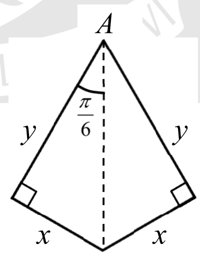
Raffles Institution
H2 Mathematics
Solution for 2013 A-Level Paper 2

Section A: Pure Mathematics

Question 1

No.	Suggested Solution	Remarks for Student
(i)	We have $D_{\mathrm{f}} = \mathbb{R}\backslash\{1\}$ and $R_{\mathrm{g}} = \mathbb{R}$. Since $R_{\mathrm{g}} \not\subset D_{\mathrm{f}}$, the composite function fg does not exist.	
(ii)	We have $\begin{aligned} \mathrm{gf}(x) &= 1 - 2\mathrm{f}(x) \\ &= 1 - 2\left(\frac{2+x}{1-x}\right) \\ &= 1 - 2\left(-1 - \frac{3}{x-1}\right) \\ &= 3 + \frac{6}{x-1}. \end{aligned}$ If $x = (\mathrm{gf})^{-1}(5)$, then $5 = \mathrm{gf}(x)$, so $5 = 3 + \frac{6}{x-1} \quad \Rightarrow \quad x = 4.$ Therefore $(\mathrm{gf})^{-1}(5) = 4$.	It is also fine to write this as $\frac{3(x+1)}{x-1}$. The form given in the answer makes the next part easier.

Question 2

No.	Suggested Solution	Remarks for Student
(i)	Consider one of the kite shapes:  We have $\tan\frac{\pi}{6} = \frac{x}{y} \quad \Rightarrow \quad y = x\sqrt{3}$. The base of the prism is an equilateral triangle with side $a - 2y = a - 2x\sqrt{3}$. The area of this base, S, is $\begin{aligned} S &= \frac{1}{2} \times (\text{base}) \times (\text{height}) \\ &= \frac{1}{2}\left(a - 2x\sqrt{3}\right)\left[\left(a - 2x\sqrt{3}\right)\sin\frac{\pi}{3}\right] \\ &= \frac{\sqrt{3}}{4}\left(a - 2x\sqrt{3}\right)^2. \end{aligned}$	

	Therefore $V = Sx = \frac{1}{4}x\sqrt{3}\left(a-2x\sqrt{3}\right)^2.$	
(ii)	We have $\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{\sqrt{3}}{4}\left[\left(a-2x\sqrt{3}\right)^2 - 4x\sqrt{3}\left(a-2x\sqrt{3}\right)\right]$ $= \frac{\sqrt{3}}{4}\left(a-2x\sqrt{3}\right)\left(a-6x\sqrt{3}\right).$ At stationary values, $\frac{\mathrm{d}V}{\mathrm{d}x} = 0 \Rightarrow \frac{\sqrt{3}}{4}\left(a-2x\sqrt{3}\right)\left(a-6x\sqrt{3}\right) = 0$ $\Rightarrow x = \frac{a}{6\sqrt{3}}$ or $x = \frac{a}{2\sqrt{3}}$ (reject). We have $\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = \frac{\sqrt{3}}{4}\left[\left(-2\sqrt{3}\right)\left(a-6x\sqrt{3}\right)+\left(-6\sqrt{3}\right)\left(a-2x\sqrt{3}\right)\right]$ $= 3\left(-2a+6x\sqrt{3}\right).$ At $x = \frac{a}{6\sqrt{3}}$, $\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = -3a < 0$, so this gives a maximum value of $V = \frac{1}{4}\left(\frac{a}{6\sqrt{3}}\right)\sqrt{3}\left(a-2\frac{a}{6\sqrt{3}}\sqrt{3}\right)^2 = \frac{a^3}{54}.$	Product rule. $x = \frac{a}{2\sqrt{3}}$ is rejected as it is too large, giving a "base" with zero length. For "real life" problems, you should use the second derivative test.

Question 3

No.	Suggested Solution	Remarks for Student
(i)	Let $y = \mathrm{f}(x) = \ln(1+2\sin x)$, so $\mathrm{e}^y = 1+2\sin x$. Differentiating implicitly gives $\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos x$ $\mathrm{e}^y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \mathrm{e}^y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = -2\sin x$ $\mathrm{e}^y\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 3\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \mathrm{e}^y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 = -2\cos x$ When $x=0$, $\mathrm{e}^y = 1 \Rightarrow y = 0$, and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 14.$ Therefore $\mathrm{f}(0) = 0$, $\mathrm{f}'(0) = 2$, $\mathrm{f}''(0) = -4$ and $\mathrm{f}'''(0) = 14$.	A direct differentiation approach gets messy real fast. For such problems, use implicit differentiation as far as possible.

	Maclaurin's Theorem gives $$\mathrm{f}(x)=\mathrm{f}(0)+\frac{\mathrm{f}'(0)}{1!}x+\frac{\mathrm{f}''(0)}{2!}x^2+\frac{\mathrm{f}'''(0)}{3!}x^3+\cdots$$ $$=2x-2x^2+\frac{7}{3}x^3+\cdots$$	You can use the standard series from MF15 to verify this answer if you have time. A faster, but less reliable way to check is to pick a small value of x, say 0.1, and verify that the value from this expression agrees with the value of f(0.1) from the GC.
(ii)	We have $$\mathrm{e}^{ax}\sin nx=\left(1+ax+\frac{(ax)^2}{2!}+\frac{(ax)^3}{3!}+\cdots\right)\left(nx-\frac{(nx)^3}{3!}+\cdots\right)$$ $$=nx+anx^2+\left(\frac{a^2n}{2}-\frac{n^3}{6}\right)x^3+\cdots,$$ so comparing with $\mathrm{f}(x)=2x-2x^2+\frac{7}{3}x^3+\cdots$ gives $n=2$ and $an=-2 \Rightarrow a=-1$. The next term in the series of $\mathrm{e}^{ax}\sin nx$ is $$\left(\frac{(-1)^2 2}{2}-\frac{2^3}{6}\right)x^3=-\frac{1}{3}x^3.$$	Thankfully this is non-zero, or we will have to find the next term.

Question 4

No.	Suggested Solution	Remarks for Student
(i)	The acute angle between p_1 and p_2 is $$\cos^{-1}\frac{\left\|\begin{pmatrix}2\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}-6\\3\\2\end{pmatrix}\right\|}{\left\|\begin{pmatrix}2\\-2\\1\end{pmatrix}\right\|\left\|\begin{pmatrix}-6\\3\\2\end{pmatrix}\right\|}=\cos^{-1}\frac{\|-12-6+2\|}{\sqrt{9}\sqrt{49}}=\cos^{-1}\frac{16}{21}=40.4^\circ.$$	
(ii)	We have $p_1: 2x-2y+z=1$ and $p_2: -6x+3y+2=-1$. Solving simultaneously with GC gives $$x=-\frac{1}{6}+\frac{7}{6}z$$ $$y=-\frac{2}{3}+\frac{5}{3}z.$$	The line of intersection can be obtained by solving the equations of the planes in Cartesian form simultaneously.

	Therefore a vector equation for l is $$\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1/6\\-2/3\\0\end{pmatrix}+\lambda\begin{pmatrix}7/6\\5/3\\1\end{pmatrix},\quad \lambda\in\mathbb{R}.$$ This can be simplifies to $$l:\ \mathbf{r}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\mu\begin{pmatrix}7\\10\\6\end{pmatrix},\quad \mu\in\mathbb{R}.$$	For this question there is no real need to simplify this equation. However, if the equation is used again later, simplifying helps. Picking $\lambda=1$ gives the point $\begin{pmatrix}1\\1\\1\end{pmatrix}$ on l, and of course the direction can be changed to a nicer multiple.
(iii)	Distance from A to p_1 is $$\frac{\left|\left(\begin{pmatrix}4\\3\\c\end{pmatrix}-\begin{pmatrix}1\\1\\1\end{pmatrix}\right)\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}\right|}{\left|\begin{pmatrix}2\\-2\\1\end{pmatrix}\right|}=\frac{1}{3}\left|\begin{pmatrix}3\\2\\c-1\end{pmatrix}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}\right|=\frac{1}{3}\lvert c+1\rvert.$$ Similarly, distance from A to p_2 is $$\frac{\left|\left(\begin{pmatrix}4\\3\\c\end{pmatrix}-\begin{pmatrix}1\\1\\1\end{pmatrix}\right)\cdot\begin{pmatrix}-6\\3\\2\end{pmatrix}\right|}{\left|\begin{pmatrix}-6\\3\\2\end{pmatrix}\right|}=\frac{1}{7}\left|\begin{pmatrix}3\\2\\c-1\end{pmatrix}\cdot\begin{pmatrix}-6\\3\\2\end{pmatrix}\right|=\frac{2}{7}\lvert c-7\rvert.$$ Therefore $$\frac{1}{3}\lvert c+1\rvert=\frac{1}{7}\lvert 2c-14\rvert \Rightarrow [7(c+1)]^2-[6(c-7)]^2=0$$ $$\Rightarrow (13c-35)(c+49)=0$$ $$\Rightarrow c=\frac{35}{13}\quad\text{or}\quad c=-49.$$	We used the point $\begin{pmatrix}1\\1\\1\end{pmatrix}$ that is on both planes. Square to remove modulus on both sides.

Section B: Statistics

Question 5

No.	Suggested Solution	Remarks for Student
(i)	Using a list of all the employees in alphabetical order, assign each employee a unique number from 1 to 100 000. We then use a random number generator to generate 90 random numbers from 1 to 100 000, and select the 90 employees assigned these numbers to form the required sample. Since the selection process is random, it might happen by chance that some countries might not have any employee in the sample. Hence the sample may not be representative.	Be concise and to the point. The time spent for each part should be proportional to the marks allocated, so don't spend too much time here.
(ii)	A more appropriate method to a get a good representation is stratified sampling. Group the employees into different strata according to the country they are based. Then, using simple random sampling, select the number of employees from each strata based on the percentage of employees in each country. For example, if r% of employees are from Singapore, select $\frac{9r}{10}$ employees from Singapore.	Either quota or stratified sampling will work to ensure that each country is represented.

Question 6

No.	Suggested Solution	Remarks for Student
	We are given $Y \sim \mathrm{N}(\mu, \sigma^2)$. $\mathrm{P}(Y < 2a) = 0.95 \Rightarrow \mathrm{P}\left(Z < \frac{2a-\mu}{\sigma}\right) = 0.95$ $\Rightarrow \frac{2a-\mu}{\sigma} = 1.644853$ $\Rightarrow \mu + 1.644853\sigma = 2a$ ----(1) $\mathrm{P}(Y < a) = 0.25 \Rightarrow \mathrm{P}\left(Z < \frac{a-\mu}{\sigma}\right) = 0.25$ $\Rightarrow \frac{a-\mu}{\sigma} = -0.674490$ $\Rightarrow \mu - 0.674490\sigma = a$ ----(2) $(1)-(2)$ gives $2.319343\sigma = a \Rightarrow \sigma = 0.431157a$. Therefore $\mu = 0.674490\sigma + a = 1.29a$ (3 s.f.), giving $k = 1.29$.	Since both mean and variance is unknown, we have to standardize. Use `invNorm` from GC. `invNorm` again. With unknown a, we have to solve (1) and (2) by hand. Note more s.f. for intermediate values to maintain accuracy. Or store the values in GC.

Question 7

No.	Suggested Solution	Remarks for Student
(i)	The assumptions are : • Whether a packet contains a free gift is independent of whether other packets contain a free gift. • The probability that a packet contains a free gift is the same for all packets.	The usual conditions to use the binomial distribution, phrased in context. Don't state the obvious, like "a packet either contains a free gift or it does not".
(ii)	When $n = 20$, we have $F \sim \mathrm{B}\left(20, \frac{1}{20}\right)$, so $\mathrm{P}(F = 1) = 0.377354 = 0.377$ (3 s.f.)	Use `binompdf`.
(iii)	When $n = 60$, $F \sim \mathrm{B}\left(60, \frac{1}{20}\right)$. Since $n = 60$ is large, $p = \frac{1}{20} = 0.05$ is small, and $np = 3 < 5$, $F \sim \mathrm{Po}(3)$ approximately. Therefore $\mathrm{P}(F \geq 5) = 1 - \mathrm{P}(F \leq 4) = 0.18474 = 0.185$ (3s.f.).	Be sure to check the conditions and state the approximate distribution used.

Question 8

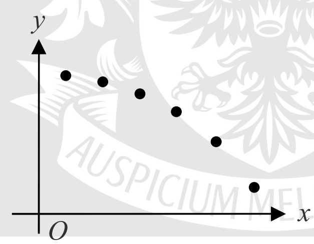
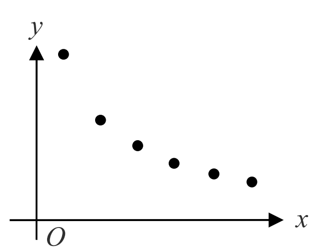
No.	Suggested Solution	Remarks for Student
(i)	$\mathrm{P}(B \mid A') = \frac{\mathrm{P}(B \cap A')}{\mathrm{P}(A')} \Rightarrow \mathrm{P}(B \cap A') = \mathrm{P}(B \mid A')\mathrm{P}(A')$ $= (0.8)(1 - 0.7)$ $= 0.24.$	Recall definition of conditional probability.
(ii)	We have $\mathrm{P}(A \cup B) = \mathrm{P}(B \cap A') + \mathrm{P}(A) = 0.24 + 0.7 = 0.94$, so $\mathrm{P}(A' \cap B') = 1 - \mathrm{P}(A \cup B) = 1 - 0.94 = 0.06.$	Use a Venn diagram to see this.
(iii)	$\mathrm{P}(A \mid B') = \frac{\mathrm{P}(A \cap B')}{\mathrm{P}(B')} = 0.88 \Rightarrow \mathrm{P}(A \cap B') = 0.88\mathrm{P}(B')$ Therefore $\mathrm{P}(A \cup B) - \mathrm{P}(B) = 0.88\left[1 - \mathrm{P}(B)\right]$ $\Rightarrow \mathrm{P}(B) = \frac{0.94 - 0.88}{0.12} = 0.5.$ This gives $\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B)$ $= 0.7 + 0.5 - 0.94 = 0.26.$	Same trick as **(ii)**.

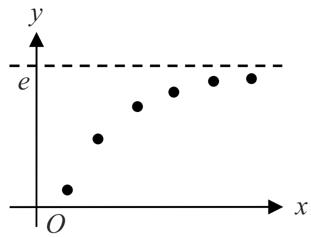
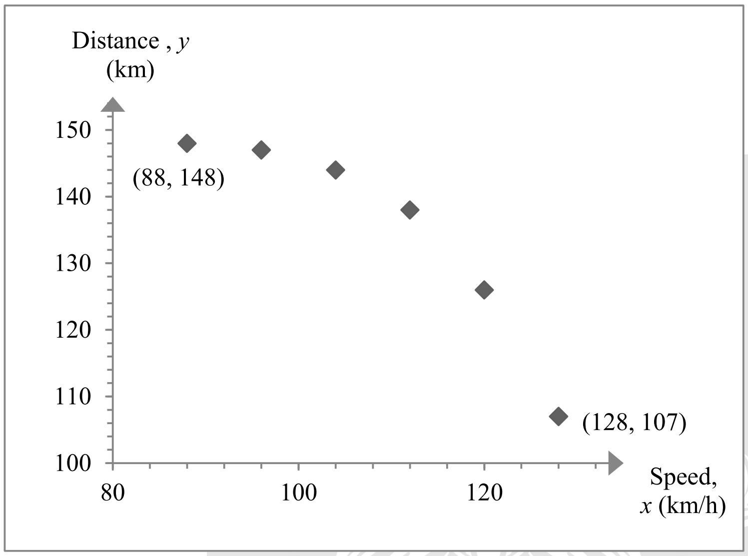
Question 9

No.	Suggested Solution	Remarks for Student
(i)	The unbiased estimates for the population mean and population variance are 12.8 and 2.31 respectively.	Straight from GC. Just remember that the GC gives s, not s^2.
(ii)	A necessary assumption is that the data is obtained from an unbiased sample. We test $H_0: \mu = 13.8$ against $H_1: \mu < 13.8$. Perform an one-tailed test at 5% significance level. Under H_0, $$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t(n-1), \text{ where } \mu_0 = 13.8.$$ From the sample, $n = 8$, $\bar{x} = 12.8$ and $s = 2.30571$. Using t-test, p-value = 0.052398. Since p-value = 0.0524 > 0.05, we do not reject H_0 and conclude that, at 5% level of significance, there is insufficient evidence to support the editor's belief that the distance travelled per litre of fuel by this model of car is less than 13.8 km.	The usual assumption of normality is given in the question, so it becomes tricky. From the context, the data is essentially obtained via an informal survey, so we need to assume that it is a random sample. Assumption aside, this is a standard t-test question.

Question 10

No.	Suggested Solution	Remarks for Student

(i)	For (A): $y = a + bx^2$, with $a > 0$ and $b < 0$.
	
	
	For (B): $y = c + d\ln x$, with $c > 0$ and $d < 0$.
		

	For (C): $y = e + \frac{f}{x}$, with $e > 0$ and $f < 0$.  	
(ii)	 	
(iii)	Since the values of y are decreasing at an increasing rate with increasing values of x, just like in the scatter plot (A), it is apparent that (A) is the most appropriate model. Using model (A), the product moment correlation coefficient is -0.939 (3 s.f.).	
(iv)	The required equation is $y = 189.7475 - 0.004619785x^2$, which is $y = 190 - 0.00462x^2$ to 3 s.f. At a speed of 110 km/h, the distance travelled is estimated to be $y = 189.7475 - 0.004619785(110)^2 = 134$ (3 s.f.).	Since this is a linearized model, be careful with the variables.

Question 11

No.	Suggested Solution	Remarks for Student
(i)	Required probability $= 1 \times \frac{25}{26} \times \frac{24}{26} \times 1 \times \frac{8}{9} = \frac{400}{507}$.	First letter can be any letter, the second cannot be the same as the first, etc.
(ii)	Consider cases: first digit is '1', first digit is '2', etc. Required probability $= \frac{1}{9}\left(\frac{8}{9} + \frac{7}{9} + \cdots + \frac{1}{9}\right) = \frac{4}{9}$.	Note that we don't have to look at the letters: we don't care what they are.

	Alternative method: No. of codes with second digit higher than the first is ${}^{9}C_2$: pick any two digits, and arrange them so that the second is larger. Required probability $= \frac{{}^{9}C_2}{9^2} = \frac{4}{9}$.	We use $\frac{\text{required cases}}{\text{total cases}}$.
(iii)	**Case 1**: Exactly two letters the same and different digits. No. of codes $= {}^{26}C_2 \times 2 \times \frac{3!}{2!} \times {}^{9}P_2 = 140400$. **Case 2**: Two digits the same and three different letters. No. of codes $= {}^{26}P_3 \times {}^{9}C_1 = 140400$. **Case 3**: Two digits the same and three letters the same. No. of codes $= {}^{26}C_1 \times {}^{9}C_1 = 234$. Therefore the required probability is $\frac{140400 + 140400 + 234}{26^3 \times 9^2} = 0.197403 = 0.197$ (3 s.f.)	This part is really tricky. The third case is often missed out. Note that answer to 3 s.f. is unchanged if the third case is missed, but students will be penalized.
(iv)	**Case 1**: One vowel and two consonants the same. No. of codes $= {}^{5}C_1 \times {}^{21}C_1 \times \frac{3!}{2!} \times {}^{4}C_1 \times {}^{5}C_1 \times 2! = 12600$. **Case 2**: One vowel and two different consonants. No. of codes $= {}^{5}C_1 \times {}^{21}C_2 \times 3! \times {}^{4}C_1 \times {}^{5}C_1 \times 2! = 252000$. Therefore the required probability is $\frac{12600 + 252000}{26^3 \times 9^2} = 0.18586 = 0.186$ (3 s.f.)	Both cases have an odd and an even digit. Note that for both **(iii)** and **(iv)**, the complement method is not recommended.

Question 12

No.	Suggested Solution	Remarks for Student
(i)	The two conditions are : • Whether an employee is absent through illness is independent of whether other employees are also absent through illness. • The average rate of employees' absence due to illness is constant over time. The first condition may fail because of infections being passed from one employee to another.	

The average rate of employees' absence due to illness may also vary over time due to seasonal factors. For example, during flu season, more employees may be taken ill.

(ii)

Let X be the number of employees from the Administration Department who are absent through illness in n days. Then $X \sim \text{Po}(1.2n)$.

$$\text{P}(X=0)<0.01 \quad \Rightarrow \quad e^{-1.2n}\frac{(1.2n)^0}{0!}<0.01$$
$$\Rightarrow \quad e^{-1.2n}<0.01$$
$$\Rightarrow \quad n>\frac{\ln 0.01}{-1.2}=3.84.$$

Therefore the smallest number of days is 4.

Formula for $\text{P}(X=k)$ is in MF15.

Remember the inequality changes direction when dividing by negative numbers.

Alternative Method:
We can use the GC to tabulate values of $\text{P}(X=0)$ for different values of n:

n	$\text{P}(X=0)$
3	0.02732
4	0.00823

Therefore 4 is the least number of days for $\text{P}(X=0)$ to drop below 0.01.

Table of values can only be used when the quantity you want to find is an integer.

(iii)

Let Y_A and Y_M be the numbers of days of absence in a 5-day period from the Administrative and Manufacturing departments respectively.
We have $Y_A \sim \text{Po}(6)$ and $Y_M \sim \text{Po}(13.5)$.
Since the absences in the two departments are independent, $Y_A+Y_M \sim \text{Po}(19.5)$.
Therefore

$$\text{P}(Y_A+Y_M>20)=1-\text{P}(Y_A+Y_M\le 20)$$
$$=0.39658=0.397 \quad (3\text{ s.f.})$$

The parameters for the Poisson distributions scale to 5 days proportionally.

The parameters add for independent Poisson distributions.

(iv)

Let W be the number of days of absence in a 60-day period.
As in (iii),

$$W \sim \text{Po}(60\times 1.2+60\times 2.7)$$
$$\Rightarrow \quad W \sim \text{Po}(234).$$

Since $\lambda=234>10$, $W \sim \text{N}(234,234)$ approximately.

$$\text{P}(200\le W\le 250)=\text{P}(199.5\le X\le 250.5)\text{(by continuity correction)}$$
$$=0.84757=0.848 \quad (3\text{ s.f.})$$