 303°/304°)

[Alternative]

$\tan x = 5 - 2\sec^2 x$

$\frac{\sin x}{\cos x} = 5 - \frac{2}{\cos^2 x}$

$\sin x \cos x = 5\cos^2 x - 2$

$5\cos^2 x - \sin x \cos x - 2 = 0$

$5\cos^2 x - \sin x \cos x - 2(\sin^2 x + \cos^2 x) = 0$ [M1]

$3\cos^2 x - \sin x \cos x - 2\sin^2 x = 0$

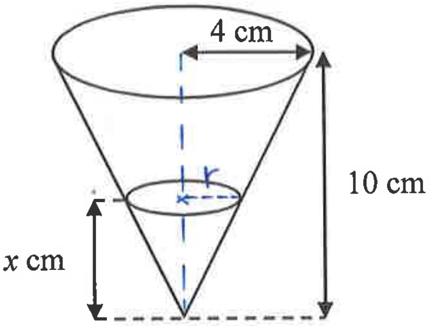
$(3\cos x + 2\sin x)(\cos x - \sin x) = 0$ [M1]

$3\cos x = -2\sin x$ or $\cos x = \sin x$

$\tan x = -\frac{3}{2}$ $\tan x = 1$

12 Water is dispensed at a constant rate into an empty paper cup in the form of an inverted cone of height 10 cm and radius 4 cm. The water dispenser is a cylindrical container with radius 12 cm. After t seconds, the depth of the water in the conical cup is x cm.

* penalise 1m overall for Q11 for absence of units in (b), (c)



(a) Show that the volume of water in the cup is $\dfrac{4\pi x^3}{75}$ cm^3. [2]

$\dfrac{r}{4} = \dfrac{x}{10}$

$r = \dfrac{2}{5}x$ [M1]

volume $= \dfrac{1}{3}\pi\left(\dfrac{2}{5}x\right)^2(x)$

$= \dfrac{2^2\pi x^3}{3\times5^2}$

$= \dfrac{4\pi x^3}{75}$

[A1] shown with substitution of r into vol. of cone formula.

$\boxed{\text{OR}}$

$\dfrac{V_1}{V_2} = \left(\dfrac{L_1}{L_2}\right)^3$

$\dfrac{V_1}{\frac{1}{3}\pi(4)^2(10)} = \left(\dfrac{x}{10}\right)^3$ [M1]

$V_1 = \dfrac{x^3}{1000} \times \dfrac{160\pi}{3}$

$= \dfrac{160\pi x^3}{3000}$

$= \dfrac{4\pi x^3}{75}$ [A1]

(b) Given that the depth of water in the cylinder dispenser decreases by 0.0035 cm/s, find the rate of increase in the volume of water dispensed in the conical cup, in terms of π. [2]

common misconception: misinterpretation of variables in qn, values happen to give same final answer but NO marks awarded.

$\dfrac{dx}{dt} \neq 0.0035$

rate of increase in the height of water in cone is not a constant.

Let h be the depth of water in cylinder.

$\dfrac{dh}{dt} = -0.0035$ cm/s

rate of increase in cup (in vol.) = rate of decrease in cylinder (in vol.)

$\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$

Volume of cylinder $= \pi(12^2)h$

$= 144\pi h$

$\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$

$= \underline{144\pi \times -0.0035}$ [M1]

$= -0.504\pi$ cm^3/s

$\Rightarrow$ rate of increase in cup

$= \underline{0.504\pi \text{ cm}^3/\text{s}}$ [A1]

(or $\dfrac{63\pi}{125}$ cm^3/s)

(u)$^{-1}$ cm^3/s

accept positive $\dfrac{dV}{dt}$ directly. $\dfrac{dV}{dt} = 144\pi \times 0.0035$

$= 0.504\pi$ cm^3/s

[Alternative]

(cylinder) $\dfrac{dV}{dt} = \pi r^2(\Delta h)$

$= \pi(12)^2(-0.0035)$ [M1]

$= -0.504\pi$ cm^3/s

$\therefore \dfrac{dV}{dt}$ (cone) $= 0.504\pi$ cm^3/s [A1]

(c) Hence, find the rate of increase in the depth of water in the conical cup when the volume of water dispensed is $\frac{5\pi}{6}$ cm^3. [4]

$$\frac{dV}{dt}_{(cup)} = \frac{dV}{dx} \times \frac{dx}{dt}$$

When $vol_{cup} = \frac{5\pi}{6}$,

$$\frac{4\pi x^3}{75} = \frac{5\pi}{6}$$

$$x^3 = \frac{5}{6}\left(\frac{75}{4}\right)$$

$$x = \sqrt[3]{\frac{125}{8}}$$

$$= 2.5 \text{ cm} \quad [M1]$$

$$\frac{dV}{dx} = \frac{3(4\pi)x^2}{75}$$

$$= \frac{4\pi x^2}{25} \quad [M1]$$

$$\therefore 0.504\pi = \frac{4\pi(2.5)^2}{25} \times \frac{dx}{dt} \quad [M1]$$

allow ecf from part (b) or incorrect x-value above.

OR $\frac{dx}{dt} = \frac{63}{20x^2}$ [M1]

$$\therefore \frac{dx}{dt} = \frac{0.504\pi(25)}{4\pi(2.5)^2}$$

$$= 0.504 \text{ cm/s} \quad [A1] \quad (\text{or } \tfrac{63}{125} \text{ cm/s})$$

u $^{-1}$ cm/s

END OF PAPER

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