Chapter 9 Complex Numbers I Tutorial (A): Algebra of Complex Numbers Solutions

Additional Practice Questions

1. Since the coefficients are all real, another root of the equation is $x = -2 - \mathrm{i}$.

$[\ x - (-2 + \mathrm{i})\ ]\ [\ x - (-2 - \mathrm{i})\ ]$
$= (x + 2 - \mathrm{i})(x + 2 + \mathrm{i})$
$= x^2 + 2x - \mathrm{i}x + 2x + \mathrm{i}x + 2^2 - \mathrm{i}^2$
$= x^2 + 4x + 5$

By comparing coefficients:
$x^4 + 4x^3 + x^2 + ax + b = (x^2 + 4x + 5)(x^2 - 4)$
$= x^4 + 4x^3 + x^2 - 16x - 20$

So $a = -16$, $b = -20$.
The other three roots of the equation are $-2 - \mathrm{i}$, 2, -2.

Alternative solution
$(-2 + \mathrm{i})^4 + 4(-2 + \mathrm{i})^3 + (-2 + \mathrm{i})^2 + a(-2 + \mathrm{i}) + b = 0$
$-12 + 16\mathrm{i} - 2a + a\mathrm{i} + b = 0$
$(b - 2a) + a\mathrm{i} = 12 - 16\mathrm{i}$
Comparing imaginary parts, $a = -16$
Comparing real parts, $b - 2a = 12$
$b = -20$
The other three roots of the equation are $-2 - \mathrm{i}$, 2, -2 from GC.

2.
$$a(k\mathrm{i})^4 + b(k\mathrm{i})^3 + c(k\mathrm{i})^2 + d(k\mathrm{i}) + e = 0$$
$$ak^4 - bk^3\mathrm{i} - ck^2 + dk\mathrm{i} + e = 0$$
$$ak^4 - ck^2 + e = 0, \qquad -bk^3\mathrm{i} + dk\mathrm{i} = 0 \Rightarrow k^2 = \frac{d}{b}$$
$$\text{Thus, } a\left(\frac{d}{b}\right)^2 - c\left(\frac{d}{b}\right) + e = 0 \Rightarrow ad^2 - cdb + eb^2 = 0.$$

$$a = 1, b = 3, c = 13, d = 27, e = 36$$
$$ad^2 + b^2e = 27^2 + 9(36) = 1053$$
$$bcd = 1053$$

$$k^2 = \frac{d}{b} = \frac{27}{3} = 9 \Rightarrow k = \pm 3$$
Thus. two roots are $\pm 3\mathrm{i}$.

3 $z = 1 + \mathrm{i}p$, $w = 1 + \mathrm{i}q$

$zw = (1 + \mathrm{i}p)(1 + \mathrm{i}q)$

$3 - 4\mathrm{i} = 1 - pq + \mathrm{i}(p + q)$

$\therefore\ 3 = 1 - pq \ldots \ldots (1)$

& $-4 = p + q \Rightarrow q = -4 - p$

Substitute into (1) $\Rightarrow 3 = 1 - p(-4 - p)$

$$p^2 + 4p - 2 = 0$$

$$p = \frac{-4 \pm \sqrt{16 - 4(1)(-2)}}{2(1)} = -2 \pm \sqrt{6}$$

Since $p > 0 \therefore\ p = -2 + \sqrt{6}$ //

$$q = -4 - (-2 + \sqrt{6}) = -2 - \sqrt{6} \quad //$$

4 $wz = \mathrm{i}$ --- (1)

$w - 2\mathrm{i}z = 2 \quad \Rightarrow \quad w = 2 + 2\mathrm{i}z$ --- (2)

Substitute (2) in (1):

$$z\left(2 + 2\mathrm{i}z\right) = \mathrm{i}$$

$$2\mathrm{i}z^2 + 2z - \mathrm{i} = 0$$

$$z = \frac{-2 \pm \sqrt{4 - 4(2\mathrm{i})(-\mathrm{i})}}{4\mathrm{i}} = \frac{-2 \pm \sqrt{-4}}{4\mathrm{i}} = \frac{-2 \pm 2\mathrm{i}}{4\mathrm{i}} = \frac{-1 \pm \mathrm{i}}{2\mathrm{i}}$$

Thus, $z = \frac{-1+\mathrm{i}}{2\mathrm{i}} \times \frac{\mathrm{i}}{\mathrm{i}} = \frac{1}{2}\left(1 + \mathrm{i}\right)$ or $z = \frac{-1-\mathrm{i}}{2\mathrm{i}} \times \frac{\mathrm{i}}{\mathrm{i}} = \frac{1}{2}\left(-1 + \mathrm{i}\right)$

$\Rightarrow\ w = \frac{2\mathrm{i}}{1+\mathrm{i}} \times \frac{1-\mathrm{i}}{1-\mathrm{i}} = 1 + \mathrm{i}$ $\qquad w = \frac{2\mathrm{i}}{-1+\mathrm{i}} \times \frac{-1-\mathrm{i}}{-1-\mathrm{i}} = 1 - \mathrm{i}$

Alternative solution

$wz = \mathrm{i}$ --- (1)

$w - 2\mathrm{i}z = 2 \qquad z = \frac{w-2}{2\mathrm{i}}$ --- (2)

Substitute (2) in (1):

$$w\left(\frac{w-2}{2\mathrm{i}}\right) = \mathrm{i}$$

$$w^2 - 2w + 2 = 0$$

$$w = \frac{2 \pm \sqrt{4 - 4(2)}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2\mathrm{i}}{2} = 1 \pm \mathrm{i}$$

i.e., $w = 1 + \mathrm{i}$ or $w = 1 - \mathrm{i}$

$$z = \frac{1+\mathrm{i}-2}{2\mathrm{i}} \qquad\qquad z = \frac{1-\mathrm{i}-2}{2\mathrm{i}}$$

$$= \frac{-1+\mathrm{i}}{2\mathrm{i}} \times \frac{\mathrm{i}}{\mathrm{i}} = \frac{1}{2}(1+\mathrm{i}) \qquad\qquad = \frac{-1-\mathrm{i}}{2\mathrm{i}} \times \frac{\mathrm{i}}{\mathrm{i}} = \frac{1}{2}(-1+\mathrm{i})$$