Chapter 1 (Statistics) : Permutations and Combinations (Teacher's Copy)

Objectives:

At the end of the chapter, you should be able to

(a) understand and use the addition and multiplication principles for counting;

(b) understand the concepts of permutation ${}^{n}\mathrm{P}_{r}$ and combination ${}^{n}\mathrm{C}_{r}$, and differentiate between permutations and combinations;

(c) understand the concept of arrangements of distinct objects in a line including cases involving restriction, and know how to calculate them

Content

1.1 BASIC COUNTING PRINCIPLES

In the process of solving a counting problem, there are two very simple but basic principles that we always apply. They are called the ***Addition Principle*** and ***Multiplication Principle***.

1.1.1 The Addition Principle (OR)

Suppose you want to buy an ice cream. You have 5 flavours to choose from: Chocolate OR strawberry OR vanilla OR oreo OR mint. How many choices do you have?

Answer: $\underline{\quad 1 \quad} + \underline{\quad 1 \quad} + \underline{\quad 1 \quad} + \underline{\quad 1 \quad} + \underline{\quad 1 \quad} = 5$

Chocolate OR strawberry OR vanilla OR oreo OR mint

In general, if there exists k non-overlapping categories of ways to perform an operation and the first category can be done in n_1 ways, the second in n_2 ways, …., the kth category in n_k ways, then the operation can be done in $n_1 + n_2 + ... + n_k$ ways.

1.1.2 The Multiplication Principle (AND)

Suppose you want to buy an ice cream on a cone. There are 5 ice cream flavours to choose from and there are 3 types of cones. How many different choices do you have?

Cone

1. Waffle
2. Biscuit
3. Cup

Flavour

1. Chocolate
2. Vanilla
3. Strawberry
4. Oreo
5. Mint

No. of ways to choose a cone = 3

No. of ways to choose a flavour = 5

Total number of ways an ice cream AND a cone = 3 × 5 = 15

In general, if one operation can be done in n_1 ways, and (when this has been done), the second operation can be performed in n_2 ways and so on for k operations, then the k successive operations can be performed in $n_1 \times n_2 \times ... \times n_k$ ways.

Example 1

Mary and John go to a restaurant that offers 5 types of pizza and 6 types of salads. Mary would like to have either a pizza or a salad. John would like to have a pizza and a salad. How many choices do Mary and John have respectively?

Solution:

Mary	John
Case 1 Chooses pizza: 5 ways Case 2 Chooses salad: + 6 ways No. of choices = 11	Stage 1 pizza Stage 2 salad No. of choices = 5 ways x 6 ways = 30

Exercise 1

A bookshelf holds 6 different English books, 8 different Malay books and 10 different Chinese books.

(a) How many ways are there to select 1 book in any of the languages?
(b) How many ways are there to select 3 books, 1 in each language?
(c) How many ways are there to select 2 books in 2 languages?

[Ans: (a) 24; (b) 480; (c) 188]

Solution

(a) No. of ways = 6 + 8 + 10 = 24

(b) No. of ways = $6\times8\times10 = 480$

(c) 3 cases: The 2 books can be English and Malay or English and Chinese or Malay and Chinese.

No. of ways = $6\times8+6\times10+8\times10 \ = 188$

1.2 PERMUTATIONS

A permutation is an arrangement of objects where the **order** is important.

Example

1. Permutations (or arrangements) that can be formed with the letters A, B, C by taking three at a time.

____ ____ ____

The first slot can have 3 choices (A, B or C) .The second slot can have 2 choices. The third slot can have 1 choice.

No of ways to arrange A, B, C = 3 × 2 × 1 = 3! = 6 ways

ABC BAC CAB

ACB BCA CBA

2. Permutations (or arrangements) that can be formed with the letters A, B, C by taking two at a time.

____ ____

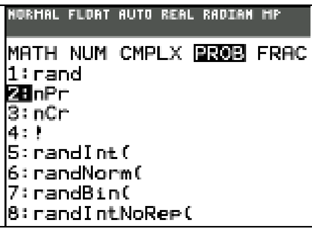
In this case, the first slot can have 3 choices (A, B or C) .The second slot can have 2 choices.
= {AB, AC, BA, BC, CA, CB} = 3 × 2 = 6 ways

Example 2

Without replacement (i.e. no repetition)
How many 5-letter code-words can be formed from the letters of
the word ***MATRICES***? ____ ____ ____ ____ ____ Number of 5-letter code-words = $8\times7\times6\times5\times4=6720$

Unless otherwise stated, a permutation of objects refers to one without replacement.

The symbol ${}^{n}\mathrm{P}_{r}$ is used to represent the number of permutations of r objects out of a total of n objects. Thus, the answer in Example 2(a) is ${}^{8}\mathrm{P}_{5}$. There is a GC command for it:

Press MATH and press ▶ to PROB, Press '2'.		

In particular, the number of permutations of n distinct objects taken all at a time is

${}^{n}\mathrm{P}_{n} = n(n-1)...(2)(1) = n!$. E.g. ${}^{8}\mathrm{P}_{8} = 8!$

The number of permutations of r distinct objects out of n distinct objects is

${}^{n}\mathrm{P}_{r} = n(n-1)...(n-r+1) = \dfrac{n!}{(n-r)!}$ E.g. ${}^{8}\mathrm{P}_{5} = 8\times7\times6\times5\times4 = \dfrac{8!}{3!}$

<u>Example 3</u>

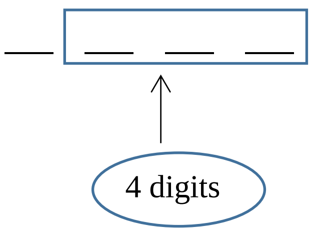
(a) How many 4-digit numbers greater than 2000 can be formed from {0,1,2,3,4} if no repetition is allowed?

(b) How many of these numbers from **(a)** will be divisible by 5 if no repetition is allowed?

(c) How many of these numbers from **(a)** are even if no repetition is allowed?

Solution:

(a)

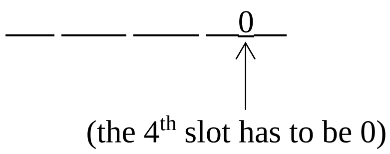


The first slot can only be taken by the numbers 2,3 or 4.

The subsequent slots can only be taken by the remaining numbers and arranged in ${}^{4}\mathrm{P}_{3}$ ways.

No. of numbers = $3\times{}^{4}\mathrm{P}_{3} = 72$

(b)

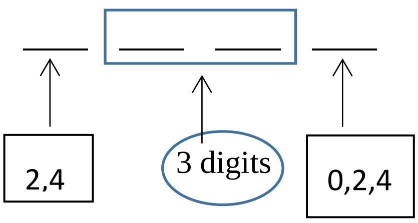


The first slot can take 2, 3 or 4, the last slot has to take 0, the remaining 3 numbers will be used to fill the remaining two slots.

No. of numbers = $3\times{}^{3}\mathrm{P}_{2}\times1 = 18$

(c)

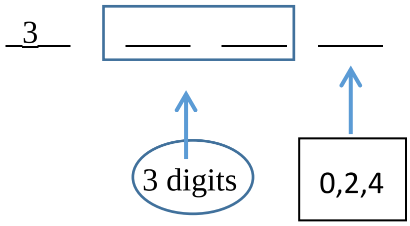
Case 1:



The first slot is taken by an even number 2 or 4. (2 ways)
The last slot has to be taken by 0 or the other even number **not taken** by first slot. (2 ways)

No. of numbers = $2\times {}^{3}P_{2}\times 2=24$

Case 2:



The first slot is taken by an odd number 3.
(1 way)
The last slot has to be taken by either 0, 2, or 4. (3 ways)

No. of numbers = $1\times {}^{3}P_{2}\times 3=18$

Total no. of numbers = 24 + 18 = 42

Exercise 2

1(i) How many 3-digit numbers can be formed from the digits { 4, 5, 6, 7, 8 } if repetitions of number are not allowed?

(ii) Find the number of ways that a 3-digit number can be formed from the digits {4, 5, 6, 7, 8 } if repetitions of number are not allowed and that it

(a) is an odd number,
(b) is an even number,
(c) contains the digit 4 at most once,
(d) contains the digit 8 twice.

[Ans : (i) 60; (ii)(a) 24 ; (ii)(b) 36 ; (ii)(c) 60 ; (ii)(d) 12]

Solution:

(i) No. of ways = 5 x 4 x 3 = 60 or ${}^{5}P_{3}$

(ii) **(a)**No. of odd numbers = ${}^{4}P_{2}$ x 2 = 24

(b) No. of even numbers = ${}^{4}P_{2}$ x 3 = 36

(c) Only 1 ‘4’ = 1 x 4 x 3 x **3** = 36

No ‘4’ = ${}^{4}P_{3}$ = 24

Total no. = 36+ 24 = 60

(e) Total no. $=1\times 1\times 4\times 3=12$

Question for (ii)(c)

Why is there a need to multiply by 3?

Because of 3 cases:

4_ _ , _ 4 _ , _ _ 4

Example 4: Grouping

2 girls and 2 boys are lining up in a row. If the 2 girls must be together, how many possible ways are there to line the children up?

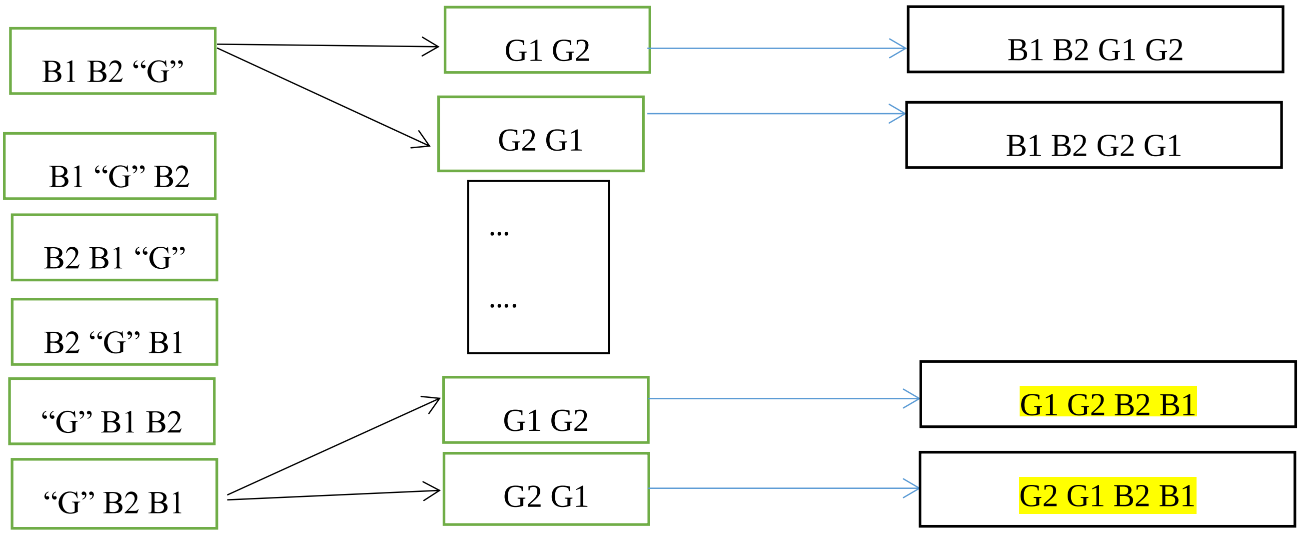
Solution:

Step 1: Treat the 2 girls as one unit. Thus, we have three units: B1, B2, "G"

Step 2: No. of ways to permute the three units B1 B2 "G" = 3! = 6

Step 3: No. of ways to permute the 2 girls within "G", i.e. G1, G2 = 2!

Final possible outcomes



Total No of ways= 3!×2!=12

Example 5

Seven people, of whom 3 are men, are to form a queue. Find the number of ways this can be done if

(a) there is no restriction

(b) all the women are always together

(c) no two people of the same gender are to stand next to each other.

Solution:

(a) No of ways = 7! = 5040

(b) Treat all the women as 1 unit. No. of ways to permute these 4 units = 4!

$\boxed{W_1\ W_2\ W_3\ W_4}\quad \boxed{M_1}\quad \boxed{M_2}\quad \boxed{M_3}$

No of ways to permute the women within the unit = 4!

Total number of ways = $4!\times 4! = 576$

(c) If no two people of the same gender stand next to each other, there is only one case:

$\boxed{W_1}\ \boxed{M_1}\ \boxed{W_2}\ \boxed{M_2}\ \boxed{W_3}\ \boxed{M_3}\ \boxed{W_4}$

Let's arrange the women first. There are 4! ways to do this.

There are 3 slots in between to slot the 3 men. There are 3! ways to do this.

Total number of ways = $4!\times 3! = 144$

Exercise 3

1 A class of 10 boys and 15 girls stand in a row for photo-taking. How many ways can the students stand if:

(a) there is no restriction on how they are arranged;
(b) the two class chairpersons must stand on the left;
(c) the two class chairpersons must stand together.

Suppose the boys and girls must be in two separate rows. How many ways can the students stand?

[Ans: (a) 25!; (b) 2! × 23!; (c) 2! × 24! ; 2! × 10! × 15!]

Solution:

(a) There are 25 students in total, so there are 25! ways for them to be arranged

(b) There are 2! ways of arranging the chairpersons.
There are 23! ways of arranging the other 23 students.
So, there are 2! × 23! ways.

(c) There are 2! ways of arranging the 2 chairpersons. Treat them as a unit.
The above unit and the 23 other students can be arranged in 24! ways.
So, there are 2! × 24! ways.

There are 10! ways to arrange the boys and 15! ways to arrange the girls.
There are 2! ways to arrange the rows (boys in front or girls in front).
So, there are 2! × 10! × 15! ways of arranging the students.

2 (a) How many 3-letter code words can be formed from the word ENGLISH?

(b) How many 3-digit numbers can be formed from the digits 1 to 7 if repetition is not allowed?

(c) A waiting room contains a row of seven chairs. In how many ways can three of these chairs be occupied by three people?

(d) In how many ways can the first, second and third prizes be awarded to a group of 7 people assuming that each person can be awarded only one prize?

[Ans: (a) 210 (b) 210 (c) 210 (d) 210]

Solution: (for all parts of the question)

No. of ways = $7 \times 6 \times 5 = {}^{7}\mathrm{P}_{3} = 210$

1.3 COMBINATIONS (or Selection)

A combination is a selection of objects in which the order of selection **does not** matter.

In English we use the word "combination" loosely, without thinking if the **order** of things is important. In other words:

"My fruit salad is a combination of apples, grapes and bananas"

We don't care what order the fruits are in, they could also be "bananas, grapes and apples" or "grapes, apples and bananas", it's the same fruit salad.

"The combination to the safe was 472".

Now we **do** care about the order. "724" would not work, nor would "247". It has to be exactly **4-7-2**.

So, in Mathematics we use more *precise* language:

- If the order **does not** matter, it is a **Combination**.
- If the order **does** matter, it is a **Permutation**.

In the above description, the fruit salad is correctly described as a combination, but the safe number should be described as a permutation instead.

Combinations	Permutations
ORDER DOES NOT MATTER	ORDER MATTERS
Number of combinations of r objects from n different objects = ${}^{n}\mathrm{C}_{r}$ or $\binom{n}{r}$	Number of permutations of r objects from n different objects = ${}^{n}\mathrm{P}_{r}$

Example

1. Combinations that can be formed with the letters A, B, C.

____ ____ ____

No of ways to select A, B, C = {A, B, C} = ${}^{3}\mathrm{C}_{3} = 1$ way

2. Combinations that can be formed with 2 letters from A, B or C.

____ ____

No of ways to select A, B, C = {AB, AC, BC} = ${}^{3}\mathrm{C}_{2} =$ 3 ways

Establishing the Formula for Combinations

$$\text{Since } {}^{n}\mathrm{P}_{r} = {}^{n}\mathrm{C}_{r} \times r!,$$
$$\begin{aligned} {}^{n}\mathrm{C}_{r} &= {}^{n}\mathrm{P}_{r} \div r! \\ &= \frac{n!}{(n-r)!} \div r! \\ &= \frac{n!}{(n-r)!r!} \end{aligned}$$

Example 6

In how many ways can a **committee** of five be formed from a group of eight people consisting of 3 men, 3 women and a married couple if

(i) there is no restriction in a selection?

(ii) the committee must include the married couple?

(iii) the married couple cannot serve in the same committee?

Solution:

(i) No. of ways = ${}^{8}\mathrm{C}_{5} = 56$

(ii) The married couple must be included in the committee. Need to select 3 more people from the remaining 6 people. Thus, no. of ways = ${}^{6}\mathrm{C}_{3} = 20$

(iii) The committee can only include the married couple or not include the married couple. Thus they are mutually exclusive.

No. of ways = no of ways without restriction – no of ways the committee includes the couple
=56 – 20 = 36

Example 7

Find the number of ways in which ten people can be divided into

(i) two groups consisting of seven and three people;

(ii) three groups consisting of four, three and two people, with one person being excluded;

(iii) Group 1 and Group 2 with five people each.

Solution:

(i) No of ways = ${}^{10}\mathrm{C}_{7}\,{}^{3}\mathrm{C}_{3} = 120$ (Note: the two groups are distinct)

(ii) No of ways = ${}^{10}\mathrm{C}_{4} \times {}^{6}\mathrm{C}_{3} \times {}^{3}\mathrm{C}_{2} = 12600$ (Note: the three groups are distinct)

(iii) No. of ways = ${}^{10}\mathrm{C}_{5}\,{}^{5}\mathrm{C}_{5} = 252$ (Note: The two groups are distinct since they are labelled.)

Exercise 4

A CCA group consists of 12 Ex-co members, 10 female ordinary members and 9 male ordinary members. A team of 4 Ex-co members, 4 female ordinary members and 3 male ordinary members is to be formed. Find the number of ways of forming the team.

[Ans: 8731800]

Solution

No. of ways = ${}^{12}C_4 \times {}^{10}C_4 \times {}^{9}C_3 = 8731800$

Exercise 5

1(a) A panel of judges in an essay competition has to select, and place in order of merit, 4 winning entries from a total entry of 20. Find the number of ways in which this can be done.

(b) As a first step in the selection, 5 finalists are selected, without being placed in order. Find the number of ways in which this can be done.

[Ans: (a) 116280; (b) 15504]

Solution

(a) Number of ways = ${}^{20}P_4 = 116280$

(b) Number of ways = ${}^{20}C_5 = 15504$

2) How many odd numbers, greater than 500 000, can be made from the digits 2, 3, 4, 5, 6, 7 without repetitions? [Ans: 168]

Solution

Case 1:

1	4	3	2	1	3

{6} {3, 5, 7}

Number of ways = $1 \times 4! \times 3 = 72$

Case 2:

2	4	3	2	1	2

{5, 7} {3, 5/7}

Number of ways = $2 \times 4! \times 2 = 96$

Total no. of ways = $96 + 72 = 168$

3) A chess team of 5 players is to be selected from 15 boys. In how many ways can the team be chosen if

(i) not more than one of the three best is to be included;
(ii) at least one of the 4 youngest players is to be included.
[Ans : (i) 2277; (ii) 2541]

Solution

(i) Case 1: 1 best player chosen

Number of ways = ${}^{3}C_1 \times {}^{12}C_4 = 1485$

Case 2: 0 best player chosen

Number of ways = ${}^{12}C_5 = 792$

Total no. of ways = 1485 + 792 = 2277

(ii) Number of ways = ${}^{15}C_5 - {}^{11}C_5 = 2541$

4) A group of diplomats is to be chosen to represent three islands, *K*, *L* and *M*. The group is to consist of 8 diplomats and is chosen from a set of 12 diplomats consisting of 3 from *K*, 4 from *L* and 5 from *M*. Find the number of ways in which the group can be chosen if it includes

(i) 2 diplomats from *K*, 3 from *L* and 3 from *M*,
(ii) diplomats from *L* and *M* only,
(iii) at least 4 diplomats from *M*,
(iv) at least 1 diplomat from each island.
[Ans: (i) 120; (ii) 9; (iii) 210; (iv) 485]

Solution:

(i) No. of ways $= {}^{3}C_2 \times {}^{4}C_3 \times {}^{5}C_3 = 120$

(ii) Method 1: No. of ways $= {}^{4}C_4 \times {}^{5}C_4 + {}^{4}C_3 \times {}^{5}C_5 = 9$

Method 2: ${}^{9}C_8 = 9$

(iii) No. of ways $= {}^{5}C_4 \times {}^{7}C_4 + {}^{5}C_5 \times {}^{7}C_3 = 210$

(iv) No. of ways = total – no *K* – no *L* – no *M* = $= {}^{12}C_8 - {}^{9}C_8 - {}^{8}C_8 - 0 = 485$

Question for (iv)

Why are there 0 ways for no *M*?

Because the total number of diplomats from *L* and *K* is less than 8, thus there must be diplomat(s) selected from *M*.

Practice Questions

1. In how many ways can 4 boys and 2 girls seat themselves in a row if

(i) there is no restriction;
(ii) the two girls are to sit next to each other;
(iii) the two girls are not to sit next to each other;
(iv) the two girls are to be separated by two boys between them;
(v) the two girls are to be separated by two particular boys between them?

[Ans: (i) 720; (ii) 240; (iii) 480; (iv) 144; (v) 24]

Solution:

(i) No. of ways = $6! = 720$

(ii)

G_1 G_2	B_1	B_2	B_3	B_4

Let us treat the two girls as one unit and the first stage is to arrange the 5 units first. Then we have to arrange the two girls within the unit.

No. of ways to arrange 5 distinct units = 5!

No. of ways to arrange the two girls among themselves = 2!

Total no. of ways = $5! \times 2! = 240$

(iii)

B_1	B_2	B_3	B_4

We arrange the boys first = 4!

We slot the girls in between the boys. Note that there are 5 possible slots for the 2 girls.

$= {}^5C_2 \times 2! = 20$

No. of ways to arrange 4 boys and 2 girls = $4! \times 20 = 480$

Alternative method:

Required no. of ways

= total no. of ways with no restriction – no. of ways where the 2 girls sit next to each other

= 720 – 240

= 480

(iv) | G_1__ ____G_2 | B_2 | B_3 |

2B

Group 2 girls and 2 boys as 1 unit.

No. of ways to arrange 3 distinct units = 3!

No. of ways to arrange 4 boys into 2 positions = ${}^4P_2$

No. of ways to arrange the two girls among themselves = 2!

Total no. of ways = $3!\times 2!\times {}^4P_2 = 144$

<u>Alternative method:</u>

If the two girls are to be separated by any two boys, we have the following 3 cases:

Case 1: GBBGBB

Case 2: BGBBGB

Case 3: BBGBBG

In each of the cases, the boys can be arranged in 4! ways and the 2 girls in 2! ways.

No. of arrangements in each case = 4!2!

Therefore, total no. of ways = $(4!2!)\times 3 = 144$

(v) | G_1__ ____G_2 | B_2 | B_3 |

Group 2 girls and 2 boys as 1 unit.

No. of ways to arrange 3 distinct units = 3!

No. of ways to arrange 2 boys and 2 girls= $2!\times 2! = 4$

Total no. of ways = $3!\times 4 = 24$

2. In how many ways can a 4-digit number greater than 3000 be formed using the digits 0, 1, 3, 4, 5 if

(a) no repetition of digits is allowed,
(b) these numbers, formed without repetition, are even?

[Ans: (a) 72; (b) 30]

Solution:

(a) Noting that we can only place the digits 3, 4, 5 as the first 3 digits,

therefore the number of ways = 3 x 4 x 3 x 2 = 72

(b) Case 1: The digit 0 is the last digit

Number of ways = 3 x 3 x 2 x 1 = 18

Case 2: The digit 4 is the last digit

Number of ways = 2 x 3 x 2 x 1 = 12

Thus total number of ways = 18 + 12 = 30

3. [2016/CJC/Promo/Q10]
A five-digit non-zero code-word is to be generated using the digits {0, 2, 3, 5, 9}. How many code-words can be formed if repetitions of numbers are not allowed, and the digits 2 and 5 are not separated by exactly one other digit?
[Ans: 84]

Solution:

Number of permutations when 2 and 5 are 1 digit apart

$= {}^{3}C_{1} \times 3! \times 2! = 36$

[${}^{3}C_{1}$ to choose a number between 2 and 5,

3! to permutate the group 2$\square$5 and the remaining 2 numbers,

2! to exchange the position of 2 and 5]

Total number of permutations without restriction

$= 5! = 120$

Number of permutations when 2 and 5 are not 1 digit apart

$= 120 - 36 = 84$

4. [2016/CJC/Promo/Q11]

10 student leaders are shortlisted to form a committee of 5 members in a school. 3 of them are house captains, 3 of them are class presidents and the remaining 4 are CCA leaders.

(i) How many different committees can be formed? [1]

(ii) Elly and Elva are twins who are shortlisted. They decide that they will either sit in the committee together, or stay out of it. How many different committees can be formed? [3]

(iii) How many different committees can be formed if the committee must contain at least one representative from each category of student leaders? [3]

[Ans: (i) 252; (ii) 112; (iii) 204]

Solution:

(i) No. of ways $= {}^{10}C_5 = 252$

(ii)

Case 1: Elly and Elva are both in committee

No. of ways $= {}^{8}C_3 = 56$

Case 2: Elly and Elva are not in committee

No. of ways $= {}^{8}C_5 = 56$

Total no. of ways = 56 + 56 = 112

(iii)

Number of committees with at least one representative from each category

= Total number of committees – Number of committees with no house captains – Number of committees with no class presidents – Number of committees with no CCA leaders

$= 252 - {}^{7}C_5 - {}^{7}C_5 - {}^{6}C_5$

$= 204$

5. [2016/MI/Promo/Q4]

(a) In a mathematics quiz, each student either wins a prize or does not win a prize.

(i) Assuming no order of merit, find the number of ways in which ten students can stand in a line at the end of the quiz. [1]

(ii) Ten students are finalists for three prizes – one for $5, one for $10 and one for $20. In how many different ways can the prizes be awarded? [1]

(iii) Six male and four female students are finalists for four prizes of $10 each. In how many different ways can the prizes be awarded such that there are at least one male and at least one female recipients? [3]

(b) Mr Lim would like to bid for a four-digit car plate license number from the Land Transport Authority for his new car. He was informed by the Authority that only numbers with the digits 1, 2, 3, 6, 7, 8, and 9 can be used.

Mr Lim prefers an even licence number with no repeated digits. Find the number of different licence numbers that he can form based on his preference. [2]

[Ans: (ai) 3628800; (aii) 720; (aiii) 194; (b) 360]

Solution:

(ai) Number of ways $= 10! = 3628800$

(aii) Number of ways $= {}^{10}P_3 = 720$

(aiii) Number of ways

$$= {}^{10}C_4 - {}^{6}C_4 - {}^{4}C_4$$
$$= 194$$

Alternative method

Number of ways

$$= {}^{6}C_3 \times {}^{4}C_1 + {}^{6}C_2 \times {}^{4}C_2 + {}^{6}C_1 \times {}^{4}C_3$$
$$= 194$$

(b) Number of ways

$$= {}^{6}P_3 \times {}^{3}C_1$$
$$= 360$$

6. [2016/PJC/Promo/Q4]

Ten JC1 students, fourteen JC2 students and six alumni of Victory College participated in a singing competition. 7 contestants are selected into the final round.

(i) Find the number of different possible selections if there is at least one JC1 student, at least two JC2 students and at least three alumni entering the final round. [3]

Two JC1 students, two JC2 students and three alumni enter the final round and the contestants are to stand in a line. Find the number of different possible arrangements if

(ii) the JC1 students stand together and the JC2 students stand together, [2]

(iii) no two alumni stand next to each other. [3]

[Ans: (i) 168350; (ii) 480; (iii) 1440]

Solution:

(i)

JC1(10)	JC2(14)	Alumni (6)	No. of arrangements
1	2	4	${}^{10}C_1 \times {}^{14}C_2 \times {}^{6}C_4 = 13650$
1	3	3	${}^{10}C_1 \times {}^{14}C_3 \times {}^{6}C_3 = 72800$
2	2	3	${}^{10}C_2 \times {}^{14}C_2 \times {}^{6}C_3 = 81900$

Total number of selections = 13650 + 72800 + 81900 = 168350

(ii) Number of arrangements = $2! \times 2! \times 5! = 480$

(iii) Number of arrangements = $4! \times {}^{5}C_3 \times 3! = 1440$

7. [2016/RVHS/Promo/Q6]

(a) Alice keeps a mathematics book library of 7 different mathematics books. Betty, Clare and Danielle visited Alice to borrow her books. Find the number of ways to distribute these 7 different books among the 3 girls if

(i) any girl can have any number of books; [2]

(ii) Betty and Clare get 3 books each. [2]

(b) MY LITTLE EVENT organisers decided to issue serialised tickets to their audience. They decided that the serial number should contain 4 digits consisting of 0, 3, 6, 8 and 9. For security reason, they also decided that the number should be greater than 6000. How many tickets can they have if no repetition of digits is allowed? [2]

[Ans: (ai) 2187; (aii) 140; (b) 72]

Solution:

(ai) Each book can go to either of the 3 girls.

$$\begin{aligned}\text{number of ways} &= 3^7 \\ &= 2187\end{aligned}$$

(aii) Out of 7 books, 3 are selected for Betty, and out of the remaining 4 books, 3 are selected for Clare, the remaining book will belong to Danielle.

(Alternatively, we can select for Danielle, then Betty, then Clare OR Betty then Danielle then Clare etc.)

$$\begin{aligned}\text{number of ways} &= {}^7C_3 \times {}^4C_3 \text{ or } {}^7C_1 \times {}^6C_3 \text{ or } {}^7C_3 \times {}^4C_1 \\ &= 140\end{aligned}$$

(b) Noting that we can only place the digits 6, 8, 9 as the first digit, therefore the number of ways = 3 x 4 x 3 x 2 = 72

8. [2016/SRJC/Promo/Q5]

A committee of 10 members consisting of 3 married couples, 2 other men and 2 other women sit in a line to conduct a first round interview of potential candidates.

(i) Find the number of possible seating arrangements. [1]

(ii) Find the number of possible seating arrangements in which each married man sits next to his wife. [2]

A further sub-committee is to be formed with 5 members to conduct the second round of interview.

(iii) Find the number of ways such that there are more men than women chosen to form the sub-committee. [3]

[Ans: (i) 3628800; (ii) 40320; (iii) 126]

Solution:

(i) No. of possible seating arrangements = 10! = 3628800

(ii) No. of possible arrangements such that the couple sits together

$= 7! \times 2! \times 2! \times 2! = 40320$

(iii) No of ways = 5M + 4M1W + 3M2W

$= {}^5C_5 + {}^5C_4 \times {}^5C_1 + {}^5C_3 \times {}^5C_2$

= 126

Summary

Addition Principle:

In general, if there exists k non-overlapping categories of ways to perform an operation and the first category can be done in n_1 ways, the second in n_2 ways, …., the kth category in n_k ways, then the operation can be done in $n_1 + n_2 + ... + n_k$ ways.

Multiplication Principle:

In general, if one operation can be done in n_1 ways, and (when this has been done), the second operation can be performed in n_2 ways and so on for k operations, then the k successive operations can be performed in $n_1 \times n_2 \times ... \times n_k$ ways.

General Formulae:	Examples
Row Arrangements: No. of permutations of r objects out of a total of n distinct objects $= {}^{n}\mathrm{P}_{r}$	No. of 4 -letter code-words from the word BIRTHDAY = ${}^{8}\mathrm{P}_{4}$
Combinations: No. of combinations of r objects from n distinct objects = ${}^{n}\mathrm{C}_{r}$ or $\begin{pmatrix} n \\ r \end{pmatrix}$	No. of ways to select a committee of 5 from a class of 15 = ${}^{15}\mathrm{C}_{5}$ or $\begin{pmatrix} 15 \\ 5 \end{pmatrix}$

Checklist

☐ I understand how to use the addition and multiplication principles for counting;

☐ I understand how to use permutation (${}^{n}\mathrm{P}_{r}$) to arrange distinct objects when order matters;

☐ I understand how to use combination (${}^{n}\mathrm{C}_{r}$) to select objects when order does not matter;

☐ I understand how to treat objects as 1 unit when I have to group them together;

☐ I understand how to use slotting method to separate objects.