2023 YIJC JC1 H2MA Promotional Examination

1 In a mathematics competition, each participant is required to answer 50 questions. For each question, a score of a is awarded for a correct answer, a penalty of b is given for a wrong answer, and a penalty of c is given if the question is unattempted. The following table shows a breakdown of the answers submitted by five participants and some of their final scores.

	Number of correct answers	Number of wrong answers	Number of unattempted questions	Final score
Alexander	24	13	13	33
Isabella	18	12	20	14
Maximilian	17	10	?	?
Rosendale	12	20	18	–23
Seraphina	25	10	15	?

Find the final scores obtained by Maximilian and Seraphina. [4]

2 Find the equations of the tangents to the curve $3x^2-2xy-4y+y^2=2$ that are parallel to the y-axis. [4]

3 **(a)** Without using a calculator, solve the inequality $\dfrac{7x-2}{x+1}\geq 2x$. [4]

(b) Hence solve the inequality $\dfrac{-7\sqrt{x}-2}{1-\sqrt{x}}\geq -2\sqrt{x}$. [2]

4 **(a)** Verify that $\dfrac{1}{2r-1}-\dfrac{1}{2r+1}=\dfrac{2}{4r^2-1}$. [1]

(b) The sum $\displaystyle\sum_{r=1}^{n}\frac{4}{4r^2-1}$ is denoted by S_n. Using the method of differences, find an expression for S_n in terms of n. [3]

(c) Using your answer in part **(b)**, find an expression for $\displaystyle\sum_{r=-2}^{n-1}\frac{4}{(2r+5)(2r+7)}$ [2]

5 (a) Find $\int \frac{1}{x}(\ln x)^3 \, dx$. [1]

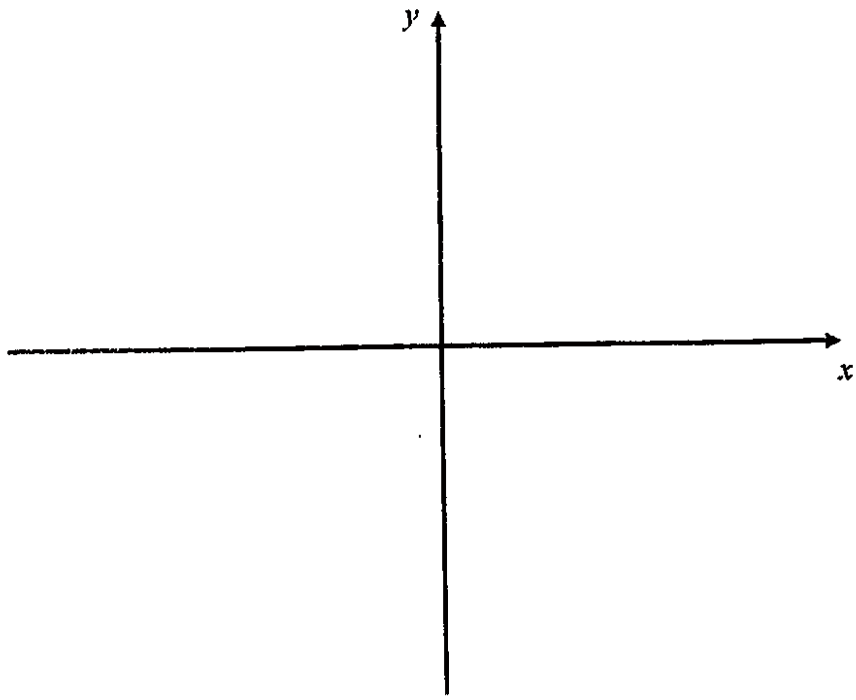
(b) Find $\int \frac{x^2}{1+x^3} \, dx$. [2]

(c) Find $\int \frac{e^{\cot 2\theta}}{2\sin^2 2\theta} \, d\theta$. [2]

6 The curve C has equation $y = \frac{x^2 - 4x + 15}{x+3}$.

(a) Use an algebraic method to find the range of values of y. [3]

(b) Sketch C on the axes. You should state the equations of the asymptotes and the coordinates of any point(s) of intersection of C with the axes. [3]



(c) On the same axes in part **(b)**, sketch the graph of $y = -|10 - 2x|$. Hence, solve the inequality

$$\frac{x^2 - 4x + 15}{x+3} \le -|10 - 2x|.$$ [3]

7 A curve C has parametric equations

$$x = t^2 - 4t \ , \ y = t^3 - 6t .$$

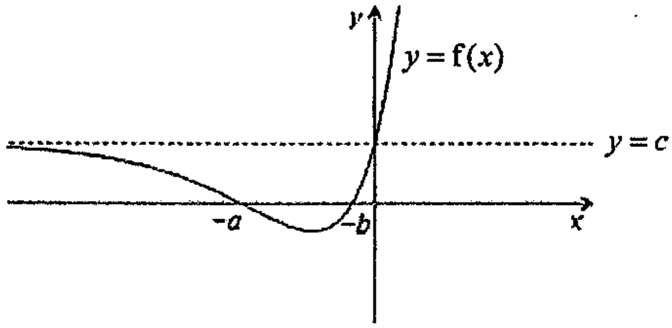
(a) Using calculus, find the equation of the normal at the point P where $t = 1$. [4]

(b) The normal at P meets the curve again at two other points. Show that t satisfies the equation $3t^3 + 2t^2 - 26t + 21 = 0$ at these two points and hence find the coordinates of the points. [3]

8 **(a)** The curve with equation $y = \mathrm{f}(x)$ is transformed by a translation of 2 units in the negative x-direction, followed by a scaling with scale factor 3 parallel to the x-axis, and followed by a translation of 4 units in the positive y-direction. The equation of the resulting curve is $y = 2e^{x^2-36}$.

Find the equation of the curve $y = \mathrm{f}(x)$. [3]

(b) The diagram shows the graph of $y = \mathrm{f}(x)$ with a horizontal asymptote $y = c$ and the curve meets the axes at $(-a, 0)$, $(-b, 0)$ and $(0, c)$.



(i) Sketch the graph of $y = 2\mathrm{f}(-x)$. [2]

(ii) Sketch the graph of $y = \dfrac{1}{\mathrm{f}(x)}$. [2]

(iii) Sketch the graph of $y = \mathrm{f}'(x)$. [2]

9 The function f is defined by

$$\mathrm{f} : x \mapsto 3 + 2x - x^2, \quad x \in \quad, x > 1.$$

(a) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(b) On the same axes, sketch the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{f}^{-1}\mathrm{f}(x)$. [3]

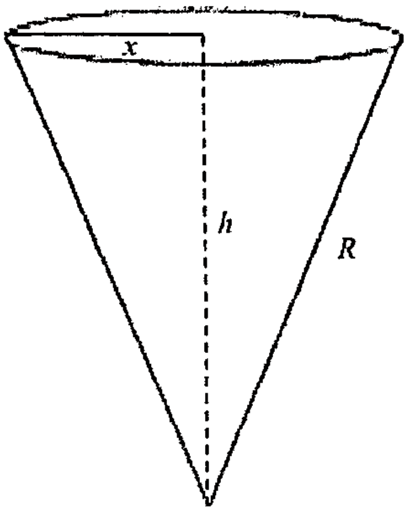
(c) Find the exact value of x that satisfies the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [2]

Another function g is defined by

$$\mathrm{g} : x \mapsto \sqrt{4 - x}, \quad x \in \quad, x \le 4.$$

(d) Explain why the composite function gf exists and find $\mathrm{gf}(x)$, simplifying your answer. [3]

10 [It is given that the volume of a cone of base radius r and height h is $V = \frac{1}{3}\pi r^2 h$.]



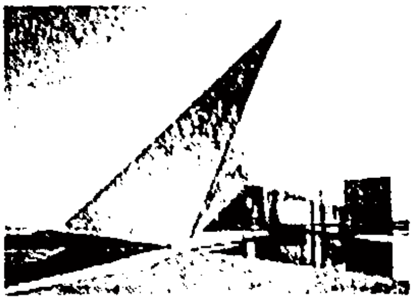
The diagram shows a water tank modelled as a right circular cone with a slant edge of fixed length R m, height h m and radius x m.

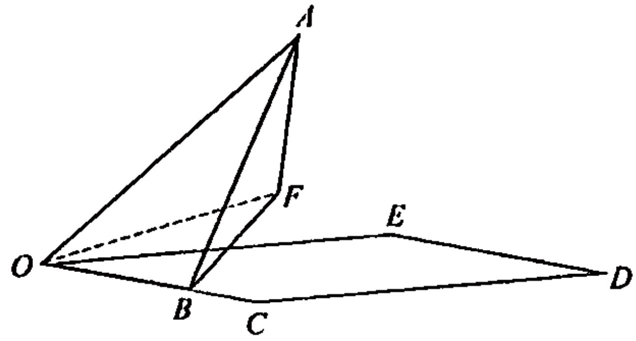
(a) Find an expression for the volume of the cone V, in terms of R and x only. [1]

(b) Show that $9V\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right) = \pi^2 x^3\left(2R^2 - 3x^2\right)$. Find the maximum value of V in terms of R. You do not need to show that this value is a maximum. [6]

(c) It is known instead that $x = \frac{h}{2}$. The tank is being filled with water at the rate of $2\,\mathrm{m^3 s^{-1}}$. Find the rate of increase of the depth of water at the instant when the volume of water in the tank is $8\,\mathrm{m}^3$. [4]

11





Source of picture: https://eventcomm.com/projects/oman-across-ages-museum

The picture on the left features the Oman Across Ages Museum located in Nizwa, Oman. The diagram on the right shows a 3-dimensional representation depicting part of the museum. The slanted rooftop is the face OAB, where O is the origin and the points O, B, C, D and E lie on the horizonal ground. Points (x, y, z) are defined relative to O where units are metres. The coordinates of A and B may be assumed to be $(30, 15, 25)$ and $(25, -4, 0)$ respectively.

(a) Find a cartesian equation of face OAB. [3]

(b) Find the acute angle between the right edge of the rooftop, AB, and the horizontal ground. [2]

It is given that the coordinates of F are $(20, 12, 8)$.

(c) Tubes of lights are attached to edge AF to form a straight line starting from A to F and extended until it hits the horizontal ground at a point G. Find a vector equation of the line AF and the exact coordinates of G. [4]

(d) Find, correct to the nearest metre, the shortest distance from F to face OAB. [3]

12 **(a)** Given that the sum of the first n terms of a series is S_n, where $S_n = 3n^2 - n$, state the value of its first term and write down an expression for S_{n-1} for $n \geq 2$. Hence show that the terms in the series form an arithmetic progression. [4]

(b) The first term of an infinite geometric progression is a and the common ratio is r, where $|r| < 1$. Given that the sum of all the terms after the nth term is equal to the nth term of the progression, show that $r = \frac{1}{2}$ and hence find the sum to infinity of the progression in terms of a. [4]

(c) A bank has an account for investors. Compound interest is added to the account at the end of each year at a fixed rate of 3% of the amount in the account at the beginning of that year.

Mr Ang decides to invest \$2 000 on 1 January 2023 and a further \$2 000 on 1 January of each subsequent year.

(i) How much compound interest has his **original** \$2 000 earned at the end of 31 December 2030? [2]

(ii) By considering the total amount in the account at the end of the nth year, find the year in which the total in the account will first exceed \$40 000. Explain whether this occurs at the beginning or the end of the year. [6]

2023 YIJC JC1 H2MA Promotional Examination Solutions

	Solution
1	$24a-13b-13c=33$ $18a-12b-20c=14$ $12a-20b-18c=-23$ Using GC, $a=3,\ b=2.5,\ c=0.5,$ Seraphina's final score $=25\times3+15\times-0.5+10\times-2.5$ $=42.5$ Number of Maximilian's blank answers $=50-17-10$ $=23$ Maximilian's final score $=17\times3+23\times(-0.5)+10\times(-2.5)$ $=14.5$ **Alternative solution (negative *b* and *c*)** $24a+13b+13c=33$ $18a+12b+20c=14$ $12a+20b+18c=-23$ Using GC, $a=3,\ b=-2.5,\ c=-0.5$ Seraphina's final score $=25\times3+15\times(-0.5)+10\times(-2.5)$ $=42.5$ Number of Maximilian's blank answers $=50-17-10$ $=23$ Maximilian's final score $=17\times3+23\times(-0.5)+10\times(-2.5)$ $=14.5$

2

$3x^2-2xy-4y+y^2=2$

$$6x-2x\frac{dy}{dx}-2y-4\frac{dy}{dx}+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}(2y-2x-4)=2y-6x$$

$$\frac{dy}{dx}=\frac{y-3x}{y-x-2}$$

For tangent parallel to y-axis, $y-x-2=0$

$$y=x+2$$

Substitute $y=x+2$ into the original equation

$$3x^2-2x(x+2)-4(x+2)+(x+2)^2=2$$

$$3x^2-2x^2-4x-4x-8+x^2+4x+4=2$$

$$2x^2-4x-6=0$$

$x=-1$ or $x=3$

The equations of tangents are $x=-1$ or $x=3$

3(a)

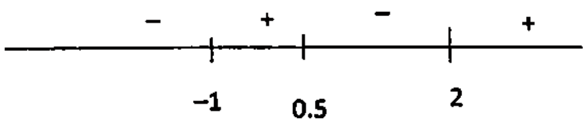
$$\frac{7x-2}{x+1}\geq 2x$$

$$\frac{7x-2-2x(x+1)}{x+1}\geq 0$$

$$\frac{-2x^2+5x-2}{x+1}\geq 0$$

$$\frac{2x^2-5x+2}{x+1}\leq 0$$

$$\frac{(2x-1)(x-2)}{x+1}\leq 0$$



$x<-1$ or $0.5\leq x\leq 2$

(b)	$\dfrac{-7\sqrt{x}-2}{1-\sqrt{x}} \geq -2\sqrt{x}$ Replacing x with $-\sqrt{x}$ in previous result, $-\sqrt{x} < -1$ *or* $0.5 \leq -\sqrt{x} \leq 2$ $\sqrt{x} > 1$ (Rejected since $\sqrt{x} \geq 0 \forall x \in \square$) $x > 1$
4(a)	$\dfrac{1}{2r-1} - \dfrac{1}{2r+1} = \dfrac{2r+1-(2r-1)}{(2r-1)(2r+1)}$ $= \dfrac{2}{(2r)^2 - 1}$ $= \dfrac{2}{4r^2 - 1}$ (verified)
4(b)	$S_n = \sum_{r=1}^{n} \dfrac{4}{4r^2-1}$ $= 2\sum_{r=1}^{n} \dfrac{2}{4r^2-1}$ $= 2\sum_{r=1}^{n} \left(\dfrac{1}{2r-1} - \dfrac{1}{2r+1}\right)$ $= 2\Big(\dfrac{1}{1} - \dfrac{1}{3}$ $+ \dfrac{1}{3} - \dfrac{1}{5}$ $+ \dfrac{1}{5} - \dfrac{1}{7}$ $+ \ldots$ $+ \dfrac{1}{2n-3} - \dfrac{1}{2n-1}\Big)$ $+ \dfrac{1}{2n-1} - \dfrac{1}{2n+1}\Big)$ $= 2\left(1 - \dfrac{1}{2n+1}\right)$

<table>
<tr><td>4(c)</td><td>

$$\sum_{r=-2}^{n-1}\frac{4}{(2r+5)(2r+7)}$$
$$=\frac{4}{(1)(3)}+\frac{4}{(3)(5)}+\frac{4}{(5)(7)}+\cdots+\frac{4}{(2n+3)(2n+5)}$$
$$=\sum_{r=1}^{n+2}\frac{4}{(2r-1)(2r+1)}$$
$$=2\left(1-\frac{1}{2(n+2)+1}\right)$$
$$=2-\frac{2}{2n+5}$$

<u>Alternative method (replacement)</u>

$$\sum_{r=-2}^{n-1}\frac{4}{(2r+5)(2r+7)}$$
$$=\sum_{r=-2}^{n-1}\frac{4}{(2(r+3)-1)(2(r+3)+1)}$$
$$=\sum_{j=1}^{n+2}\frac{4}{(2j-1)(2j+1)}, \text{using } j=r+3$$
$$=2\left(1-\frac{1}{2(n+2)+1}\right)$$
$$=2-\frac{2}{2n+5}$$

</td></tr>
<tr><td>5(a)</td><td>

$$\int\frac{1}{x}(\ln x)^3\,dx=\frac{1}{4}(\ln x)^4+C$$

</td></tr>
<tr><td>5(b)</td><td>

$$\int\frac{x^2}{1+x^3}\,dx=\frac{1}{3}\int\frac{3x^2}{1+x^3}\,dx$$
$$=\frac{1}{3}\ln\left|1+x^3\right|+C$$

</td></tr>
<tr><td>5(c)</td><td>

$$\int\frac{e^{\cot 2\theta}}{2\sin^2 2\theta}\,d\theta=-\frac{1}{4}\int-2\operatorname{cosec}^2 2\theta\, e^{\cot 2\theta}\,d\theta$$
$$=-\frac{1}{4}e^{\cot 2\theta}+C$$

</td></tr>
</table>

6(a)

$$y=\frac{x^2-4x+15}{x+3}$$

$$xy+3y=x^2-4x+15$$

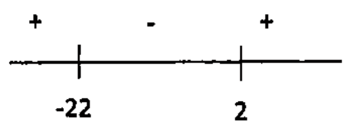
$$x^2-(4+y)x+(15-3y)=0$$

$$b^2-4ac\geq 0$$

$$(4+y)^2-4(15-3y)\geq 0$$

$$y^2+20y-44\geq 0$$

$$(y+22)(y-2)\geq 0$$



$y\leq -22$ or $y\geq 2$

6(b)

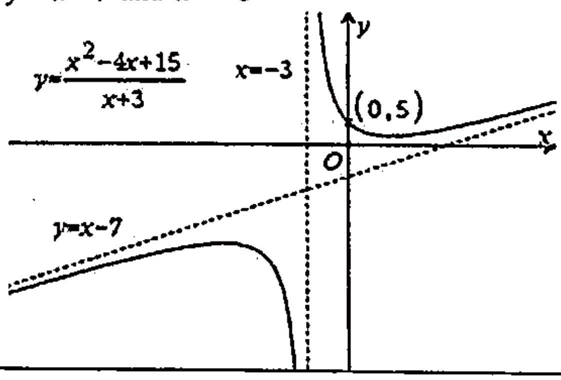
$$y=\frac{x^2-4x+15}{x+3}$$

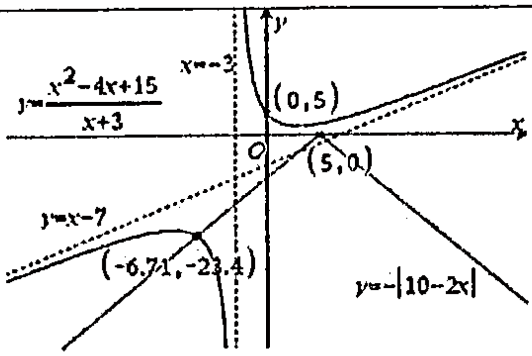
$$=\frac{(x+3)(x-7)+36}{x+3}$$

$$=x-7+\frac{36}{x+3}$$

Equations of asymptotes are

$y=x-7$ and $x=-3$



6(c)	 From the graph, $\frac{x^2-4x+15}{x+3} \le -\|10-2x\|$ $-6.71 \le x < -3$
7(a)	$\frac{dy}{dt} = 3t^2 - 6, \quad \frac{dx}{dt} = 2t - 4$ $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$ $= \frac{3t^2-6}{2t-4}$ At P, $t=1$ and $x=-3$, $y=-5$, $\frac{dy}{dx} = \frac{3}{2}$ gradient of normal at $P = -\frac{2}{3}$ Equation of normal at P: $y-(-5) = -\frac{2}{3}(x-(-3))$ $y+5 = -\frac{2}{3}(x+3)$ $y = -\frac{2}{3}x - 7$

7(b)	$x = t^2 - 4t$, $y = t^3 - 6t$ $y = -\frac{2}{3}x - 7$ ----- (*) Substitute the parametric equations into (*) $t^3 - 6t = -\frac{2}{3}\left(t^2 - 4t\right) - 7$ $-3t^3 + 18t = 2t^2 - 8t + 21$ $3t^3 + 2t^2 - 26t + 21 = 0$ (shown) Using GC, $t = -3.6072$ or $t = 1.9406$ or $t = 1$ When $t = -3.6072$ $x = 27.441, y = -25.294$ When $t = 1.9406$ $x = -3.9965, y = -4.3357$ The coordinates of the points are $(27.4, -25.3)$ and $(-4.00, -4.34)$
8 (a)	**Method 1 (Backward method)** $y = 2e^{x^2-36} \xrightarrow[\text{Undo } C]{} y = 2e^{x^2-36} - 4$ $\xrightarrow[\text{Undo } B]{} y = 2e^{(3x)^2-36} - 4$ $= 2e^{9x^2-36} - 4$ $\xrightarrow[\text{Undo } A]{} y = 2e^{9(x-2)^2-36} - 4$ $= 2e^{9x^2-36x} - 4$ $= 2e^{9x(x-4)} - 4$ Statements for student reference. Undo C: Translation of 4 units in the negative y-direction Undo B: Scaling of scale factor $\frac{1}{3}$ parallel to x-axis Undo A: Translation of 2 units in the positive x-direction

Method 2 (Forward method)

$$f(x) \to f(x+2) \to f\left(\frac{1}{3}x+2\right) \to f\left(\frac{1}{3}x+2\right)+4$$

$$f\left(\frac{1}{3}x+2\right)+4 = 2e^{x^2-36}$$

$$f\left(\frac{1}{3}x+2\right) = 2e^{x^2-36}-4$$

Let $m = \frac{1}{3}x+2$

$$x = 3(m-2)$$

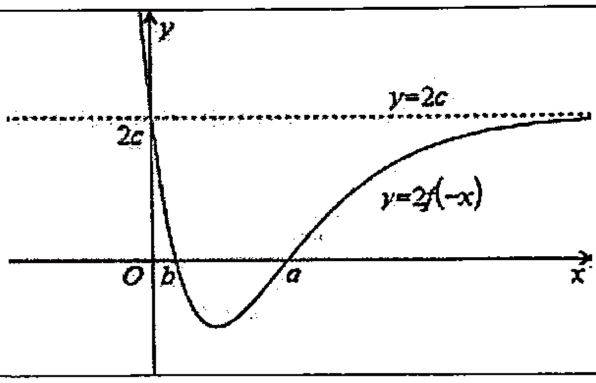
$$f(m) = 2e^{9(m-2)^2-36}-4$$

i.e. $f(x) = 2e^{9(x-2)^2-36}-4$

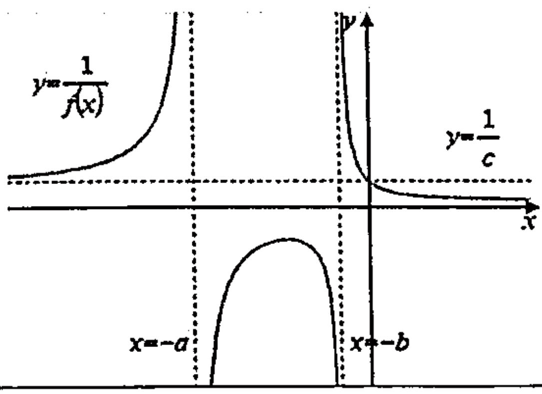
$$= 2e^{9x^2-36x}-4$$

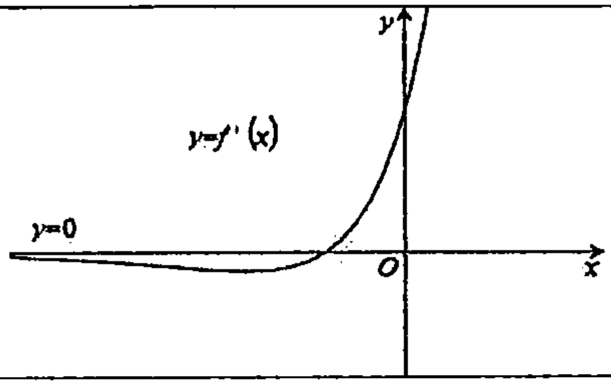
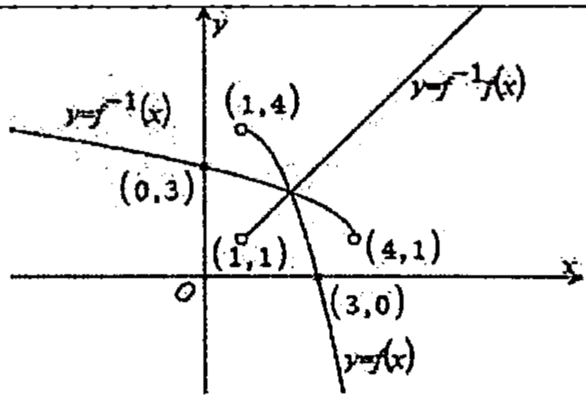
$$= 2e^{9x(x-4)}-4$$

8(b) (i)



8(b) (ii)



8(b) (iii)	
9(a)	Let $y = 3 + 2x - x^2$ $y = -(x^2 - 2x) + 3$ $= -[(x-1)^2 - 1] + 3$ $= 4 - (x-1)^2$ $(x-1)^2 = 4 - y$ $x - 1 = \pm\sqrt{4-y}$ Since $x > 1$, $x = 1 + \sqrt{4-y}$ $f^{-1}(x) = 1 + \sqrt{4-x}$ $D_{f^{-1}} = R_f = (-\infty,\ 4)$
9 (b)	

9 (c)	From the graph, the root of $f(x) = f^{-1}(x)$ is the same as the root of $f(x) = x$. $3 + 2x - x^2 = x$ $x^2 - x - 3 = 0$ $x = \frac{1 \pm \sqrt{1-4(1)(-3)}}{2}$ $= \frac{1 \pm \sqrt{13}}{2}$ Since $x > 1$, reject $x = \frac{1-\sqrt{13}}{2}$ $\therefore x = \frac{1+\sqrt{13}}{2}$
9 (d)	$g(x) = \sqrt{4-x}, \quad x \le 4$ $D_g = (-\infty, 4]$ $R_f = (-\infty, 4) \subseteq D_g$ Thus gf exists. $gf(x) = g(3+2x-x^2)$ $= \sqrt{4-(3+2x-x^2)}$ $= \sqrt{x^2-2x+1}$ $= \sqrt{(x-1)^2}$ $= \lvert x-1 \rvert$ $= x-1$ since $x > 1$.
10(a)	$h = \sqrt{R^2 - x^2}$ $V = \frac{1}{3}\pi x^2 h$ $= \frac{1}{3}\pi x^2 \sqrt{R^2 - x^2}$

10(b)	**<u>Method 1: Square both sides</u>**

$$9V^2 = \pi^2 x^4\left(R^2 - x^2\right)$$
$$= \pi^2\left(R^2x^4 - x^6\right)$$

Using implicit differentiation,

$$18V\frac{\mathrm{d}V}{\mathrm{d}x} = \pi^2\left(4R^2x^3 - 6x^5\right)$$
$$= \pi^2 x^3\left(4R^2 - 6x^2\right)$$
$$9V\frac{\mathrm{d}V}{\mathrm{d}x} = \pi^2\left(2R^2x^3 - 3x^5\right)$$

<u>Method 2: Direct differentiation</u>

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}\pi\sqrt{R^2x^4 - x^6}\right)$$
$$= \frac{\pi}{6}\times\left(R^2x^4 - x^6\right)^{-\frac{1}{2}}\times\left(4R^2x^3 - 6x^5\right)$$
$$= \frac{\pi\left(4R^2x^3 - 6x^5\right)}{6\sqrt{\left(R^2x^4 - x^6\right)}}$$
$$= \frac{\pi x^3\left(2R^2 - 3x^2\right)}{3x^2\sqrt{\left(R^2 - x^2\right)}}$$
$$= \frac{\pi x^3\left(2R^2 - 3x^2\right)}{3\left(\frac{3V}{\pi}\right)}$$
$$3\left(\frac{3V}{\pi}\right)\frac{\mathrm{d}V}{\mathrm{d}x} = \pi x^3\left(2R^2 - 3x^2\right)$$
$$9V\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right) = \pi^2 x^3\left(2R^2 - 3x^2\right) \text{ ------(*)}$$

Method 3: Using product rule

$$\frac{dV}{dx}=\frac{d}{dx}\left(\frac{1}{3}\pi x^2\sqrt{R^2-x^2}\right)$$

$$=\frac{2}{3}\pi x\sqrt{R^2-x^2}+\frac{1}{2}\times\frac{1}{3}\pi x^2\times\frac{-2x}{\sqrt{R^2-x^2}}$$

$$=\frac{2}{3}\pi x\sqrt{R^2-x^2}-\frac{\pi x^3}{3\sqrt{R^2-x^2}}$$

$$=\frac{2\pi x\left(R^2-x^2\right)-\pi x^3}{3\sqrt{R^2-x^2}}$$

$$=\frac{2\pi xR^2-3\pi x^3}{3\sqrt{R^2-x^2}}$$

$$=\frac{\pi x\left(2R^2-3x^2\right)}{3\sqrt{R^2-x^2}}$$

$$9V\frac{dV}{dx}=9\left(\frac{1}{3}\pi x^2\sqrt{R^2-x^2}\right)\times\frac{\pi x\left(2R^2-3x^2\right)}{3\sqrt{R^2-x^2}}$$

$$=\pi^2x^3\left(2R^2-3x^2\right)------(*)$$

When the volume is maximum, $\frac{dV}{dx}=0$

$4R^2x^3-6x^5=0$

$2x^3\left(2R^2-3x^2\right)=0$

$-4x^2+2R^2=0$ or $x=0$ (reject $\because x>0$)

$x^2=\frac{2}{3}R^2$

Maximum $V=\frac{1}{3}\pi\left(\frac{2}{3}R^2\right)\sqrt{R^2-\left(\frac{2}{3}R^2\right)}$

$$=\frac{2}{9}\pi R^2\sqrt{\frac{1}{3}R^2}$$

$$=\frac{2}{9\sqrt{3}}\pi R^3$$

10(c)	Let r, H, W be the radius of water surface, depth of water and volume of water respectively. By similar triangles, $r=\frac{H}{2}$ $W=\frac{1}{3}\pi r^2 H$ $=\frac{1}{3}\pi\left(\frac{H}{2}\right)^2 H$ $=\frac{1}{12}\pi H^3$ When $W=8$, $H=\left(\frac{8}{\frac{1}{12}\pi}\right)^{1/3}=\left(\frac{96}{\pi}\right)^{1/3}$ $\frac{\mathrm{d}W}{\mathrm{d}H}=\frac{1}{4}\pi H^2$ Since $\frac{\mathrm{d}W}{\mathrm{d}t}=2$, $\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{\mathrm{d}W}{\mathrm{d}t}\times\frac{\mathrm{d}H}{\mathrm{d}W}$ $=2\times\frac{1}{\frac{1}{4}\pi\left(\frac{96}{\pi}\right)^{2/3}}$ $=0.26053$ m/s The rate of increase of depth of water is 0.26053 m/s.
11 (a)	$\vec{OA}=\begin{pmatrix}30\\15\\25\end{pmatrix}$, $\vec{OB}=\begin{pmatrix}25\\-4\\0\end{pmatrix}$ $\vec{OA}\times\vec{OB}=5\begin{pmatrix}6\\3\\5\end{pmatrix}\times\begin{pmatrix}25\\-4\\0\end{pmatrix}=5\begin{pmatrix}20\\125\\-99\end{pmatrix}$ Face OAB: $\underset{\sim}{r}\cdot\begin{pmatrix}20\\125\\-99\end{pmatrix}=0$ *OR* $\underset{\sim}{r}\cdot\begin{pmatrix}100\\625\\-495\end{pmatrix}=0$ $20x+125y-99z=0$ or $100x+625y-495z=0$

11 (b)	$\overrightarrow{AB}=\begin{pmatrix}25\\-4\\0\end{pmatrix}-\begin{pmatrix}30\\15\\25\end{pmatrix}=\begin{pmatrix}-5\\-19\\-25\end{pmatrix}$ The normal of the horizontal ground is $\begin{pmatrix}0\\0\\1\end{pmatrix}$ Let θ be the required acute angle. $\sin\theta=\dfrac{\left\lvert\begin{pmatrix}-5\\-19\\-25\end{pmatrix}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert}{\sqrt{5^2+19^2+25^2}\,(1)}$ $=\dfrac{25}{\sqrt{1011}}$ $\theta\approx 51.8^\circ$
11 (c)	$\overrightarrow{AF}=\begin{pmatrix}20\\12\\8\end{pmatrix}-\begin{pmatrix}30\\15\\25\end{pmatrix}=\begin{pmatrix}-10\\-3\\-17\end{pmatrix}$ Line AF: $\underset{\sim}{r}=\begin{pmatrix}30\\15\\25\end{pmatrix}+\lambda\begin{pmatrix}10\\3\\17\end{pmatrix}$, $\lambda\in\Box$ or equivalent such as Line AF: $\underset{\sim}{r}=\begin{pmatrix}20\\12\\8\end{pmatrix}+\mu\begin{pmatrix}10\\3\\17\end{pmatrix}$, $\mu\in\Box$ **<u>Method 1 :</u>** Vector equation of the horizontal ground: $\underset{\sim}{r}\Box\begin{pmatrix}0\\0\\1\end{pmatrix}=0$ At G, $\left[\begin{pmatrix}30\\15\\25\end{pmatrix}+\lambda\begin{pmatrix}10\\3\\17\end{pmatrix}\right]\Box\begin{pmatrix}0\\0\\1\end{pmatrix}=0$ $25+17\lambda=0$ $\lambda=-\dfrac{25}{17}$

	$\vec{OG}=\begin{pmatrix}30\\15\\25\end{pmatrix}-\frac{25}{17}\begin{pmatrix}10\\3\\17\end{pmatrix}=\frac{1}{17}\begin{pmatrix}260\\180\\0\end{pmatrix}$ Coordinates of G are $\left(\frac{260}{17},\frac{180}{17},0\right)$ **<u>Method 2 :</u>** When line AF intersects the horizontal ground at G, $z=0$. At G, $25+17\lambda=0$ $\lambda=-\frac{25}{17}$ $\vec{OG}=\begin{pmatrix}30\\15\\25\end{pmatrix}-\frac{25}{17}\begin{pmatrix}10\\3\\17\end{pmatrix}=\frac{1}{17}\begin{pmatrix}260\\180\\0\end{pmatrix}$ Coordinates of G are $\left(\frac{260}{17},\frac{180}{17},0\right)$
11 (d)	**<u>Method 1: Using length of projection</u>** Perpendicular distance from F to face OAB = Length of projection of $\vec{OF}$ onto normal $=\frac{\left\|\begin{pmatrix}20\\12\\8\end{pmatrix}\cdot\begin{pmatrix}20\\125\\-99\end{pmatrix}\right\|}{\sqrt{20^2+125^2+99^2}}$ $=\frac{1108}{\sqrt{25826}}$ ≈ 6.89463 $=7\text{ m}$ (correct to the nearest m) (can also be length of projection of $\vec{AF}$ or $\vec{BF}$ onto the normal to plane OAB) **<u>Method 2: Using distance between parallel planes</u>** Vector equation of plane parallel to OAB passing through F is $\underset{\sim}{r}\cdot\begin{pmatrix}20\\125\\-99\end{pmatrix}=\begin{pmatrix}20\\12\\8\end{pmatrix}\cdot\begin{pmatrix}20\\125\\-99\end{pmatrix}=1108$

	Shortest distance from F to OAB $=\dfrac{\lvert 1108-0\rvert}{\sqrt{20^2+125^2+99^2}}$ $=\dfrac{1108}{\sqrt{25826}}$ ≈ 6.89463 $=7\text{ m}$ (correct to the nearest m)
12(a)	$S_n=3n^2-n$ $u_1=S_1=2$ $S_{n-1}=3(n-1)^2-(n-1)$ $=3n^2-6n+3-n+1$ $=3n^2-7n+4$ $u_n=S_n-S_{n-1}=6n-4$ $u_{n-1}=6(n-1)-4=6n-10$ $u_n-u_{n-1}=-4+10=6,$ a constant. Thus the series is an AP.
12(b)	$u_n=ar^{n-1},\ u_{n+1}=ar^n$ Sum to infinity from the term after u_n is $\dfrac{ar^n}{1-r}$ (Alternatively it may be expressed as $S_\infty-S_n=\dfrac{a}{1-r}-\dfrac{a(1-r^n)}{1-r}=\dfrac{ar^n}{1-r}$) $\dfrac{ar^n}{1-r}=ar^{n-1} \Rightarrow \dfrac{r}{1-r}=1$ $r=1-r$ $2r=1$ $r=\dfrac{1}{2}$ (Shown) $S_\infty=\dfrac{a}{1-\frac{1}{2}}=2a$
12(c) (i)	At the end of 31 Dec 2030, the compound interest earned $=\$(2000\times1.03^8-2000)$ $=\$533.54$ (2 d.p.)

12(c)(ii)	Let the number of years required be n.

Method 1: Considering how much each \$2000 invested yields at the end of nth year

Total amount in the account

$$\left[2000\times1.03^{n}+2000\times1.03^{n-1}+\cdots+2000\times1.03\right]>40000$$

$$2000\left(1.03+1.03^{2}+\cdots+1.03^{n}\right)>40000$$

$$2000\left[\frac{1.03\left(1.03^{n}-1\right)}{1.03-1}\right]>40000$$

$$\text{(optional step)}\quad \frac{103}{3}\left(1.03^{n}-1\right)>20$$

Using GC, $n\geq16$

Smallest value of n is 16

The total amount first exceeds \$40 000 in 2038.

When $n=15$ on 31st December 2037,

$$\text{Amount}=2000\left[\frac{1.03\left(1.03^{15}-1\right)}{1.03-1}\right]$$

$$=38313.76$$

On 1 January 2038,

$38313.76+2000=40313.76>40000$

This occurs at the beginning of the year.

Method 2: Considering amount at the end of each year

Yr	Amt on 1 Jan (\$)	Amt on 31 Dec (\$)
1	\$2000	2000×1.03
2	$2000\times1.03+2000$	$2000\times1.03^{2}+2000\times1.03$
3	$2000\times1.03^{2}+2000\times1.03+2000$	$2000\times1.03^{3}+2000\times1.03^{2}+2000\times1.03$
⋮	⋮	⋮
n		$2000\times1.03^{n}+2000\times1.03^{n-1}+\cdots+2000\times1.03^{2}+2000\times1.03$

$$2000\times1.03^{n}+2000\times1.03^{n-1}+\cdots+2000\times1.03>40000$$

$$2000\left(1.03+1.03^{2}+\cdots+1.03^{n}\right)>40000$$

$$2000\left[\frac{1.03\left(1.03^{n}-1\right)}{1.03-1}\right]>40000$$

$$\text{(optional step)}\quad \frac{103}{3}\left(1.03^{n}-1\right)>20$$

Using GC, $n\geq16$

Smallest value of n is 16

The total amount first exceeds \$40 000 in 2038.

	When $n=15$ on 31st December 2037, $\text{Amount} = 2000\left[\frac{1.03(1.03^{15}-1)}{1.03-1}\right]$ $=38313.76$ On 1 January 2038, $38313.76+2000=40313.76>40000$ This occurs at the beginning of the year.