F08: Further Complex Numbers

1 **(a)** Show that if $z = \mathrm{e}^{\mathrm{i}\theta}$, then

$$z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta,$$

where n is a positive integer. [1]

Hence, or otherwise, show that $\sin^5\theta$ can be expressed in the form

$$a\sin\theta + b\sin 3\theta + c\sin 5\theta,$$

where the numbers a, b and c are to be determined. [3]

Deduce that $\cos^5\theta$ can be expressed in the form

$$p\cos\theta + q\cos 3\theta + r\cos 5\theta,$$

where the numbers p, q and r are to be determined. [2]

(b) On the same Argand diagram sketch the loci of points given by each of the following equations:

$$L_1: \ |z + 3 - 3\mathrm{i}| = 3\sqrt{2},$$

$$L_2: \ \arg\left(z - 3\sqrt{2} + 3 - 3\mathrm{i}\right) = \frac{5\pi}{6}.$$

Find, in the form $x + \mathrm{i}y$, the exact complex number which represents the point on both L_1 and L_2 in the Argand diagram. [5]

(2016 RI / JC1 / MYE / Q6)

2 **Do not use a calculator in answering this question.**

The complex number z is given by $1 + (\sqrt{3})\mathrm{i}$.

(a) Let $\mathrm{P}(w) = 2w^3 + aw^2 + 10w + b$, where a and b are real. If $\mathrm{P}(z) = 0$, find the values of a and b and determine all the roots of the equation $\mathrm{P}(w) = 0$. [4]

(b) Find the smallest positive integer n such that $(2z + z^*)^n$ is a positive real number. [3]

(2016 ACJC / JC1 / Promo / Q2)

3 Consider the polynomial $\mathrm{P}(z) = z^4 + z^3 + z^2 + z + 1$.

(i) By considering $(z-1)\mathrm{P}(z)$, find the solutions to the equation $\mathrm{P}(z) = 0$, expressing the solutions in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [3]

(ii) Show that $\dfrac{\mathrm{P}(z)}{z^2} = w^2 + w - 1$, where $w = z + \dfrac{1}{z}$. Determine the exact values of w such that $\mathrm{P}(z) = 0$. [3]

(iii) Using the results from parts **(i)** and **(ii)**, find the exact value of $\cos\left(\dfrac{2\pi}{5}\right)$ in surd form. [3]

(2016 ACJC / JC1 / Promo / Q7)

4 Two complex numbers z_1 and z_2, where $0<\arg(z_1)<\arg(z_2)<\frac{\pi}{2}$, are roots to the equation $z^6-\left(32\sqrt{2}+32\sqrt{2}\mathrm{i}\right)=0$.

(i) Show that $z_1=2\mathrm{e}^{\mathrm{i}\frac{\pi}{24}}$ and $z_2=2\mathrm{e}^{\mathrm{i}\frac{3\pi}{8}}$. [3]

(ii) Show all the roots to the equation on an Argand diagram. [2]

(iii) Given that z_1 and z_2 satisfy the equation $|z-w|=r$, state, in exact forms, the cartesian equation of the line that the point corresponding to w lies on and the minimum value of r. [2]

(2016 CJC / JC1 / Promo / Q4)

5 It is given that the complex number $z=1+\cos\theta+\mathrm{i}\sin\theta$, where $-\pi<\theta\le\pi$.

(i) By considering appropriate trigonometric identities, or otherwise, show that the argument of z is $\frac{\theta}{2}$ and find the modulus of z in terms of θ. [3]

(ii) Hence, find the real and imaginary parts of $(1+\cos\theta+\mathrm{i}\sin\theta)^n$, where $n\in\mathbb{Z}^+$. [3]

(iii) By considering the binomial expansion of $\left[1+(\cos\theta+\mathrm{i}\sin\theta)\right]^n$, show that

$$1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos(2\theta)+\cdots+\binom{n}{n}\cos(n\theta)=\left[2\cos\left(\frac{\theta}{2}\right)\right]^n\cos\left(\frac{n\theta}{2}\right),$$

where $\binom{n}{r}=\frac{n!}{r!(n-r)!}$. [3]

(2016 CJC / JC1 / Promo / Q7)

6 **(i)** Express $\frac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $0\le\theta<2\pi$. [2]

(ii) Hence find the smallest positive value of n for which $\left(\frac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}}\right)^n$ is real and positive. [3]

(2016 IJC / JC1 / Promo / Q3)

7 Write down, in the form $r\mathrm{e}^{\mathrm{i}\theta}$, the five roots of the equation $w^5-1=0$.

Hence show that the roots of the equation $(1+z)^5-(1-z)^5=0$ are $\mathrm{i}\tan\left(\frac{k\pi}{5}\right)$, where $k=0,\pm1,\pm2$. [6]

(2016 IJC / JC1 / Promo / Q4)

8 The complex numbers z_1 and z_2 are given by $1+\sqrt{3}\,\mathrm{i}$ and $1+\mathrm{i}$ respectively.

(i) Find $\frac{z_1}{z_2}$ in the form $x+y\,\mathrm{i}$, giving x and y in the exact form. [3]

(ii) By considering the exponential forms of z_1 and z_2, show that $\tan\left(\frac{\pi}{12}\right)=2-\sqrt{3}$. [4]

(2016 IJC / JC1 / Promo / Q8)

9 **(a)** **(i)** Solve $z^5 = \mathrm{i}$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0, -\pi \le \theta < \pi$. [3]

(ii) Hence, solve the equation $\left(\frac{z}{\sqrt{2}+\sqrt{2}\mathrm{i}}\right)^5 = \mathrm{i}$, giving your answers in a similar form. [3]

(b) If z is a non-zero complex number, we define $K(z)$ by the equation

$$K(z) = \ln|z| + \mathrm{i}\arg(z), \quad -\pi < \arg(z) \le \pi.$$

Show that $K(z_1 z_2) = K(z_1) + K(z_2)$. [2]

(2016 JJC / JC1 / Promo / Q6)

10 **(a)** On the same Argand diagram, sketch the loci of points given by each of the following equations:

$$L_1 : |z + 2 - \mathrm{i}| = \sqrt{5},$$
$$L_2 : \arg(z + 3 + \mathrm{i}) = \alpha, \text{ where } \alpha = \tan^{-1} 2.$$

Find, in the form $x + \mathrm{i}y$, the complex number which represents the point in the Argand diagram which is on both L_1 and L_2, giving the exact values of x and y. [5]

(b) **(i)** Show that $\frac{1}{1-\mathrm{e}^{\mathrm{i}\theta}\cos\theta} = 1 + \mathrm{i}\cot\theta$, where $0 < \theta < \frac{\pi}{2}$. [2]

(ii) Given that $|z| \le 1$ and $0 < \theta < \frac{\pi}{2}$, state the sum of the infinite series

$$z + z^2\cos\theta + z^3\cos^2\theta + \cdots.$$ [1]

(iii) By putting $z = \mathrm{e}^{\mathrm{i}\theta}$ in **(ii)** and using the result in **(i)**, find

$$\sin\theta + \sin 2\theta\cos\theta + \sin 3\theta\cos^2\theta + \sin 4\theta\cos^3\theta + \cdots,$$

simplifying your answer. [3]

(2016 RI / JC1 / Promo / Q6)

11 **(i)** Show that if $z = \mathrm{e}^{\mathrm{i}\theta}$, then

$$z^k - \frac{1}{z^k} = 2\mathrm{i}\sin k\theta,$$

where k is a positive integer. [1]

(ii) Show that $\sin^5\theta$ can be expressed in the form

$$A\sin\theta + B\sin 3\theta + C\sin 5\theta,$$

where the values of A, B and C are to be determined. [4]

(iii) Find the particular solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\mathrm{e}^x \operatorname{cosec} y\right)^5$, given that $y = 0$ when $x = 0$. [3]

(2016 TJC / JC1 / Promo / Q6)

12 **(i)** Sketch the locus of z that satisfies $|z-2-\mathrm{i}|\le 2$ and $\mathrm{Im}(z)>1$. [3]

(ii) Find the maximum and minimum values of $|z+2+\mathrm{i}|$. [3]

(iii) Find z in the form $x+\mathrm{i}y$ where $x, y\in\mathbb{R}$, such that $\arg(z+2+\mathrm{i})$ is a maximum. [6]

(2016 TJC / JC1 / Promo / Q10)

13 **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z which satisfy both the inequalities

$$|z-1-4\mathrm{i}|^2\le 5 \text{ and } |z+2|\ge|z-2-12\mathrm{i}|.$$ [5]

(ii) Determine exactly the greatest and least possible values of $|z+3|$ for points in this region. [5]

(2017 ACJC / JC2 / BT / Q1)

14 The complex number z satisfies the inequalities

$$|z|\ge|z+2+2\mathrm{i}| \text{ and } \frac{5\pi}{4}<\arg(-2-2\mathrm{i}-z)\le\frac{3\pi}{2}.$$

(i) Sketch the locus of z on an Argand diagram. [5]

(ii) Find the exact range of $\arg(z+3\mathrm{i})$. [3]

(2017 TJC / JC2 / BT / Q6)

15 **(a)** Solve $z^3=-4\sqrt{2}(1+\mathrm{i})$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [3]

The complex number z_1 is a root of $z^3=-4\sqrt{2}(1+\mathrm{i})$ with $-\frac{\pi}{2}<\arg(z_1)<0$.

Given that $|w-z_1|=w$ for some complex number w, find w, showing your working clearly. [2]

(b) On the same Argand diagram, sketch the loci of the points given by each of the following equations:

$$L_1: \left|z-1-\mathrm{i}\sqrt{3}\right|=2,$$

$$L_2: \arg\left(z+1-\mathrm{i}\left(2+\sqrt{3}\right)\right)=-\frac{\pi}{4}.$$ [3]

The complex numbers z_1 and z_2 represent the two points in the Argand diagram which are on both L_1 and L_2.

Given that $\mathrm{Re}(z_2)>\mathrm{Re}(z_1)$, find $\arg(z_2)$ without using a calculator, leaving your answer in terms of π. [3]

(2017 ACJC / JC2 / MYE / Q2)

16 Show that

$$zz^* + c(z + z^*) + d = 0,$$

where c, d are real numbers and $c^2 > d$, represents a circle in the Argand diagram, stating its centre and radius. [4]

Shade in an Argand diagram the region for which

$$zz^* + 3(z + z^*) + 5 < 0.$$ [2]

(2017 CJC / JC2 / MYE / Q1)

17 The variable complex number z satisfies the following inequalities :

$$\left|\mathrm{i}z - \sqrt{3}\mathrm{i} - 1\right| \le 2 \text{ and } |z| \ge \left|\mathrm{i}z - \sqrt{3}\mathrm{i} - 1\right|.$$

(i) On an Argand diagram, sketch the region R satisfied by the point P which represents z. [4]

(ii) Find the range of $\left|z + \sqrt{3} + \mathrm{i}\right|$. [2]

(iii) Find the exact value of z where $\arg(z + \sqrt{3} + \mathrm{i})$ is least. [2]

(2017 DHS / JC2 / MYE P1 / Q5)

18 **(a)** **(i)** Find the roots of the equation $z^8 + 1 = 0$ in the form $r\mathrm{e}^{\mathrm{i}\theta}, r > 0, -\pi < \theta \le \pi$. [2]

(ii) On the Argand diagram, the points which represent the roots in **(i)** are rotated about the origin O in the clockwise sense by $\frac{\pi}{32}$ followed by scaling with a factor of $\sqrt{2}$ to obtain the points A, B, C, D, E, F, G and H.
Find the equation in the form $z^8 = a + \mathrm{i}b,\ a, b \in \mathbb{R}$, whose roots are represented by A, B, C, D, E, F, G and H on the Argand diagram. [3]

(b) **(i)** Show that $1 + \cos\theta + \mathrm{i}\sin\theta = 2\cos\left(\frac{1}{2}\theta\right)\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}$. [2]

(ii) Use de Moivre's theorem to show that

$$\left(\frac{1}{1 + \cos\theta + \mathrm{i}\sin\theta}\right)^n + \left(\frac{1}{1 + \cos\theta - \mathrm{i}\sin\theta}\right)^n = \frac{\cos\left(\frac{n\theta}{2}\right)}{2^{n-1}\cos^n\left(\frac{\theta}{2}\right)}.$$ [3]

(iii) Given that $\left(\frac{1}{1 + \cos\theta + \mathrm{i}\sin\theta}\right)^4 + \left(\frac{1}{1 + \cos\theta - \mathrm{i}\sin\theta}\right)^4 = 1$, find the value of $\cos\theta$. [3]

(2017 DHS / JC2 / MYE P1 / Q9)

19 **(a)** Solve $\left(1-z\right)^{6}+\left(1+z\right)^{6}=0$, giving your answers in the form $a\,\mathrm{i}\tan b\pi$, where a and b are real constants to be determined, $0\le b\le 1$. [6]

(b) On an Argand diagram, sketch the locus of points of z given by

$$\left|z-k-\mathrm{i}\right|=\left|\sqrt{3}+\mathrm{i}\right|,\ k\in\mathbb{R};$$

showing the axial intercepts on the real axis.

On the same Argand diagram, sketch the locus of points of w given by

$$\arg(w+1)=\arg(\sqrt{3}+\mathrm{i}).$$ [3]

Hence find the range of values of k such that the two loci intersect at exactly one point. [3]

(2017 HCI / JC2 / MYE P1 / Q6)