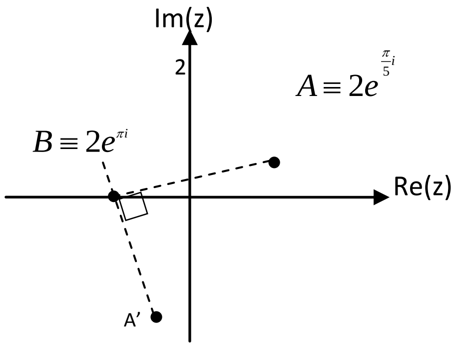
Anderson Junior College
Preliminary Examination 2015
H2 Mathematics Paper 2 (9740/02)

1a (i)	$\frac{\mathrm{d}w}{\mathrm{d}x} = -2(\mathrm{e}^{-2x} + A)^{-2}(-2\mathrm{e}^{-2x}) = \frac{4\mathrm{e}^{-2x}}{(\mathrm{e}^{-2x} + A)^2}$ $\therefore$ LHS = $\mathrm{e}^{2x}\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{4}{(\mathrm{e}^{-2x} + A)^2} = w^2$ = RHS (verified)
(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{\mathrm{e}^{-2x} + A}$ $y = \int \frac{2}{\mathrm{e}^{-2x} + A}\,\mathrm{d}x$ $= \frac{1}{A}\int \frac{2A\mathrm{e}^{2x}}{1 + A\mathrm{e}^{2x}}\,\mathrm{d}x$ $= \frac{1}{A}\ln\left\|1 + A\mathrm{e}^{2x}\right\| + B,$ where B is an arbitrary constant.
1b	Let S be the amount of salt (in grams) at time t minutes. $\frac{\mathrm{d}S}{\mathrm{d}t}$ = rate of salt entering tank $-$ rate of salt leaving tank $= 2(5) - \frac{S}{100}(5)$ $= \frac{200 - S}{20}$ g/min
	$\frac{1}{200 - S}\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{1}{20}$ $\int \frac{1}{200 - S}\,\mathrm{d}S = \int \frac{1}{20}\,\mathrm{d}t$ $-\ln\left\|200 - S\right\| = \frac{1}{20}t + c$ $200 - S = A\mathrm{e}^{-\frac{1}{20}t}$ $S = 200 - A\mathrm{e}^{-\frac{1}{20}t}$ When $t = 0$, $S = 50 \Rightarrow 50 = 200 - A \Rightarrow A = 150$ $\therefore S = 200 - 150\mathrm{e}^{-\frac{1}{20}t}$ When concentration is 1 g/litre, $S = 100$. $\Rightarrow 100 = 200 - 150\mathrm{e}^{-\frac{1}{20}t} \Rightarrow \mathrm{e}^{-\frac{1}{20}t} = 2/3$ $\therefore t = -20\ln(2/3) = 8.11$ min (3 s.f.)
2	$z^5 + 32 = 0$ $z^5 = -32 = 32e^{(\pi + 2k\pi)i}$ $z = 2e^{(\frac{\pi}{5} + \frac{2k\pi}{5})i}, \quad k = 0, \pm1, \pm2$ $= 2e^{-\frac{3\pi}{5}i}, 2e^{-\frac{\pi}{5}i}, 2e^{\frac{\pi}{5}i}, 2e^{\frac{3\pi}{5}i}, 2e^{\pi i}$
2i	<u>Method 1</u> $\left(\frac{z_1}{z_2{*}}\right)^n = \frac{2^n e^{\left(\frac{n\pi}{5}\right)i}}{2^n e^{\left(-\frac{3n\pi}{5}\right)i}} = e^{\left(\frac{4n\pi}{5}\right)i}$

	For $\left(\dfrac{z_1}{z_2*}\right)^n$ to be real and positive, smallest $n = 5$ <u>Method 2</u> $\arg\left(\dfrac{z_1}{z_2*}\right)^n = n\left[\arg(z_1) - \arg(z_2*)\right]$ $= n\left[\arg(z_1) - \left(-\arg(z_2)\right)\right]$ $= n\left[\dfrac{\pi}{5} + \dfrac{3\pi}{5}\right]$ $= \dfrac{4n\pi}{5}$ For $\left(\dfrac{z_1}{z_2*}\right)^n$ to be real and positive, $\dfrac{4n\pi}{5} = 2k\pi, \quad k \in \mathbb{Z}$ $n = \dfrac{5}{2}k, \quad k \in \mathbb{Z}$, so smallest $n = 5$
2iii	 Let the complex number represented by A' be x+iy BA rotates 90^o about B to get BA': $(x+iy) - 2e^{i\pi} = (-i)\left(2e^{i\frac{\pi}{5}} - 2e^{i\pi}\right)$ $x+iy = (-2) - i\left[2e^{\frac{1}{5}\pi i} - (-2)\right]$ since $e^{i\pi} = -1$ $x+iy = -2 - i\left[2\cos\dfrac{\pi}{5} + 2i\sin\dfrac{\pi}{5} + 2)\right]$ Real part = $-2 - 2i^2\sin\dfrac{\pi}{5} = -2 + 2\sin\dfrac{\pi}{5}$
3(i)	$\overrightarrow{AB} = \begin{pmatrix}4\\2\\4\end{pmatrix} - \begin{pmatrix}6\\0\\0\end{pmatrix} = \begin{pmatrix}-2\\2\\4\end{pmatrix} = 2\begin{pmatrix}-1\\1\\2\end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix}3\\2\\6\end{pmatrix} - \begin{pmatrix}6\\0\\0\end{pmatrix} = \begin{pmatrix}-3\\2\\6\end{pmatrix}$ Normal to plane *ABC* = $\overrightarrow{AB} = \begin{pmatrix}-1\\1\\2\end{pmatrix} \times \begin{pmatrix}-3\\2\\6\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix}$ Equation of plane *ABC* is $\mathrm{r} \bullet \begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix} \bullet \begin{pmatrix}6\\0\\0\end{pmatrix} = 12$

3(ii)	Since S lies on the perpendicular from (3,-1, 4) to plane $x-y+z=1$, $\therefore \overrightarrow{OS} = \begin{pmatrix}3\\-1\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$ (Note S lies along perpendicular from D to p_1. $\therefore$ Line SD: $\underset{\sim}{r} = \overrightarrow{OD} + \lambda \underset{\sim}{n}_{p_1}$) S lies on plane $ABC \Rightarrow \begin{pmatrix}3+\lambda\\-1-\lambda\\4+\lambda\end{pmatrix}\bullet\begin{pmatrix}2\\0\\1\end{pmatrix}=12$ $\Rightarrow 6+2\lambda+4+\lambda=12$ $\Rightarrow \lambda=\frac{2}{3}$ $\therefore \overrightarrow{OS} = \begin{pmatrix}3\\-1\\4\end{pmatrix}+\frac{2}{3}\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}\frac{11}{3}\\-\frac{5}{3}\\\frac{14}{3}\end{pmatrix}$. So S = $(\frac{11}{3}, -\frac{5}{3}, \frac{14}{3})$.
3(iii)	(iii) Let M be the mid-point of DS. (Note a pt. on plane required is the mid pt of D and S) $\overrightarrow{OM} = \frac{\overrightarrow{OS}+\overrightarrow{OD}}{2} = \frac{1}{2}\left[\begin{pmatrix}\frac{11}{3}\\-\frac{5}{3}\\\frac{14}{3}\end{pmatrix}+\begin{pmatrix}3\\-1\\4\end{pmatrix}\right]=\begin{pmatrix}\frac{10}{3}\\-\frac{4}{3}\\\frac{13}{3}\end{pmatrix}$ Notice $\overrightarrow{OM}$ is NOT $\frac{1}{2}\overrightarrow{SD}$. Normal to $p_2 = \begin{pmatrix}1\\-1\\1\end{pmatrix}$ Equation of p_2: $r\bullet\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\-1\\1\end{pmatrix}\bullet\begin{pmatrix}\frac{10}{3}\\-\frac{4}{3}\\\frac{13}{3}\end{pmatrix}=9$ $\Rightarrow r\bullet\begin{pmatrix}1\\-1\\1\end{pmatrix}=9$
4(i)	Let n be the number of intervals between the first hook to the last. $\frac{n}{2}\left[2(50)+(n-1)(-2)\right]\le 500$ $n^2-51n+500\ge 0$ $n\le 13.2$ or $n\ge 37.8$ Thus smallest $n=13$ $n\ge 37.8$ is rejected since any additional intervals beyond 13 will give a total length greater than 500. Since the smallest number of intervals is 13, number of hooks = 13+1 = 14

4(ii)

$$80+80(0.9)+80(0.9)^2+\ldots+80(0.9)^n=\frac{80\left(1-0.9^n\right)}{1-0.9}$$

$$\frac{80\left(1-0.9^n\right)}{1-0.9}>500$$

$$0.9^n<0.375$$

$$n\ln(0.9)<\ln(0.375)$$

$$n>9.31$$

Smallest n = 10

4(iii)

Length of the ribbon between the n^{th} and $(n+1)^{\text{th}}$ hook must satisfy the condition:

$$80\left(0.9\right)^{n-1}\geq 50-2\left(n-1\right)$$

4(iv)

Using GC to solve the inequality in (iii),
$n\leq 9.28$ or $n\geq 21.3$ (N.A since there are only 14 hooks from (i))
Thus the number of hooks with ribbons will be 10.

5(i)

The interviewer will station beside the auditorium exit and interview the first 4 Chinese, 4 Malays, 4 Indians, and 3 from other races that exits the auditorium after the film screening has ended.

[Appropriate strata <u>must be suggested</u> (male/female, race, age group, etc.), and <u>quota must be set for each strata</u>.]

5(ii)

To obtain a stratified sample of size 30 from a population of size 650, draw **random samples** (using simple random sampling) from each mutually exclusive subgroup, divided based on age group, with sample sizes as follows:

Subgroup (age group)	below 21 years old	21 – 35 years old	36 – 50 years old	above 50 years old	Total
Sample size	$\frac{30}{650}\times 130$ $=6$	$\frac{30}{650}\times 195$ $=9$	$\frac{30}{650}\times 260$ $=12$	$\frac{30}{650}\times 65$ $=3$	30

After selecting the 30 people, the surveyor will contact them through telephone using the contact numbers in the registration list.

Difficulty could be encountered in contacting some of the selected people after the event. For stratified sampling, the **person selected that is hard to contact cannot be simply replaced** by another person (this is allowed if it is quota sampling).

6

No of ways $=(3-1)!\times\left(3!\right)^3$ = 432

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6 last part

Method 1 (Direct)
Case 1 : Charles is with Father & his mother is separated.

$$\underbrace{3!}_{\text{3 units}}\times\underbrace{\left(2!\right)^2}_{\text{Andy \& Ben}}\times\underbrace{2!}_{\text{C \& Fc}}\times\underbrace{2}_{\text{Mc}}=96$$

Mc | FaAMa | Mc | FbBMb | C Fc

Case 2 : Charles is with Mother & his father is separated.
No of ways = 96 (same as case 1)

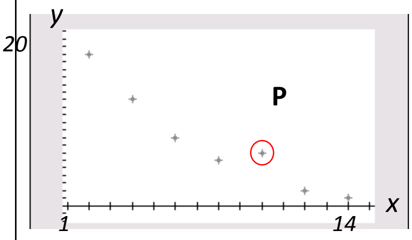
Case 3: Charles ‘s family is together:

FaAMa | FbBMb | CFcMc

$$\underbrace{3!}_{\text{3 units}}\times\underbrace{\left(2!\right)^2}_{\text{Andy \& Ben}}\times\underbrace{3!}_{\text{C family}}=144$$

Total no of ways = 96 x 2 + 144 = 336

	Method 2 (Complement method) Total no of ways if Andy and Ben stand between their parents $= 5!\times(2!)^2 = 480$ $\boxed{\text{FaAMa}}\ \boxed{\text{FbBMb}}\ \boxed{\text{C}}\ \boxed{\text{Mc}}\ \boxed{\text{Fc}}$ No. of ways where Andy and Ben stand between their parents and charles family all separated $= (2!)^2\times 2!\times 3\times 2\times 1 = 48$ $\uparrow\ \boxed{\text{FaAMa}}\ \uparrow\ \boxed{\text{FbBMb}}\ \uparrow$ No. of ways where Andy and Ben stand between their parents, charles parents are together but not with Charles $= 3!\times(2!)^2\times 2!\times 2 = 96$ $\uparrow\ \boxed{\text{FaAMa}}\ \uparrow\ \boxed{\text{FbBMb}}\ \boxed{\text{FcMc}}$ Total = 480 – 48 – 96 = 336
	Method 3: Case 1: Charlie's family are together $\boxed{\text{FaAMa}}\ \boxed{\text{FbBMb}}\ \boxed{\text{C Fc Mc}}$ No of ways = $\underbrace{3!}_{\text{3 units}}\times\underbrace{(2!)^2}_{\text{Andy \& Ben}}\times\underbrace{3!}_{\text{Charlie family}}=144$ Case 2 : Charlie's parents are separated $\uparrow\ \boxed{\text{FaAMa}}\ \uparrow\ \boxed{\text{FbBMb}}\ \uparrow$ No of ways = $\underbrace{2!}_{\text{2 units}}\times\underbrace{(2!)^2}_{\text{Andy \& Ben}}\times\underbrace{3\times 2\times 4}_{\text{Slot in Fc, Mc first, then Charlie has 4 choices to be beside his parents}}=192$ Total number of ways = 144+192 = 336
7	X: Heights of plants supplied. $X \sim \mathrm{N}\left(\mu,\sigma^2\right)$ $P(X<25)=0.06 \Rightarrow P\left(Z<\frac{25-\mu}{\sigma}\right)=0.06 \Rightarrow \frac{25-\mu}{\sigma}=-1.55477$ -- (1) $P(X>91)=0.04 \Rightarrow P\left(Z<\frac{91-\mu}{\sigma}\right)=0.96 \Rightarrow \frac{91-\mu}{\sigma}=1.750686$---- (2) Solving (1) and (2), $\mu=56.044=56.0$, $\sigma=19.967=20.0$ Let m be the required median height of plants accepted by Evergreen. $P(X<m)=0.06+\frac{1-0.06-0.04}{2}=0.51$ $m=56.5$ cm
	Y: number of plants supplied, out of 20, that are rejected. $Y \sim \mathrm{B}(20, 0.1)$ $P(Y\le 4)=0.957$
	$P\left(\frac{X_1+X_2+X_3}{3}\ge 2\left(\frac{X_1+X_2}{2}\right)\right)$ $=P(2X_1+2X_2-X_3\le 0)$ $2X_1+2X_2-X_3 \sim \mathrm{N}\left(168.132, 9\left(20^2\right)\right)$ $=0.00254$ (Answer uses sigma = 20, $\sigma=$5sf, Use 5 sf for both parameters = 0.00250, use mu = 168, $\sigma=20$, get 0.00256)
8i	$r=-0.9550961661=0-0.955$ (to 3 sig.fig) Since r value is close to -1, it suggests a strong negative **linear** correlation **between *x* and *y***, hence a linear model is appropriate.

8ii	
8iii	With point P removed, the remaining points lie close to an exponential curve, as x increases, y decreases at a decreasing rate, hence consistent with a model of the form $y = Ae^{bx}$.
8iv	Since <u>x is the controlled variable</u>, we use the regression line of $\ln y$ on x. From GC, $\ln y = -0.24888x + 3.6276$ When $y = 7$, $\ln 7 = -0.24888x + 3.6276$ $x = 6.75 = 7$ (nearest whole no)
9(i)	P(Mr Red wins in 3rd draw) $= \frac{2}{5} \times \frac{3}{5} \times \frac{2}{5} \times \frac{3}{5} \times \frac{3}{5}$ $= \frac{108}{3125}$ $= 0.03456$
9(ii)	P(Mr Blue wins in n^{th} draw) $= \left(\frac{2}{5} \times \frac{3}{5}\right)^{n-1} \left(\frac{2}{5} \times \frac{2}{5}\right)$ $= \frac{4}{25}\left(\frac{6}{25}\right)^{n-1}$
9(iii)	Required Prob $= \frac{\text{P(Mr Red wins in 3rd draw)}}{\text{P(winner wins in his 3rd draw)}}$ $= \dfrac{\frac{108}{3125}}{\frac{4}{25}\left(\frac{6}{25}\right)^{3-1} + \frac{108}{3125}}$ $= \frac{15}{19}$
9(iv)	P(Mr Blue wins) $= \sum_{n=1}^{\infty} \frac{4}{25}\left(\frac{6}{25}\right)^{n-1}$ $= \dfrac{\frac{4}{25}}{1 - \frac{6}{25}}$ $= \frac{4}{19}$

10	$H_0 : \mu = 1.2$ $H_1 : \mu < 1.2$ Test statistic : $T = \dfrac{\overline{X} - \mu}{S/\sqrt{n}} \sim t(9)$ Reject H_0 if p-value < 0.10 $\overline{x} = 1.1021,\ s = 0.179503,\quad n = 10$ By using GC, p-value = 0.059335 < 0.10. Reject H_0. There is sufficient evidence to say that the mean weight of melons is less than 1.2 kg, i.e. the farmer's claim is incorrect. $H_0 : \mu = 1.2$ $H_1 : \mu \neq 1.2$ H_0 is not rejected $\Rightarrow \alpha < 5.9335 \times 2$ $\Rightarrow \alpha < 11.9$ (3 s.f.)
10(i)	1st sample : $n = 10$, $\sum x = 11.021$ and $\sum x^2 = 12.436237$ 2nd sample : $n = 40$, $\sum x = 45.738$ and $\sum x^2 = 72.576$ Combined : $n = 50$, $\sum x = 56.759$ and $\sum x^2 = 85.012237$ Unbiased estimate of population mean $= \dfrac{56.759}{50} = 1.13518$ Unbiased estimate of population variance $= \dfrac{1}{49}\left[85.012237 - \dfrac{(56.759)^2}{50}\right] = 0.42001$
10ii	$H_0 : \mu = m$ $H_1 : \mu < m$ Since $n = 50$, we conduct Z-test. Test statistic : $Z = \dfrac{\overline{X} - \mu}{s/\sqrt{n}} \sim N(0,1)$ Do not reject H_0 if $\dfrac{1.13518 - m}{\sqrt{0.42001}/\sqrt{50}} > -1.64485$ $\Rightarrow 0 \le m < 1.29$ (3 sf.)
11a	Let X and Y denotes the number of chocolate and banana muffins sold respectively. In 1 hour, $X \sim P_o(3)$, $Y \sim P_o(\lambda)$, $X + Y \sim P_o(3+\lambda)$ $P(X + Y = 2) = P(Y = 4)$ $\dfrac{e^{-(3+\lambda)}(3+\lambda)^2}{2!} = \dfrac{e^{-\lambda}\lambda^4}{4!}$ $\dfrac{e^{-3}(3+\lambda)^2}{2} = \dfrac{\lambda^4}{24}$ Or simply use GC to sketch y=P(X+Y=2) – P(Y=4) to find λ at the x-intercept. $12e^{-3}(3+\lambda)^2 - \lambda^4 = 0$ Using GC, $\lambda = 1.95752 = 1.96 (3sf)$

	For 8 hours, $X \sim P_o(24), \quad Y \sim P_o(15.66016)$ Since both means >10, Using normal approximation, $X \sim N(24, 24) \quad and \quad Y \sim N(15.66016, 15.66016)$ $X - Y \sim N(8.33984, 39.66016)$ $P(X > Y) = P(X - Y > 0) = P(X - Y > 0.5)$ cc $= 0.8934 = 0.893(3sf)$
11b (i)	$p > 0.6$ $\Rightarrow \frac{2}{65}(40 - d) > 0.6$ $\Rightarrow 40 - d > 19.5$ $\Rightarrow d < 20.5$ Maximum distance of d = 20 metres.
11b (ii)	Let W denote the number of kicks that hit the net $W \sim B(15, p)$ $P(W \geq 2) = 0.9$ $1 - P(W \leq 1) = 0.9$ $P(W \leq 1) = 0.1$ $(1-p)^{15} + 15p(1-p)^{14} = 0.1$ $(1-p)^{14}(1+14p) - 0.1 = 0$ Using GC, p=0.23557 $0.23557 = \frac{2}{65}(40 - x)$ $\Rightarrow x = 32.343975$ $\Rightarrow x = 32\ metres\,(nearest\ metres)$
11b (iii)	Let S be the number of kicks that hit into the net out of 100 kicks at a distance of 24 metres from the goalpost. $S \sim B(100, 0.4923077)$ $E(S) = 100 \times 0.4923077 = 49.231$ $Var(S) = 100 \times 0.4923077 \times 0.5076923 = 24.9941$ Since sample size is large (60 days), by CLT, $\bar{S} \sim N(49.231, \frac{24.9941}{60})$ Required prob = $P(\lvert \bar{S} - 50 \rvert < 1) = P(49 < \bar{S} < 51) = 0.637(3sf)$